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Theory of initially twisted, composite, thin-walled beams

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Abstract

An asymptotically correct theory for initially twisted, thin-walled, composite beams has been constructed by the variational asymptotic method. The strain energy of the original, three-dimensional structure is first rigorously reduced to be a two-dimensional energy expressed in terms of shell strains. Then the two-dimensional strain energy is further reduced to be expressed in terms of the classical beam strain measures. The resulting theory is a classical beam model approximating the three-dimensional energy through the first-order of the initial twist. Consistent use of small parameters that are intrinsic to the problem allows a natural derivation for all thin-walled beams within a common framework, regardless of whether the section is open, closed, or strip-like. Several examples are studied using the present theory and the results are compared with a general cross-sectional analysis, VABS, and other published results.

Keywords: Composite beams; Thin-walled; Initial twist; Asymptotic method; VABS

1. Introduction

A thin-walled beam is characterized as a flexible body that has different magnitudes for all three of its characteristic dimensions [1]. To be classified as a beam, c, the characteristic dimension of the cross-section, must be much smaller than l, the wavelength

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of the deformation along the beam, i.e. $c/l \ll 1$. Moreover, for a beam to be classified as thin-walled implies that the maximum thickness of the walls, h, is much smaller than c, so that $h/c \ll 1$. Although one can analyze thin-walled beams using three-dimensional (3D) elasticity theory, thin-walled beam theories take advantage of the small parameters, h/c and c/l, to derive a one-dimensional (1D) model. This model consists of 1D constitutive equations (cross-sectional elastic constants) and 'recovery relations'. The former are used in the 1D equilibrium and kinematic equations to analyze the original 3D structure, and the latter provide approximate values of the 3D displacement, strain, and stress from the 1D solution.

Thin-walled beam theories strive to present closed-form expressions for cross-sectional stiffness constants and stresses (or stress flows). There are mainly two types of thin-walled beam theories. The first type can be classified as ad hoc models [1–5]. In these models, assumptions are invoked based on engineering intuition. These can be assumptions that the beam deforms in specific modes or that certain components of the displacement/strain/stress are negligible. Usually, these assumptions are based on experience with thin-walled beams made with isotropic materials, which can be justified by some exact solutions. However, for anisotropic materials, various modes of deformation can be coupled, and these theories might fail for some special cases which cannot be properly represented by the invoked assumptions [6]. Nevertheless, some of these models such as [4,5] can provide a good prediction for many cases, and it is straightforward to refine the model by incorporate additional deformation such as transverse shear to remedy possible errors introduced by ad hoc assumptions.

The second type encompasses asymptotic models [6–9]. Therein, the original 3D elasticity equations are mathematically reduced to a 1D model using small parameters inherent to the problem. While application of traditional asymptotic methods is possible, the authors prefer the Variational Asymptotic Method (VAM) [10]. In these models, the material anisotropy is accounted for in a consistent and systematic manner, and those deformation modes that contribute most significantly to the energy emerge naturally. In our formulation, elastic couplings among all deformations are accounted for by using the 3D material law, which uses 21 elastic constants for anisotropic materials. However, the refined models constructed directly using the VAM are of little practical use, perse. Usually, some transformation, which might detract from the asymptotical correctness, has to be carried out to convert such models into a form that is of practical use for engineers [11].

The present paper was originally planned to serve as a natural extension of the work in [9] to enhance the capability of that theory to accommodate initial twist, so that more realistic problems (such as pretwisted composite rotor blades or wind turbines) can be analyzed. It was later found out that it is very complicated, if not impossible, to incorporate the initial twist into that, already complex, formulation. Instead, the present formulation is cast in an intrinsic form and the derivation departs from previous work at the outset. First, the 3D elasticity representation is rationally reduced to the classical shell approximation of Berdichevsky [10] with geometric correction by considering h/c as the main small parameter and taking into account all first-order corrections from the initial twist of the thin-walled beam. Then the two-dimensional (2D) variables are expressed in terms of intrinsic beam variables and unknown warping functions. Substituting these relations back into the 2D strain energy, which is an asymptotic approximation of the original 3D energy, one can use the VAM to solve for the unknown warping functions to minimize the 2D strain energy. The final result is a strain energy for the thin-walled beam with first-order correction from initial twist. For validation, several examples of thin-walled beams are studied; and the results are compared with some available in the literature and VABS [11], a general-purpose finite element program for arbitrary cross-sections that does not take advantage of the smallness of the wall thickness.

