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A generalized Vlasov theory for composite beams

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Abstract

A generalized Vlasov theory for composite beams with arbitrary geometric and material sectional properties is developed based on the variational asymptotic beam sectional analysis. Instead of invoking *ad hoc* kinematic assumptions, the variational-asymptotic method is used to rigorously split the geometrically-nonlinear, three-dimensional elasticity problem into a linear, two-dimensional, cross-sectional analysis and a nonlinear, one-dimensional, beam analysis. The developed theory is implemented into VABS, a general-purpose, finite-element based beam cross-sectional analysis code. Several problems are studied to compare the present theory with published results and a commercial three-dimensional finite element code. The present work focuses on the issues concerning the use of the Vlasov correction in the context of the accuracy of the resulting beam theory. The systematic comparison with three-dimensional finite element analysis results helps to quantitatively demonstrate both the advantages and limitations of the Vlasov theory.

Keywords: Vlasov; Composite beams; Asymptotic method; Thin-walled; VABS

1. Introduction

For thin-walled beams with open sections it is well known that the classical beam theory, which relies on four generalized strain measures associated with stretching of the reference line (γ_{11}), twist (κ_1), and bending in two mutually orthogonal directions (κ_2 and κ_3), does not suffice; and a refined beam theory becomes necessary. There are several ways

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to explain such a phenomenon. Perhaps the most revealing one is from the standpoint of the St. Venant principle. One implication of the principle for beams is that any two statically equivalent systems of forces at the end of a long beam provide practically identical stress distributions far away from that end. More precisely, the difference between the two stress distributions exponentially decays along the axis of the beam. This provides validity for the classical beam theory since its generalized strain measures adequately describe the non-decaying part of the three-dimensional (3-D) elasticity solution. As a result, for static and low-frequency behavior of slender beams that are not thin-walled with open section (what we will call 'regular beams'), classical theory is adequate. Technically speaking, the principle remains valid even for *t*hin-*w*alled, *o*pen-*s*ection (TWOS) beams. However, application of a certain system of forces (usually referred to as the 'bi-moment') at an end of a TWOS beam leads to a deformation mode that decays away from that end much more slowly than any system of forces in a regular beam. This implies that in most practical applications, even for relatively long TWOS beams with slenderness ratios of 50 or more, the importance of this additional decaying mode may remain significant [1].

Engineering theories that adequately model this effect for isotropic beams existed for a good part of the last century. They rely on incorporation of the derivative of twist (κ_1') as an independent generalized strain measure. Most commonly such a refinement is referred to as Vlasov theory [2,3]; however, alternative names, such as Wagner theory [4], are in use as well. Because the resulting governing equation for torsion in Vlasov theory is of fourth order, rather than second order as in the St. Venant treatment of torsion, an additional boundary condition is required at each end of the beam. The geometric form of this boundary condition, *i.e.* specifying $\kappa_1 = 0$ at a boundary, is often referred to as 'restrained warping.' Indeed, within the context of this theory, only warping out of the cross-sectional plane is present and its magnitude is proportional to κ_1. This leads to other common names of such a refinement: 'torsional theory with restrained warping' (as opposed to the St. Venant torsional theory where the warping is free) and 'nonuniform torsion' (as opposed to uniform torsion in the St. Venant case). While those theories were based primarily on engineering intuition, it was later rigorously shown that the slowly decaying deformation mode in question is indeed related to torsion [5,6].

Primarily because the rotorcraft industry uses composite, TWOS beam structures in such parts as bearingless rotor flexbeams, extension of the Vlasov theory to anisotropic beams has attracted significant attention from researchers [7–10]. Such theories construct beam models based on the classical, laminated plate/shell theory in conjunction with the kinematic assumptions that were originally used in Vlasov theory for isotropic beams. In particular, the beam cross-section is assumed to be rigid in its own plane, and the transverse shear strains are neglected [7,8]. As discussed in detail in [6], such assumptions lead to certain contradictions even for isotropic beams, while the consequences are even less predictable for the generally anisotropic case. There exists an alternative approach to constructing thin-walled beam theories that avoids *ad hoc* kinematic assumptions and relies instead on equilibrium equations. This in general leads to more rigorous thin-walled beam theories [11,12]. Some attempts to apply this method to the development of Vlasov theory have been made [10], but the procedure is not straightforward because there are not enough equilibrium equations to solve for all the necessary quantities.

A third approach, and the one advocated by the authors, relies on the direct use of certain small parameters inherent to slender structures, which we would like to model as beams, by applying the variational-asymptotic method (VAM) [13]. This rigorous mathematical procedure does not invoke any *ad hoc* kinematic assumptions, and its results are fully consistent with those from the equilibrium approach where appropriate comparisons can be made. However, it provides more flexibility and allows one to circumvent many of the problems that would be faced in use of the equilibrium approach. The first attempt to apply this method to TWOS beams [14] provided some insights into the nature of the Vlasov phenomenon, but the paper contains certain inconsistencies. These are corrected in later works where a fully consistent theory for TWOS beams was developed using the VAM [6,15]. Therein it was shown that, compared to the classical terms, the Vlasov effect is caused by correction terms of the second order with respect to the beam small parameter $a/\lambda l$, where a and λl are the characteristic cross-sectional dimension and the wavelength of elastic deformation along the beam axis, respectively. However, for TWOS beams these terms become very important due to the presence of the inverse of another small parameter, h/a, where h is the wall thickness. Remarkably, terms involving the inverse of this small parameter are absent when one considers a closed section, which leads to the conclusion that the Vlasov correction is relatively unimportant for slender beams with closed cross-sections.

The TWOS beam theory presented in [6,15] did not include Timoshenko corrections, which are also second-order corrections with respect to $a/\lambda l$. In order to obtain a meaningful Vlasov model, a Timoshenko model has to be first constructed, from which the shear center location is deduced. This is followed by moving the origin to the shear center and finally constructing a generalized Vlasov theory using appropriate terms from the second-order approximation. This procedure can be incorporated into the versatile cross-sectional analysis code VABS (variational asymptotic beam sectional analysis) [16,17] so that the Vlasov model can be constructed for general composite beams with arbitrary geometry shape (not necessarily thin-walled) and materials.

The goal of this paper is to illustrate the procedure of constructing a generalized Vlasov model for composite beams as a component of VABS, the mathematical foundation of which is the VAM, and provide some benchmark results with which other theories can be compared. This task of providing a comprehensive set of benchmark problems is of particular importance due to the false sense of security that often follows when there are relatively few published results that practically all analyses predict accurately. That is, a theory might provide quite reasonable results for these few cases but totally erroneous ones for other (untested) configurations.