2. Kinematics

A general thin-walled beam can be depicted as in Fig. 1; note that this picture does not show the initial twist. Here, O is a fixed point in space, $\bar{O}$ is on the beam axis specified by the position vector $\mathbf{r}_o$, and $\hat{O}$ is on the contour intersecting the reference surface (considering the thin-walled structure as a shell) with the beam section cut through the point $\bar{O}$. Here, two dextral coordinate systems x_i and y_i are introduced; y_1 is the running length coordinate along the beam axis with $\mathbf{b}_1$ as the unit vector; y_a are the local Cartesian coordinates of the beam section with $\mathbf{b}_a$ as the unit vectors; x_1 is parallel to y_1, x_2 is the arc length along the contour, and x_3 is the outward normal of the reference surface. (Here and throughout the paper, Greek indices assume values 1 and 2 while Latin indices $i, j, \ldots, z$ assume 1, 2, and 3 and $a, b, \ldots, h$ assume 2 and 3. Repeated indices are summed over their range except where explicitly indicated.)

The position vector of the shell reference surface is

$$\mathbf{r}(x_1, x_2) = \mathbf{r}_0(y_1) + y_a(x_2)\mathbf{b}_a(y_1) \tag{1}$$

The covariant base vectors of the reference surface are defined by

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} \tag{2}$$

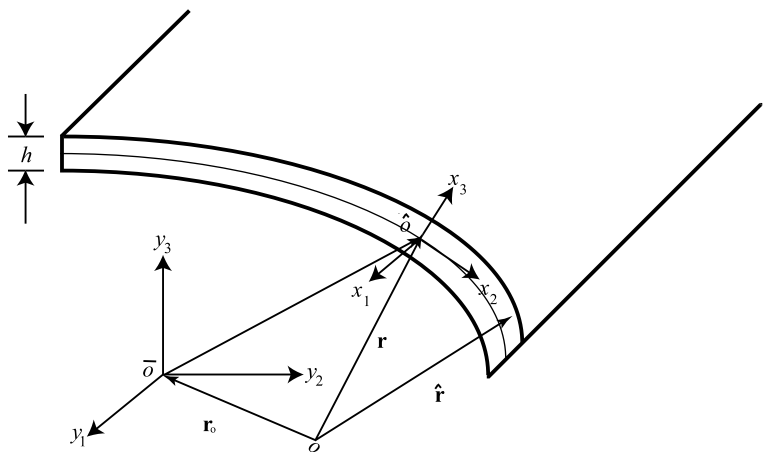

Fig. 1. Schematic of a thin-walled beam.

where $(\ )_{,\alpha} = \partial(\ )/\partial x_\alpha$. Using Eq. (1), the covariant base vectors can be written explicitly as

$$\mathbf{a}_1 = \mathbf{b}_1 + y_2 k_1 \mathbf{b}_3 - y_3 k_1 \mathbf{b}_2 \qquad \mathbf{a}_2 = \dot{y}_a \mathbf{b}_a$$
$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} = \frac{-k_1 R_\tau \mathbf{b}_1 - \dot{y}_3 \mathbf{b}_2 + \dot{y}_2 \mathbf{b}_3}{\sqrt{1 + k_1^2 R_\tau^2}} \tag{3}$$

with $R_\tau = y_a \dot{y}_a$, $(\dot{\ }) = \partial(\ )/\partial x_2$, and k_1 as the initial twist. The first fundamental form of the surface is obtained from $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, such that

$$a_{11} = 1 + k_1^2 R^2 \qquad a_{12} = a_{21} = k_1 R_n \qquad a_{22} = 1 \tag{4}$$

with $R^2 = y_a y_a$ and $R_n = y_2 \dot{y}_3 - y_3 \dot{y}_2$. The determinant of the first fundamental form is

$$a = 1 + k_1^2 R_\tau^2 \tag{5}$$

The second fundamental form of the surface can be calculated by the definition

$$b_{\alpha\beta} = \mathbf{r}_{,\alpha\beta} \cdot \mathbf{a}_3 = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 \tag{6}$$

such that

$$b_{11} = \frac{k_1^2 R_n}{\sqrt{a}} \qquad b_{12} = b_{21} = \frac{k_1}{\sqrt{a}} \qquad b_{22} = \frac{\dot{y}_2 \ddot{y}_3 - \dot{y}_3 \ddot{y}_2}{\sqrt{a}} \tag{7}$$

Any point in the undeformed 3D structure can be described as

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1, x_2) + x_3 \mathbf{a}_3 \tag{8}$$

The 3D covariant base vectors are defined using $\mathbf{g}_i = \hat{\mathbf{r}}_{,i}$ and can be obtained as

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha - x_3 b_\alpha^\lambda \mathbf{a}_\lambda \qquad \mathbf{g}_3 = \mathbf{a}_3 \tag{9}$$

The 3D metric tensor can then be calculated as

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 b_\alpha^\lambda b_{\lambda\beta} \qquad g_{i3} = \delta_{i3} \tag{10}$$

where δ is the Kronecker symbol. From Eq. (10) one can calculate the determinant of the metric tensor

$$g = [1 - 2x_3 H + x_3^2 K]^2 a \tag{11}$$

with $H = (1/2) b_\alpha^\alpha$ as the mean curvature and $K = \det(b_\alpha^\beta)$ as the Gaussian curvature of the surface.

3. Dimensional reduction from 3D to 2D

The dimensional reduction from the original 3D thin-walled structure to a 1D beam model can be carried out in two steps due to the existence of two different small parameters h/c and c/l. Firstly, making use of h/c, one can approximate the original 3D energy with a 2D energy defined in the shell reference surface. Secondly, making use of c/l, one can approximate the above-found 2D energy with 1D energy defined along the beam axis.

After deformation, the position of any material point in the 3D structure can be described by the vector

$$\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{R}(x_1, x_2) + x_3\mathbf{A}_3(x_1, x_2) + w_i(x_1, x_2, x_3)\mathbf{A}^i(x_1, x_2) \tag{12}$$

where the $\mathbf{A}^i$ are the contravariant base vectors of $\mathbf{A}_i$, which are, in turn, the base vectors of the deformed reference surface given by

$$\mathbf{A}_\alpha = \mathbf{R}_{,\alpha} \qquad \mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|} \tag{13}$$

Here, we constrain $\mathbf{A}_3$ to be normal to the deformed reference surface by including all possible deformations of the transverse normal into the 3D warping functions $w_i(x_1, x_2, x_3)$. One can define $\mathbf{R}$ as

$$\mathbf{R} = \frac{1}{h}\langle\hat{\mathbf{R}}\rangle \tag{14}$$

which implies three constraints on the warping functions, so that

$$\langle w_i(x_1, x_2, x_3)\rangle = 0 \tag{15}$$

where the angle-brackets denote the definite integral through the thickness of the shell.

From Eq. (12), one can obtain the 3D covariant base vectors and the corresponding metric tensor according to the following formulas

$$\mathbf{G}_i = \hat{\mathbf{R}}_{,\alpha} \qquad G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j \tag{16}$$

The 3D strain tensor is defined as

$$\Gamma_{ij} = \tfrac{1}{2}(G_{ij} - g_{ij}) \tag{17}$$

Making use of Eqs. (17), (16), (12), and (10), one can derive the linearized 3D strain field as

$$\begin{aligned}
&\Gamma_{\alpha\beta} = \varepsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta} - x_3 b^\lambda_{(\alpha}\varepsilon_{\lambda\beta)} - x_3^2 b^\lambda_{(\alpha}\kappa_{\lambda\beta)} + w_{(\alpha;\beta)} - b_{\alpha\beta}w_3 - x_3 b^\lambda_{(\alpha}w_{\lambda;\beta)} + x_3 b^\lambda_\alpha b_{\lambda\beta}w_3 \\
&2\Gamma_{\alpha 3} = w_{3,\alpha} + w_\lambda b^\lambda_\alpha + w_{\alpha,3} - x_3 b^\beta_\alpha w_{\beta,3} \qquad \Gamma_{33} = w_{3,3}
\end{aligned} \tag{18}$$

where the semicolon preceding an index denotes the covariant derivative with respect to the coordinate, and the parentheses in the subscripts denote the symmetrization operation [12], meaning $a_{(\alpha\beta)} = \left(\frac{1}{2}\right)(a_{\alpha\beta} + a_{\beta\alpha})$. The 2D generalized strain measures $\epsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are defined as [13,14]