2. Construction of a generalized Vlasov model

The Vlasov beam theory can be considered as a truncation of a beam theory that is asymptotically correct to the second order. Hence, the first step of constructing a Vlasov theory requires the determination of the second-order energy, which has been done in

Ref. [17] and can be expressed as

$$2U_1 = \varepsilon^T A \varepsilon + 2\varepsilon^T B \varepsilon' + \varepsilon' T C \varepsilon' + 2\varepsilon^T D \varepsilon'' \tag{1}$$

where A, B, C, and D are matrices carrying the geometry and material information of the cross-section, $\varepsilon = \lfloor \gamma_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3 \rfloor^T$ are the generalized strain measures defined in the classical beam theory, and $()'$ means a derivative with respect to the beam axial coordinate x_1.

A generalized Timoshenko model [18] can be constructed from this energy expression, such that

$$\begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{12} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{2}$$

here $2\gamma_{12}$ and $2\gamma_{13}$ are the generalized strain measures associated with two shear deformations; F_i and M_i, are beam stress resultants and moment measures, expressed in the deformed beam cross-sectional frame basis. (Here and throughout all the paper, Greek indices assume values 2 and 3 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated.) The shear center can be obtained based on the flexibility matrix in Eq. (2) as

$$e_2 = -\frac{S_{34}}{S_{44}} \quad e_3 = \frac{S_{24}}{S_{44}} \tag{3}$$

Finally, the origin is moved to the shear center and the second-order energy, Eq. (1), is sought based on the new coordinate system. In the framework of the Vlasov theory, it is assumed that κ_1' is much larger than the derivatives of the other classical generalized strain measures γ_{11}', κ_2', and κ_3'. By setting the latter quantities to zero, a strain energy expression can be expressed in terms of the five 'degrees of freedom' of the Vlasov beam theory, and a Vlasov model can be constructed as

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \\ M_\omega \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_1' \end{Bmatrix} \tag{4}$$

here M_ω is the bi-moment (conjugate to the κ_1' strain measure). Eq. (4) can be used as input in a one-dimensional (1-D) Vlasov beam analysis to solve for the global deformation, 1-D strain measures, and stress resultants along the beam axis. This generalized Vlasov model is constructed within the framework of VABS [17] to take advantage of its versatility and generality. We will term the model in Eq. (4) as the VABS generalized Vlasov model in the rest of the development.

One of the main applications of the VABS generalized Vlasov model is thus exhibited in Eq. (4). These constitutive relations obtained from VABS can be used as input for various 1-D beam analyses (such as static, dynamic, buckling, aeroelastic, and etc.) considering the restrained warping effects. It is worthy to emphasize that, although VABS casts the strain energy into a form similar to that of Vlasov theory, it does not invoke any kinematic assumptions of this theory as what is usually done in the literature. In fact, the VABS generalized Vlasov theory considers all possible 3-D deformation but still creates a seamless connection to traditional beam theories so that the 1-D beam analysis remains essentially the same. Any general 1-D Vlasov beam analysis that can make use of a fully populated 5×5 stiffness matrix should be able to directly use the VABS generalized Vlasov model, as long as the 1-D analysis uses generalized strain measures equivalent to the standard ones used in Eq. (4) and defined in Ref. [17].

3. One-dimensional Vlasov beam analysis

Having obtained the constitutive model, Eq. (4), corresponding to the Vlasov beam theory, we can derive the governing equations for the 1-D beam analysis. Following [19], we can derive the 1-D beam static equilibrium equations for anisotropic, initially curved and twisted TWOS beams using an intrinsic formulation, such that

$$\begin{aligned}
&F_1' - F_2K_3 + F_3K_2 + f_1 = 0\\
&F_2' - F_3K_1 + F_1K_3 + f_2 = 0\\
&F_3' - F_1K_2 + F_2K_1 + f_3 = 0\\
&M_1' - M_2K_3 + M_3K_2 - M_\omega'' + m_1 = 0\\
&M_2' - M_3K_1 + M_1K_3 - (1+\gamma_{11})F_3 - K_3M_\omega' + m_2 = 0\\
&M_3' - M_1K_2 + M_2K_1 + (1+\gamma_{11})F_2 + K_2M_\omega' + m_3 = 0
\end{aligned} \tag{5}$$

where f_i and m_i are applied distributed forces and moments, respectively, and $K_i=k_i+\kappa_i$ with k_i being the initial twist and curvature measures in the undeformed beam cross-sectional frame basis. Eq. (5) is as general as a geometrically exact nonlinear Vlasov beam theory allows. General analytical solutions of these equations are not known; hence, numerical solutions based on 1-D finite element methods are usually employed.

For later purpose of validating the present theory, we consider only prismatic beams with no distributed applied loads, so that $f_i=m_i=k_i=0$. The load comes in through the boundary conditions, such that $F_1(L)=F_2(L)=F_3(L)=M_2(L)=M_3(L)=M_\omega(L)=0$ and $M_1(L)=T$. The rotation and the warping displacement of the root are restricted so that $\theta_1(0)=\kappa_1(0)=0$. Under these specialized conditions Eq. (5) can be solved once the cross-sectional constants of Eq. (4) are determined. The cross-sectional constants for the strip and the I-beam have analytical solutions from the VAM for the stiffnesses based on the assumption of thin-walled geometry [6,15]. The stiffnesses without the thin-walled assumption must be determined numerically using VABS.

So that an analytical solution can be obtained, let us consider the case of small rotations, so that the equations become linear. For the case of loading outlined above, one finds the torsional equation reducing to

$$M_1' - M_\omega'' = 0 \tag{6}$$

For the case of, say, elastic coupling between κ_1 and κ_2

$$M_1 = C_{22}\kappa_1 + C_{23}\kappa_2; \quad M_\omega = C_{55}\kappa_1' \tag{7}$$

where for the isotropic case C_{22} is typically denoted as GJ and C_{55} as EC. The loading gives rise to zero bending moment, so that $C_{23}\kappa_1 + C_{33}\kappa_2 = 0$, thus allowing κ_2 to be eliminated. Finally, κ_1 is written as θ_1' so that Eq. (6) becomes

$$\frac{d\theta_1}{dx} - \gamma\frac{d^3\theta_1}{dx^3} - \tau = 0 \tag{8}$$

where $x = x_1/L$ and

$$\tau = \frac{TL}{C_{22}\left(1 - \frac{C_{23}^2}{C_{22}C_{33}}\right)}; \quad \gamma = \frac{C_{55}}{L^2C_{22}\left(1 - \frac{C_{23}^2}{C_{22}C_{33}}\right)} \tag{9}$$

the solution of which can be written as

$$\frac{\theta_1}{\tau} = x - \sqrt{\gamma}\sinh\left(\frac{x}{\sqrt{\gamma}}\right) + \sqrt{\gamma}\tanh\left(\frac{1}{\sqrt{\gamma}}\right)\left[\cosh\left(\frac{x}{\sqrt{\gamma}}\right) - 1\right] \tag{10}$$

Although a more comprehensive numerical solution based on Eq. (5) could be used, this simplified analytical solution will be used exclusively for the examples studied in this paper to calculate the 1-D variables from Vlasov analysis. For the anisotropic case, the coupling terms are not zero; hence, deformations other than twist may present.