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(A_{\alpha\beta} - a_{\alpha\beta}) \qquad \kappa_{\alpha\beta} = b_{\alpha\beta} - B_{\alpha\beta} + b^\lambda_{(\alpha}\varepsilon_{\lambda\beta)} \tag{19}$$

where $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are the first and second fundamental forms of the deformed surface, respectively, and defined as

$$A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta \qquad B_{\alpha\beta} = \mathbf{A}_{\alpha,\beta} \cdot \mathbf{A}_3 \tag{20}$$

Note that the 3D strains, as in Eq. (18), are obtained based on the assumption that the 2D generalized strains are small and the warping functions are of the order of the generalized strains. This means all the products between warping functions and generalized strains are neglected. Other than this, all the 3D deformation is accounted in this strain field. In view of the smallness of h/c and initial twist k_1, the main terms of the 3D strain field which will contribute to the zeroth-order energy are

$$\Gamma^0_{\alpha\beta} = \epsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta} \quad 2\Gamma^0_{\alpha 3} = w_{\alpha,3} \quad \Gamma^0_{33} = w_{3,3} \tag{21}$$

Now, the 3D strain energy of the thin-walled structure can be expressed as

$$J = \frac{1}{2}\int_v \Gamma^{\mathrm{T}} D\Gamma \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \frac{1}{2}\int_s \langle \Gamma^{\mathrm{T}} D\Gamma(1 - 2x_3 H + x_3^2 K)\rangle \mathrm{d}s \tag{22}$$

where v is the volume occupied by the 3D body in the undeformed configuration, s is the surface stretched by the undeformed reference surface and

$$\begin{aligned} &\Gamma = \lfloor \Gamma_e \; 2\Gamma_s \; \Gamma_t \rfloor^{\mathrm{T}} \quad \Gamma_e = \lfloor \Gamma_{11} \; 2\Gamma_{12} \; \Gamma_{22} \rfloor^{\mathrm{T}} \\ &2\Gamma_s = \lfloor 2\Gamma_{13} \; 2\Gamma_{23} \rfloor^{\mathrm{T}} \quad \Gamma_t = \Gamma_{23} \end{aligned} \tag{23}$$

The strain energy per unit area (which is the same as the strain energy for the deformation of the normal-line element) is

$$U = \frac{1}{2}\langle \Gamma^{\mathrm{T}} D\Gamma(1 - 2x_3 H + x_3^2 K)\rangle \tag{24}$$

where D is the 3D 6×6 material matrix transformed from the material system into the general curvilinear system x_i and is generally fully populated for composite materials. D is a function of initial twist k_1 which may be expanded asymptotically as

$$D = D_0 + D_1 + o(k_1\mu) \tag{25}$$

where D_0 is of the order of the elastic constants μ, D_1 is of the order $k_1\mu$, and $o(k_1\mu)$ represents terms of order higher than $k_1\mu$. For isotropic materials, D as a fourth-order tensor can be found in a typical elasticity book, which is an isotropic tensor formed by two material parameters and the 3D metric tensor of the undeformed configuration. What one needs to do next is to put this fourth-order tensor to a 6×6 matrix according to the engineering notation used in Eq. (23). For an anisotropic material, we need to perform the transformation between the material system and an orthonormal system. Usually, the orthonormal system is defined in terms of the lamina plane orientation angle and the ply angle, yielding the conventional transformation for composite materials. Then we need to transform the material properties in the aforementioned orthonormal system into the curvilinear system. It is convenient to derive such relations using general tensorial notation. If D^{ijkl} represents the components of material tensor in the curvilinear system and $\hat{D}^{mnpq}$ represents those in the orthonormal system, we have

$$D^{ijkl}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k\mathbf{g}_l = \hat{D}^{mnpq}\mathbf{s}_m\mathbf{s}_n\mathbf{s}_p\mathbf{s}_q \tag{26}$$

with $\mathbf{s}_i$ as the unit vectors associated with the orthonormal system. From the above equation, one can easily obtain the material properties in the curvilinear system such that:

$$D^{ijkl} = \hat{D}^{mnpq}(\mathbf{s}_m \cdot \mathbf{g}_i)(\mathbf{s}_n \cdot \mathbf{g}_j)(\mathbf{s}_p \cdot \mathbf{g}_k)(\mathbf{s}_q \cdot \mathbf{g}_l) \tag{27}$$

According to the VAM, it is sufficient to find the zeroth-order warping for the purpose of obtaining an energy asymptotically correct through the first-order of k_1, which is the focus of the present work. The zeroth-order energy per unit area can be written as

$$2U_0 = \left\langle \begin{Bmatrix} \Gamma_e^0 \\ 2\Gamma_s^0 \\ \Gamma_t^0 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} D_e & D_{es} & D_{et} \\ D_{es}^{\mathrm{T}} & D_s & D_{st} \\ D_{et}^{\mathrm{T}} & D_{st}^{\mathrm{T}} & D_t \end{bmatrix} \begin{Bmatrix} \Gamma_e^0 \\ 2\Gamma_s^0 \\ \Gamma_t^0 \end{Bmatrix} \right\rangle \tag{28}$$

where D_e, D_{es}, D_{et}, D_s, D_{st}, D_t are the corresponding partition matrices of D_0. The leading terms of warping functions in Eq. (28) are

$$2U_0^* = \langle 2\Gamma_e^{0\mathrm{T}} D_{es} 2\Gamma_s^0 + 2\Gamma_e^{0\mathrm{T}} D_{et} \Gamma_t^0 + 2\Gamma_s^{0\mathrm{T}} D_s 2\Gamma_s^0 + 4\Gamma_s^{0\mathrm{T}} D_{st} \Gamma_t^0 + \Gamma_t^{0\mathrm{T}} D_t \Gamma_t^0 \rangle \tag{29}$$

Minimizing Eq. (29) by following the general procedure from the calculus of variations, one can solve for $2\Gamma_s^0$ and Γ_t^0 as

$$2\Gamma_s^0 = -D_s^{*-\mathrm{T}} D_{es}^{*\mathrm{T}} \Gamma_e^0 \quad \Gamma_t^0 = -D_t^{-\mathrm{T}} D_{et}^{*\mathrm{T}} \Gamma_e^0 \tag{30}$$

with

$$\begin{aligned} &D_s^* = D_s - D_{st} D_t^{-1} D_{st}^{\mathrm{T}} \quad D_{es}^* = D_{es} - D_{et} D_t^{-1} D_{st}^{\mathrm{T}} \\ &D_{et}^* = D_{et} - D_{es}^* D_s^{*-1} D_{st} \end{aligned} \tag{31}$$

From Eqs. (15), (21) and (30), one can solve for the zeroth-order warping field. Substituting the solved warping functions into Eq. (18), one can obtain the 3D strains asymptotically correct through the first order of k_1, which can be symbolically written as

$$\Gamma = \Gamma^0 + \Gamma^1 \tag{32}$$

The strain energy per unit area asymptotically correct through the zeroth-order of k_1 can be calculated from Eq. (24) as

$$2U_0 = \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^{\mathrm{T}} & \mathcal{D} \end{bmatrix} \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix} \tag{33}$$

where

$$\epsilon = \lfloor \epsilon_{11} \;\; 2\epsilon_{12} \;\; \epsilon_{22} \rfloor^{\mathrm{T}} \quad \kappa = \lfloor \kappa_{11} \;\; 2\kappa_{12} \;\; \kappa_{22} \rfloor^{\mathrm{T}} \tag{34}$$

$$\mathcal{A} = \langle D_{||} \rangle \quad \mathcal{B} = \langle x_3 D_{||} \rangle \quad \mathcal{D} = \langle x_3^2 D_{||} \rangle \tag{35}$$

and

$$D_{||} = D_e - D_{es} D_s^{*-\mathrm{T}} D_{es}^{*\mathrm{T}} - D_{et} D_t^{-\mathrm{T}} D_{et}^{*\mathrm{T}} \tag{36}$$