4. Recovery relations

The uniqueness of the present Vlasov beam theory is that the original 3-D, nonlinear elasticity problem is reduced to a 2-D, linear, cross-sectional analysis and a 1-D, nonlinear, Vlasov beam analysis, which have been dealt with in the above two sections, respectively. To compare the present theory with standard solutions to the 3-D problem, a final step is needed to assemble results from the above two analyses to recover the original 3-D fields, including displacements, stresses, and strains.

Although it is necessary for a Vlasov model to provide accurate results for the various types of beam global behavior (*i.e.*, static deflections, natural frequencies, mode shapes, nonlinear transient behavior, buckling loads, etc.), this is not sufficient. Indeed, it is misleading to focus only on the 1-D behavior, per se, because an insufficiently detailed study of published results may lead one to believe that differences among the various composite beam theories are insignificant. Actually, a composite beam model should be judged by how well it predicts 3-D behavior of the original 3-D structure. Therefore, recovery relations should be provided to complete the modeling. By recovery relations we mean expressions for the 3-D displacements, strains and stresses in terms of 1-D beam quantities and the local cross-sectional coordinates, x_α.

For an initially curved and twisted beam, the warping that is asymptotically correct through the first order of h/R and h/l can be expressed as

$$w(x_1, x_2, x_3) = (V_0 + V_{1R})\varepsilon + V_{1S}\varepsilon' \tag{11}$$

where $w(x_1, x_2, x_3)$ is a column matrix of the components of the 3-D warping functions, V_0, V_{1R}, and V_{1S} are the asymptotically correct warping functions of the zeroth-order approximation, the correction due to initial curvatures/twist, and the refined warping of the order of h/l, respectively. Among the derivatives of components of ε in the generalized Vlasov beam model, κ_1' dominates the others. This allows us to take

$$\varepsilon' = \lfloor 0 \quad \kappa_1' \quad 0 \quad 0 \rfloor^T \tag{12}$$

The recovered 3-D displacement field of the VABS generalized Vlasov theory can be expressed as

$$U_i(x_1, x_2, x_3) = u_i(x_1) + x_\alpha[C_{\alpha i}(x_1) - \delta_{\alpha i}] + C_{ij}w_j(x_1, x_2, x_3) \tag{13}$$

where $U_i(x_1, x_2, x_3)$ are the 3-D displacements, $u_i(x_1)$ are the 1-D beam displacements, $C_{ij}(x_1)$ are components of the direction cosine matrix representing the rotation of the beam cross-sectional triad caused by deformation, and $\delta_{\alpha i}$ is the Kronecker symbol. According to the VAM, a second-order asymptotically correct energy requires the warping field asymptotically correct only through first order, and consequently the 3-D fields can only be recovered through the first order. To recover the 3-D fields that are accurate through second order requires calculation of the second-order warping field, which means additional complexity and computation. Here the 3-D results are recovered based on the first-order warping and all the other information we have. Numerical examples show that such recovery relations yield accurate results without introducing additional computational cost.

The 3-D strain field can be expressed symbolically in terms of the 1-D strain measures and the warping functions obtained in the modeling process as

$$\Gamma = \Gamma_h w + \Gamma_\varepsilon \varepsilon + \Gamma_R w + \Gamma_l w' \tag{14}$$

where Γ is the column matrix representing the 3-D strain components, Γ_h, Γ_ε, Γ_l, and Γ_R are operators that are functions of the cross-sectional geometry, and w is a column matrix of the cross-sectional warping functions, with both in- and out-of-plane components. Let us recall [20] that w actually consists of the warping for the classical approximation and a first-order correction, expressed in terms of ε and ε'. Therefore, once the beam problem is solved and ε is known as a function of the axial coordinate, all the terms in the 3-D strain field are known.

Expressing the solution for the warping in terms of ε and ε', one finds the 3-D strain field to be

$$\Gamma = [(\Gamma_h + \Gamma_R)(V_0 + V_{1R}) + \Gamma_\varepsilon]\varepsilon + [(\Gamma_h + \Gamma_R)V_{1S} + \Gamma_l(V_0 + V_{1R})]\varepsilon' + \Gamma_l V_{1S}\varepsilon'' \tag{15}$$

where the V terms are coefficients of ε and ε' in the warping expresssion, Eq. (11). Here the 3-D strain field is

$$\Gamma = \lfloor \Gamma_{11} \quad 2\Gamma_{12} \quad 2\Gamma_{13} \quad \Gamma_{22} \quad 2\Gamma_{23} \quad \Gamma_{33} \rfloor^T \tag{16}$$

and ε'' is obtained by the derivative of Eq. (12). All the operators in Eq. (15) are defined in [17]. Although it is easier and more convenient to recover the 3-D strain and stress using 1-D stress resultants for the VABS generalized Timoshenko model [18], such an advantage is not found for the Vlasov model because the derivative of stress resultants, particularly the bi-moment M_ω, cannot be expressed explicitly in terms of the stress resultants themselves. For an analytical 1-D beam analysis, the availability of κ_1' and κ_1'' is not an issue. However, if the 1-D Vlasov beam analysis is finite-element based, shape functions of sufficiently high order should be used so that the two differentiations of κ_1 do not introduce large errors.

Finally, the 3-D stress field can be obtained from the 3-D strain field using the 3-D constitutive law.

5. Numerical results

In this section, results obtained from a 1-D beam solution that uses the VABS cross-sectional constants as input are compared with 3-D finite element results for loaded strips and I-beams. The loading considered here is a twisting moment applied at the beam tip, where $x_1 = L$, and all displacement is constrained to be zero at the beam root, where $x_1 = 0$.

5.1. Strips

A strip is a beam with a thin, rectangular cross-section with height h, width b, and length L, and $h \ll b \ll L$. End effects (or boundary-layer effects) have been noted to exist for strips, but their importance for engineering analysis is not generally agreed upon in the literature. Here we consider a strip with $b = 0.953$ in., $h = 0.03$ in., and $L = 10$ in. A schematic of the strip is shown in Fig. 1.

5.1.1. Analysis

The 2-D cross-sectional model for VABS was meshed with 2 elements through the thickness and 60 elements across width. The 3-D model maintained the same mesh for the cross-section as the 2-D model, but with 100 elements along the length, giving a total of

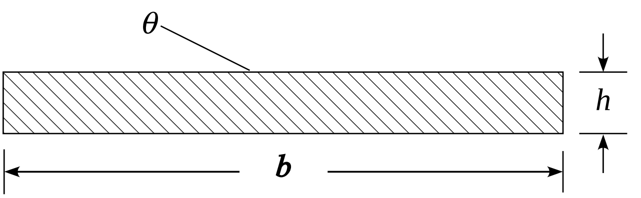

Fig. 1. Schematic of the strip.

Table 1
Sectional properties for isotropic strip

Source	Property	Value	Units
VABS	GJ	32.3418	lb.-in.^{2}
Vlasov	GJ	32.3317	lb.-in.^{2}
Analytical (VAM)	GJ	32.3317	lb.-in.^{2}
VABS	EC	1.78635	lb.-in.^{4}
Vlasov	EC	2.10971	lb.-in.^{4}
Analytical (VAM)	EC	1.78336	lb.-in.^{4}

12,000 elements. The aspect ratio of each element then becomes 6.35:1.06:1 (L:b:h). The strip was loaded with two opposite point loads (F) at each end of the horizontal plane of symmetry of the cross-section, creating a twisting moment of $M_1 = F_b$. Displacements were extracted from the 3-D model and used to compute the rotation angle, which was computed by assuming a rigid-body rotation for each cross-section.

5.1.2. Results

For the isotropic case, we take $E = 10 \times 10^6$ psi and $\nu = 0.3$. Sectional properties are given in Table 1. 1-D results for the isotropic case are shown in Fig. 2 with a blow-up of the portion around the restrained end in Fig. 3. Results indicate reasonable agreement among VABS, the analytical asymptotic results, Vlasov's original theory, and the 3-D finite element results from ABAQUS. The result for the warping rigidity from Vlasov's original theory differs slightly from VABS and the analytical asymptotic theory [15]. Vlasov's original theory gives

$$EC = \frac{Eh^3b^3}{144} \tag{17}$$

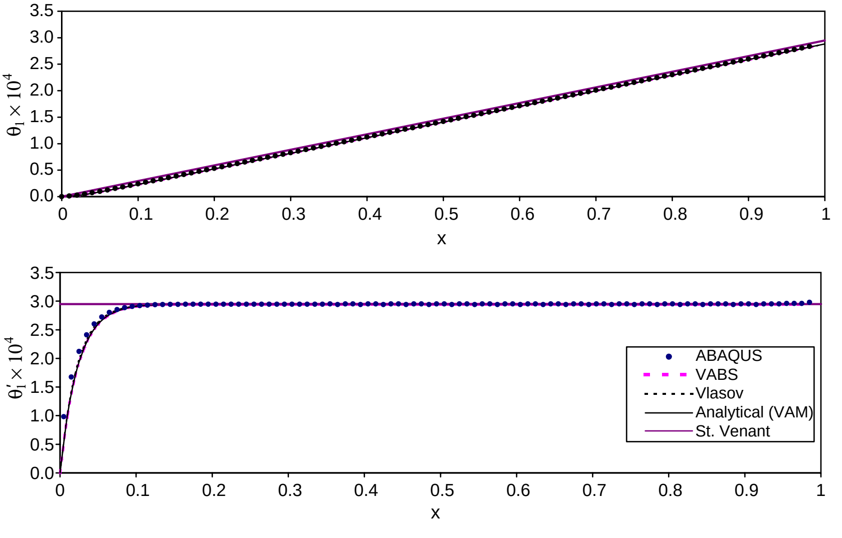

Fig. 2. Twist angle for isotropic strip versus nondimensional axial coordinate.

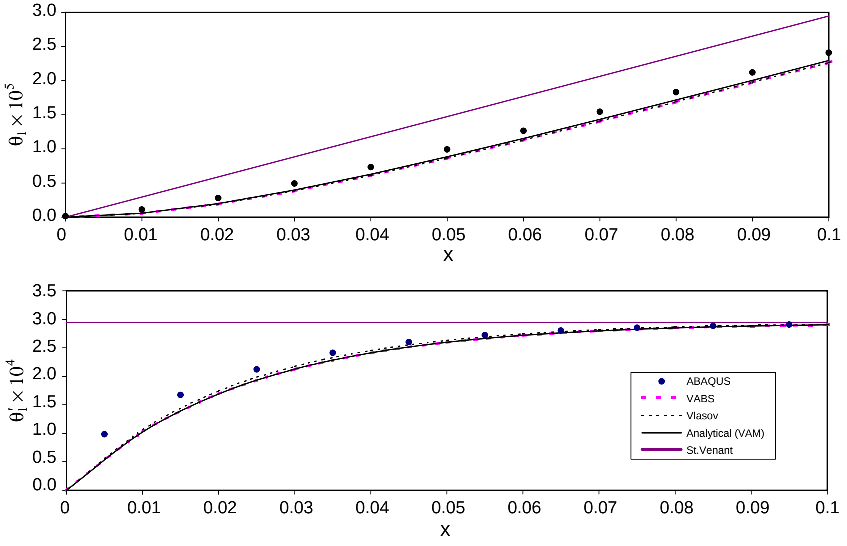

Fig. 3. Twist angle for isotropic strip versus nondimensional axial coordinate—blow-up of end-zone effect.

while the asymptotic result can be found as

$$EC = \frac{Eh^3 b^3}{144(1-\nu^2)} \tag{18}$$

The difference between the asymptotic results and Vlasov's approximation is practically negligible.