The strain energy per unit area including the first-order correction from initial twist can be expressed as

$$2U_1 = \langle \Gamma^{0\mathrm{T}} D_1 \Gamma^0 - 2x_3 H \Gamma^{0\mathrm{T}} D_0 \Gamma^0 + 2\Gamma^{1\mathrm{T}} D_0 \Gamma^0 \rangle = \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{B}_1^{\mathrm{T}} & \mathcal{D}_1 \end{bmatrix} \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix} \tag{37}$$

Up to this point, we have successfully reduced the original 3D strain energy to a 2D shell strain energy, which has an accuracy asymptotically correct through the first order of the initial twist k_1.

4. Dimensional reduction from 2D to 1D

The previously obtained model in Eqs. (33) and (37) is an asymptotically correct classical shell model with geometric correction through the first order due to initial curvature of the reference surface. However, engineering practice often requires a more simplified model to carry out the relevant analysis and design of thin-walled beams. We need to proceed further to reduce the shell model to a 1D beam model.

The deformed reference surface can be expressed in terms of beam quantities such that

$$\mathbf{R}(x_1, x_2) = \mathbf{R}_0(y_1) + y_a(x_2)\mathbf{B}_a(y_1) + v_i(y_1, y_2, y_3)\mathbf{B}_i(y_1) \tag{38}$$

where $\mathbf{B}_i$ are the unit vectors associated with y_i for the deformed configuration, and v_i is the warping field subject to the constraints

$$\langle\langle v_i \rangle\rangle = 0 \qquad \langle\langle y_3 v_2 - y_2 v_3 \rangle\rangle = 0 \tag{39}$$

with the double angle-brackets denoting the definite integral along the contour of beam sections.

From Eq. (38), one can calculate the base vectors of the deformed shell surface $\mathbf{A}_i$ based on their definitions, Eq. (13). Then one can obtain the fundamental forms of this surface from Eq. (20) and finally derive the shell strain measures in terms of beam quantities and warping functions from Eqs. (19). By neglecting all nonlinear terms with respect to the beam strain measures and warping functions v_i, one can find the 2D shell strain measures asymptotically correct through the first order of initial twist as

$$
\begin{aligned}
\epsilon_{11} &= \gamma_{11} + \kappa_2 y_3 - \kappa_3 y_2 + k_1 \kappa_1 R^2 \\
2\epsilon_{12} &= \dot{v}_1 + \kappa_1 R_n + k_1 (y_2 \dot{v}_3 - y_3 \dot{v}_2 + v_2 \dot{y}_3 - v_3 \dot{y}_2) \\
\epsilon_{22} &= \dot{y}_a \dot{v}_a \\
\kappa_{11} &= \kappa_a \dot{y}_a + \frac{1}{2} k_1 (\dot{v}_1 - 3\kappa_1 R_n) \\
2\kappa_{12} &= -2\kappa_1 + \frac{b_{22}}{2}(\dot{v}_1 + \kappa_1 R_n) + k_1 [2R_\tau(\kappa_2 \dot{y}_3 - \kappa_3 \dot{y}_2) - \dot{y}_a \dot{v}_a \\
&\quad + \dot{R}_\tau(\gamma_{11} + \kappa_2 y_3 - \kappa_3 y_2) + \frac{b_{22}}{2}(y_2 \dot{v}_3 - y_3 \dot{v}_2 + v_2 \dot{y}_3 - v_3 \dot{y}_2)] \\
\kappa_{22} &= (\dot{y}_3 \dot{v}_2 - \dot{y}_2 \dot{v}_3)_{,2} + k_1 \left[\kappa_1 R_\tau^2 b_{22} + R_\tau \ddot{v}_1 + \frac{\dot{R}_\tau}{2}(\dot{v}_1 + \kappa_1 R_n) \right]
\end{aligned}
\tag{40}
$$

where γ_{11} is the extensional strain and κ_1 the torsional strain and κ_a bending strains in the y_a direction.