For the orthotropic and anisotropic cases, the material properties are given in Table 2. The anisotropic strip has one layer with ply angle $\theta = 15°$, whereas the orthotropic case has $\theta = 0$. Sectional properties are given in Tables 3 and 4. In Table 4 $\bar{C}_{22} = C_{22} - C_{23}^2/C_{33}$ corresponds to the effective torsional rigidity that takes bending-twist coupling into consideration. 1-D results for the orthotropic case are shown in Fig. 4 and for the anisotropic case in Fig. 5. The 1-D results with VABS properties lie right on top of those extracted from ABAQUS 3-D finite element results.

It is interesting to note that strips present a unique class of thin-walled sections that is neither open nor closed. While the Vlasov effect is not as significant as for open sections, the results presented indicate that Vlasov's correction can correctly describe the end effect associated with the torsional deformation.

Table 2
Material properties for anisotropic strip and I-beam

Property	Value
Material properties:	$E_l = 20.59 \times 10^6$ psi
$E_t = 1.42 \times 10^6$ psi	$G_{lt} = 8.7 \times 10^5$ psi
$G_{tn} = 6.96 \times 10^5$ psi	$\nu_{lt} = \nu_{tn} = 0.42$

Table 3
Sectional properties for orthotropic strip

Source	Property	Value	Units
VABS	C_{22}	7.31562	lb.-in.^{2}
VABS	C_{55}	3.39251	lb.-in.^{4}

Table 4
Sectional properties for anisotropic strip

Source	Property	Value	Units
VABS	$\bar{C}_{22}$	7.96547	lb.-in.^{2}
VABS	C_{55}	1.82499	lb.-in.^{4}

5.2. I-Beams

Consider an I-beam with width $a=0.5$ in., thickness $h=0.04$ in., and height $b=1.0$ in. Both isotropic and anisotropic cases are considered. A schematic of the anisotropic I-beam is shown in Fig. 6 and the material properties in Table 2. For those layers that have a variable ply angle, it is chosen to be $\theta=15°$.

5.2.1. Analysis

The I-beam was loaded with two opposite point loads (F) at each end of the vertical plane of symmetry of the cross-section, creating a twisting moment of $M_1=F_{\mathrm{b}}$. Once

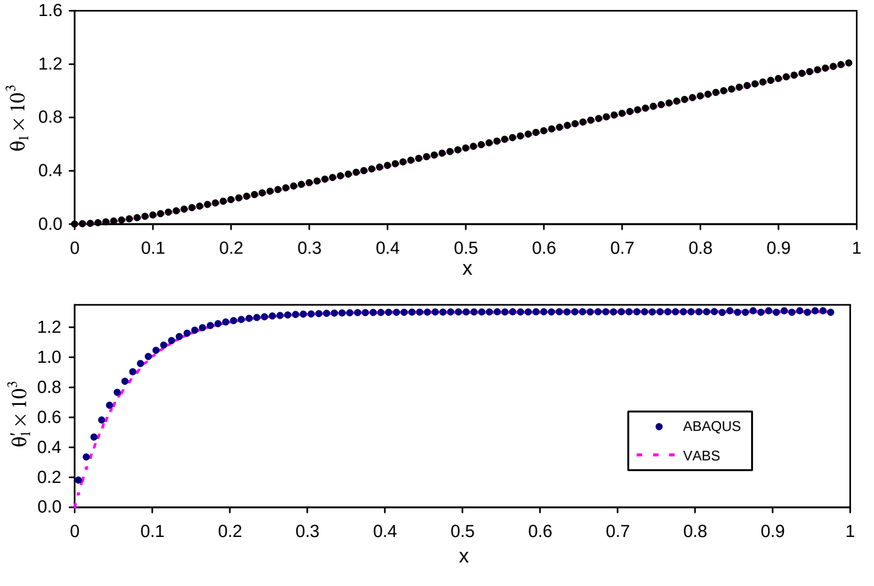

Fig. 4. Twist angle for orthotropic strip versus nondimensional axial coordinate.

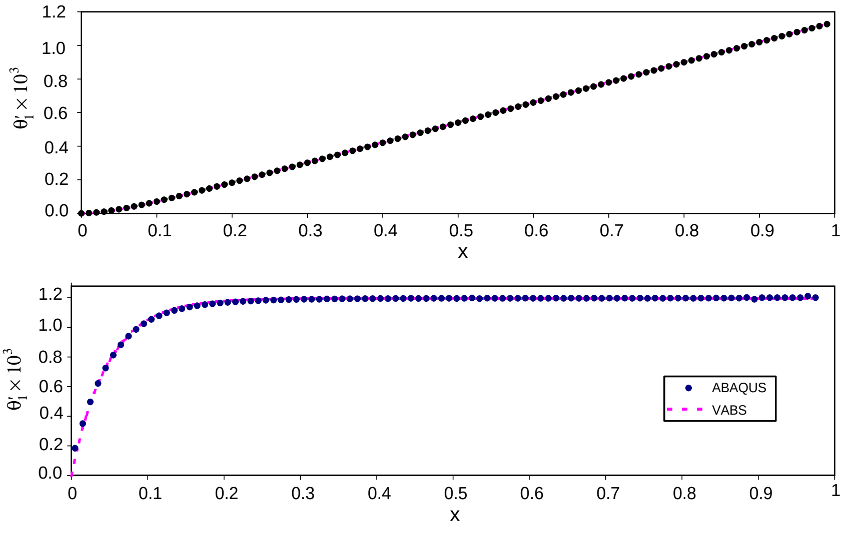

Fig. 5. Twist angle for anisotropic strip versus nondimensional axial coordinate.

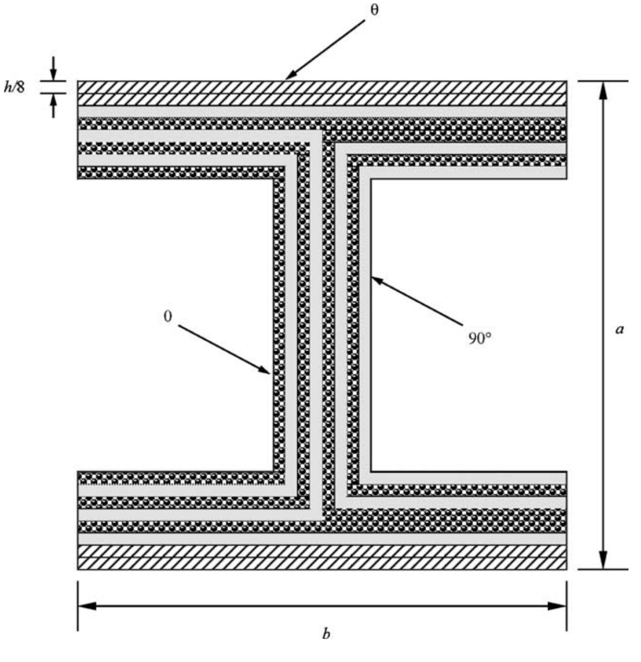

Fig. 6. Schematic of I-beam.

again, the displacements were extracted from the 3-D model and used to compute the rotation angle, which was computed by assuming a rigid-body rotation of each of the cross-section.