To obtain the strain energy defined along the beam axis through the first-order of initial twist, one needs to solve for the v_i of the zeroth-order approximation. Using the 2D strain field in Eqs. (40) without the terms related with k_1 and substituting it into the zeroth-order shell energy, Eq. (33), one can solve for the warping functions subject to the constraints in Eqs. (39). For the convenience of calculation, one can express the zeroth-order 2D shell strains in matrix form as

$$
\left\{ \begin{array}{c} \epsilon \\ \kappa \end{array} \right\} = P\varepsilon + T\psi \tag{41}
$$

with

$$
P = \begin{bmatrix} 1 & 0 & y_3 & -y_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{y}_2 & \dot{y}_3 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \frac{b_{22}}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{42}
$$

$$
\varepsilon = \lfloor \gamma_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3 \rfloor^{\mathrm{T}} \qquad \psi = \left\lfloor 2\epsilon_{12}^0 \quad \epsilon_{22}^0 \quad \kappa_{22}^0 \right\rfloor^{\mathrm{T}}
$$

The only unknowns exist in ψ and the constraints in Eqs. (39) are only needed to recover v_i and have no effect on the arbitrariness of ψ. Denoting the stiffness obtained in Eq. (33) as K, one can express the zeroth-order energy per unit length of the beam axis using Eq. (41) as

$$
2\Pi_0 = \langle\langle (P\varepsilon + T\psi)^{\mathrm{T}} K (P\varepsilon + T\psi) \rangle\rangle \tag{43}
$$

For open sections, there are no additional constraints on ψ. The minimization problem can be carried out in a straightforward manner, yielding

$$\psi = -(T^{\mathrm{T}}KT)^{-1}T^{\mathrm{T}}KP\varepsilon \tag{44}$$

Then the zeroth-order energy can be expressed in terms of the beam strains as

$$2\Pi_0 = \varepsilon^{\mathrm{T}}\langle\langle P^{\mathrm{T}}[K - KT(T^{\mathrm{T}}KT)^{-1}T^{\mathrm{T}}K]P\rangle\rangle\varepsilon \tag{45}$$

The zeroth-order warping functions can be solved from Eqs. (39) along with Eq. (39). Substituting the obtained warping functions into Eq. (40) and then into Eq. (37), one should be able to obtain the strain energy per unit length expressed in terms of beam strain measures of the order k_1.

For closed sections, four additional constraints should be applied to ensure the uniqueness of the displacement field [9]. They are

$$\langle\langle v_{i,2}\rangle\rangle = 0 \qquad \langle\langle \kappa_{22}^0\rangle\rangle = 0 \tag{46}$$

These constraints can be transformed into the matrix form

$$\langle\langle \phi\psi - L\varepsilon\rangle\rangle = 0 \tag{47}$$

with

$$\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dot{y}_2 & -y_3 \\ 0 & \dot{y}_3 & y_2 \\ 0 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & R_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

Introducing Lagrange multipliers, the functional to be minimized has the form

$$2\Lambda = \langle\langle (P\varepsilon + T\psi)^{\mathrm{T}}K(P\varepsilon + T\psi) + 2\lambda^{\mathrm{T}}\phi\psi\rangle\rangle \tag{49}$$

where one can solve for ψ as

$$\psi = -(T^{\mathrm{T}}KT)^{-1}(T^{\mathrm{T}}KP\varepsilon + \phi^{\mathrm{T}}\lambda) \tag{50}$$

Substituting the above equation into the constraints in Eq. (47), one can solve for λ as

$$\lambda = -\langle\langle \phi(T^{\mathrm{T}}KT)^{-1}\phi^{\mathrm{T}}\rangle\rangle^{-1}\langle\langle \phi(T^{\mathrm{T}}KT)^{-1}T^{\mathrm{T}}KP + L\rangle\rangle\varepsilon \tag{51}$$

Substituting Eq. (51) back into Eq. (50), one obtains ψ and then recovers the zeroth-order warping v_i with the help of Eq. (39). Finally, one obtains the strain energy per unit length asymptotically correct through the first-order of initial twist k_1 from Eqs. (33) and (37).