For the isotropic case, the 2-D cross-sectional model for VABS was meshed with 2 elements through the thickness, 20 elements across the flange and 10 elements along the web. The 3-D model maintained the same cross-sectional mesh as the 2-D cross-sectional model, but with 100 elements along the length, giving a total of 10,000 elements. The aspect ratio of each element then becomes 5:2.5:1 (L:b:h). In an attempt to reduce end effects at the free end, two cases were run with distinct b/L ratios, case 1 with 1/10, and case 2 with 1/20.

For the anisotropic case, the 2-D cross-sectional model for VABS was meshed with 8 elements through the thickness, 20 elements across flange and 10 elements along the web. The 3-D model maintained the same cross-sectional mesh as the 2-D model, but with 100 elements along the length, giving a total of 40,000 elements. The aspect ratio of each element then becomes 20:10:1 (L:b:h). Only the case with $b/L=1/10$ was run for the anisotropic I-beam.

5.2.2. Results

For the isotropic case, sectional properties are given in Table 5. As was demonstrated in [6,15], Vlasov's original theory is asymptotically correct for isotropic TWOS beams. As a result, VABS and the analytical asymptotically correct theory (Vlasov's) agree well for the torsional rigidity although they provide somewhat different warping rigidities. 1-D results for the two cases are shown in Figs. 7 and 8. One can clearly observe an improvement of the VABS results over the TWOS analytical approximation when compared to the results derived from 3-D modelling.

For the anisotropic I-beam, sectional properties are given in Table 6 where, as in the case of the anisotropic strip, $\bar{C}_{22}=C_{22}-C_{23}^2/C_{33}$ corresponds to the effective torsional rigidity that takes bending-twist coupling into consideration. 1-D results are shown in Fig. 9.

Next, let us look into stress recovery. From the point of view of Vlasov theory, stresses in the axial direction provide the best test of the correction. In recovering these stresses first one needs to calculate θ_1''. Fig. 10 demonstrates a good correlation with 3-D results for two isotropic I-beams. Figs. 11 and 12 demonstrate recovered axial stresses as predicted by ABAQUS and VABS, respectively. Although different visualization tools somewhat obscure the similarities, correlation is very good, especially if the VABS mesh is refined.

Table 5
Sectional properties for isotropic I-beam

Source	Property	Value	Units
VABS	GJ	199.944	lb.-in.^{2}
Analytical (VAM)	GJ	198.901	lb.-in.^{2}
VABS	EC	3553.43	lb.-in.^{4}
Analytical (VAM)	EC	2083.33	lb.-in.^{4}

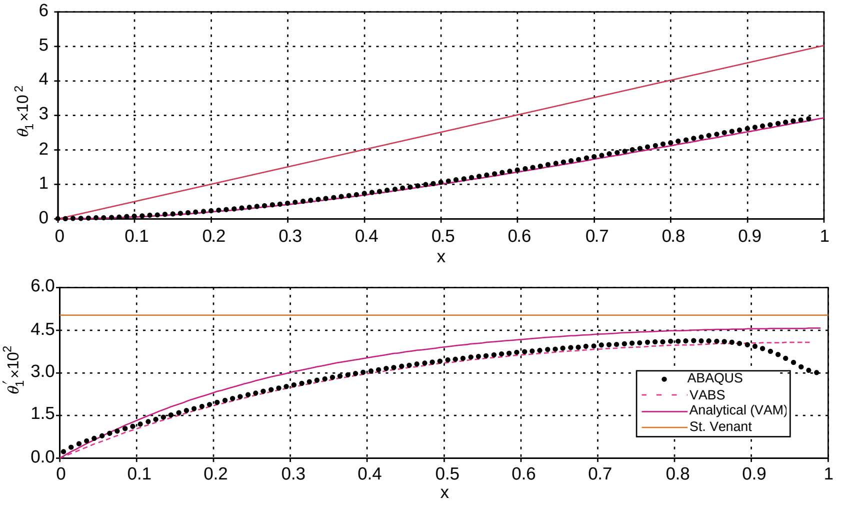

Fig. 7. Twist angle for isotropic I-beam versus nondimensional axial coordinate, $L/b=10$.

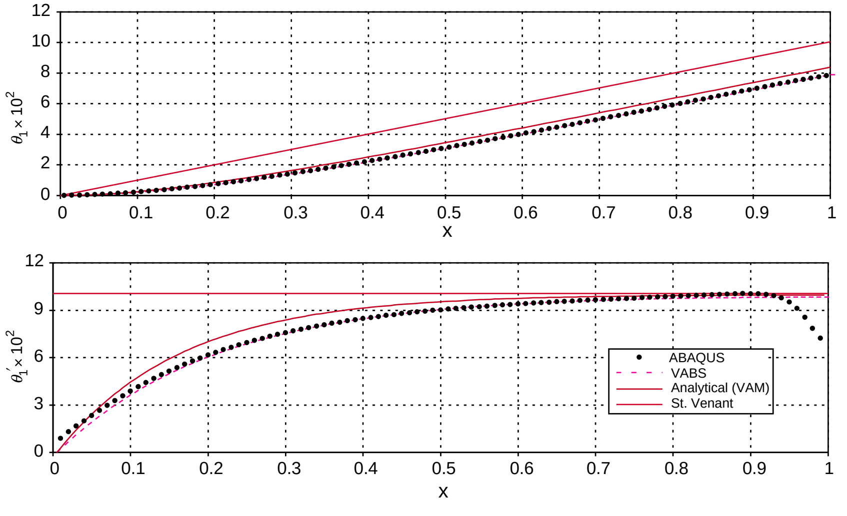

Fig. 8. Twist angle for isotropic I-beam versus nondimensional axial coordinate, $L/b=20$.

Table 6
Sectional properties for anisotropic I-beam

Source	Property	Value	Units
VABS	$\bar{C}_{22}$	55.8658	lb.-in.^{2}
VABS	C_{55}	4232.17	lb.-in.^{4}

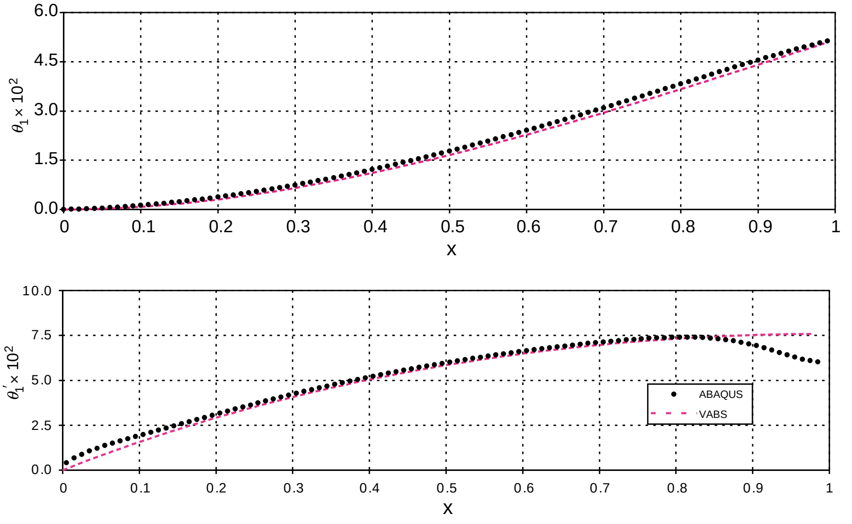

Fig. 9. Twist angle for anisotropic I-beam versus nondimensional axial coordinate, $L/b = 10$.

Figs. 13 and 14 demonstrate recovered axial stresses as predicted by ABAQUS and VABS, respectively, for the anisotropic I-beam. As with the isotropic case, the different visualization tools somewhat obscure the similarities, but correlation is still very good, especially when the VABS mesh is refined. We note here that interlaminar stresses can be recovered by VABS.

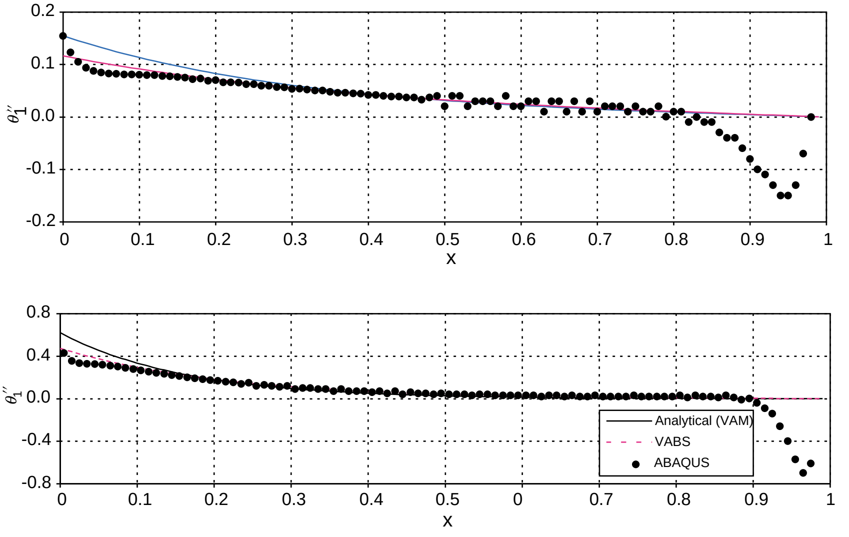

Fig. 10. θ_1'' for isotropic I-beam versus nondimensional axial coordinate, $L/b = 10$ (top) and $L/b = 20$ (bottom).

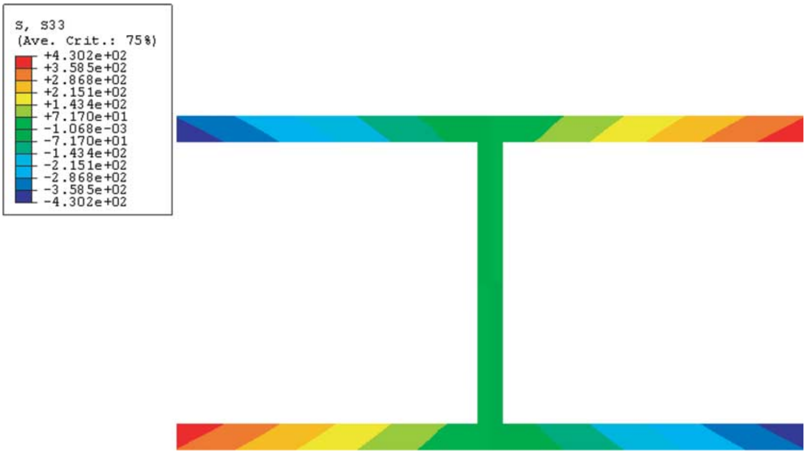

Fig. 11. σ_{11} for isotropic I-beam at the midspan of the beam from ABAQUS (3-D).

5.3. Closed-section beams

Finally, let us briefly address closed sections. The box-beam configuration of [20] is analyzed by both the VABS generalized Vlasov theory and 3-D finite elements. While it is not expected that the theory will correctly predict the end-zone behavior of a closed-section beam, such an application will no doubt be attempted by VABS users. Fig. 15

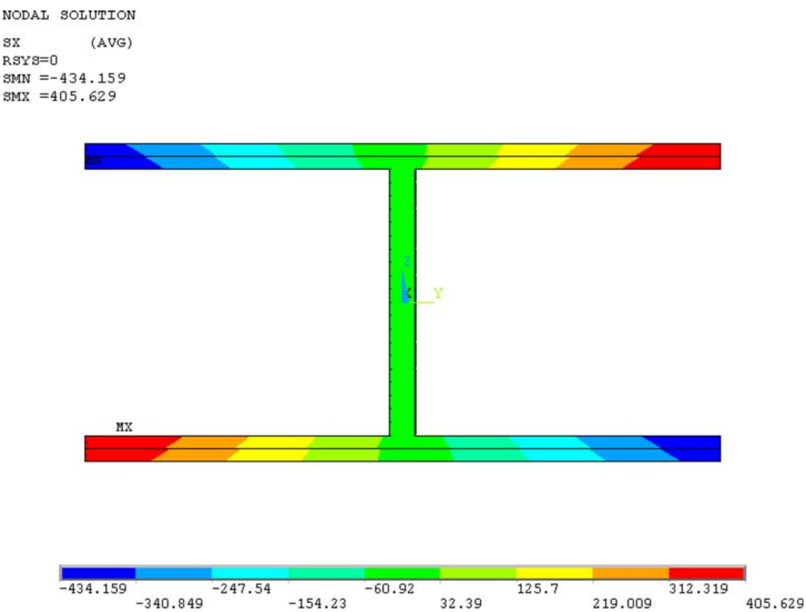

Fig. 12. σ_{11} for isotropic I-beam at the midspan of the beam from VABS.

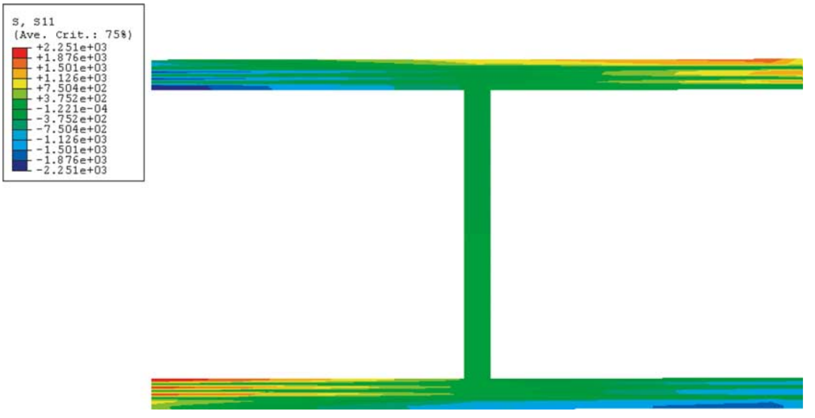

Fig. 13. σ_{11} for anisotropic I-beam at the mid-span of the beam, ABAQUS (3-D).

shows that the results are not satisfactory. The reason is that the mode of deformation for TWOS beams contains a significant amount of the torsional mode of deformation. On the other hand, the mode of deformation in the end-zone of a closed-section beam is completely different and must be determined by other means, such as the use of dispersion equations [21]. In fact, the VABS classical theory (here equivalent to St. Venant theory) provides a better correlation with 3-D results than the VABS generalized Vlasov theory. It is important to realize that the Vlasov mode for a box-beam is a fast decaying one and therefore cannot be picked up using asymptotic considerations. Clearly, the VABS generalized Vlasov theory should not be used for closed-section beams.

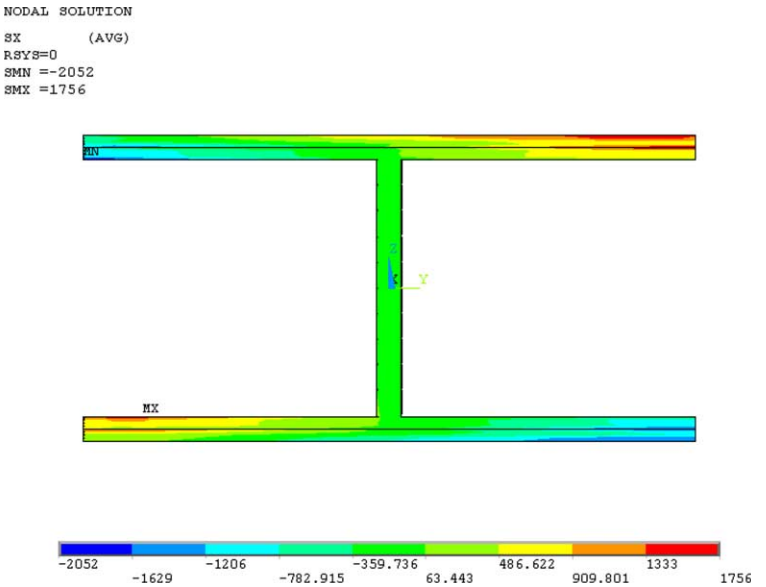

Fig. 14. σ_{11} for anisotropic I-beam at the mid-span of the beam, VABS.

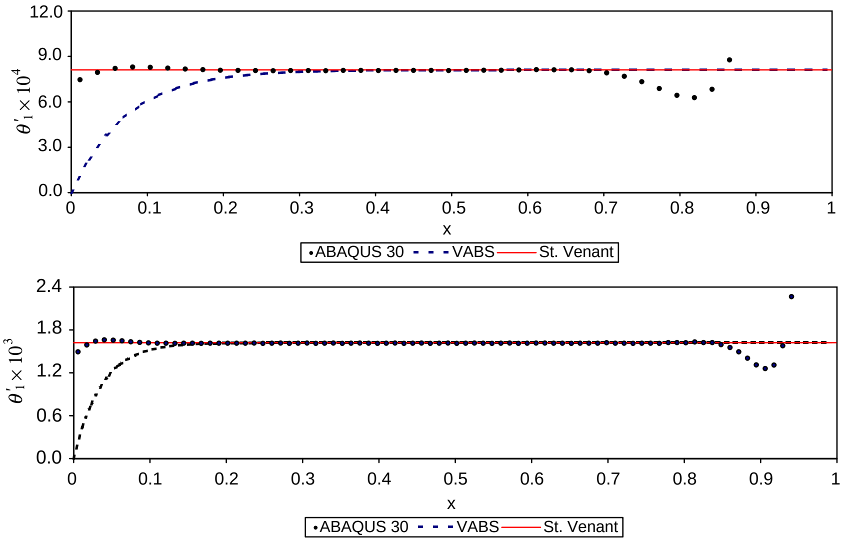

Fig. 15. Twist rate for isotropic box-beam versus nondimensional axial coordinate, $L=30$ in. (top) and $L=60$ in. (bottom).

6. Concluding remarks

A generalized Vlasov theory for composite beams has been developed based on the variational asymptotic beam sectional analysis without invoking any ad hoc kinematic assumptions. As long as the VABS generalized Vlasov theory is applied to beams with thin-walled, open cross-sections, one finds excellent agreement between the developed theory and the 3-D finite element results from ABAQUS in the regime near a fixed boundary. Both the decay length and the end-zone behavior are predicted accurately. When a box-beam is analyzed in this fashion, the 'restrained warping effect' for this closed-section beam is not insignificant. However, the generalized Vlasov theory is based on a particular mode of deformation, which is not significant in closed-section beams. The end-zone effect for the closed-section beam is caused by another type of deformation mode, which requires additional analysis of the boundary layer. The VABS generalized Vlasov theory has been shown to be extremely useful for beams with open cross-sections. However, it should not be used for closed-section beams. Instead, the VABS classical or generalized Timoshenko theory should be used in such cases.

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