It is worthy to emphasize that the present theory asymptotically reduces the original 3D energy into a 1D energy similar to those of the traditional classical beam theory. Hence, the 1D beam analysis remains essentially the same including the governing differential equations and boundary conditions as long as the beam analysis uses 1D strain measures equivalent to those of the present theory.

5. Numerical examples

To demonstrate the usage and accuracy of the present theory, we study some simple examples such as an isotropic strip, a composite strip, an isotropic box-beam, and a composite box-beam.

The first example is a strip with width c, thickness h and initial twist k_1. This strip is made with isotropic material with Young's modulus as E and Poisson's ratio ν. The warping functions in Eq. (12) can be solved according to Eq. (30) yielding

$$w_\alpha = 0 \qquad w_3 = \frac{\nu}{\nu - 1}\left\{x_3(\epsilon_{11} + \epsilon_{22}) + \left[\frac{(x_3)^2}{2} - \frac{h^2}{24}\right](\kappa_{11} + \kappa_{22})\right\} \tag{52}$$

The warping functions in Eq. (38) can be solved according to Eq. (40), yielding

$$v_1 = 0 \qquad v_2 = \frac{1}{2}\left(y_2^2 - \frac{c^2}{12}\right)\kappa_3\nu - y_2\gamma_{11}\nu \qquad v_3 = \frac{1}{2}\left(y_2^2 - \frac{c^2}{12}\right)\kappa_2\nu \tag{53}$$

Substituting the above warping functions into the 2D shell strain measures in Eq. (40), one can obtain the strain measures for the strip with the first-order correction of initial twist expressed in terms of beam strains as

$$\begin{aligned} &\epsilon_{11} = \gamma_{11} - \kappa_3 y_2 + k_1\kappa_1 y_2^2 \qquad 2\epsilon_{12} = \frac{1}{2}k_1\left(y_2^2 + \frac{c^2}{12}\right)\kappa_2\nu \\ &\epsilon_{22} = y_2\kappa_3\nu - \gamma_{11}\nu \qquad \kappa_{11} = \kappa_2 \\ &2\kappa_{12} = -2\kappa_1 + k_1\gamma_{11}(\nu + 1) - k_1(3 + \nu)y_2\kappa_3 \qquad \kappa_{22} = -\kappa_2\nu \end{aligned} \tag{54}$$

Substituting Eq. (54) into Eqs. (33) and (37), one can calculate beam stiffness with first-order correction of k_1 as

$$\begin{aligned} &S_{11} = Ech \qquad S_{12} = \frac{Ec^3h}{12}\left[1 - 3\left(\frac{h}{c}\right)^2\right]k_1 \qquad S_{22} = \frac{Ech^3}{6(\nu + 1)} \\ &S_{33} = \frac{Ech^3}{12} \qquad S_{44} = \frac{Ec^3h}{12} \end{aligned} \tag{55}$$

where S_{11} is the extensional stiffness, S_{22} torsional stiffness, S_{33} bending stiffness in the x^2 direction, S_{44} bending stiffness in the x^3 direction, and S_{12} is the extension–twist coupling from initial twist. This value is the same as that for an initially twisted isotropic solid beam [15]

$$S_{12} = [S_{33} + S_{44} - 2(\nu + 1)S_{22}]k_1 \qquad = \frac{Ec^3h}{12}\left[1 - 3\left(\frac{h}{c}\right)^2\right]k_1 \tag{56}$$

Eq. (56) is derived using the VAM without taking advantage of the smallness of wall thickness and is asymptotically correct though the first-order of the initial twist. The value

Table 1
Stiffness for pretwisted composite strip

S	VABS	Present	Difference (%)
S_{11}	0.85985931×10^7	0.85985931×10^7	0
S_{12}	0.27792311×10^6	0.28043542×10^6	0.9
S_{22}	0.27868182×10^5	0.29968253×10^5	7.5
S_{13}	0.24939907×10^2	0.61395754×10^3	
S_{23}	0.70870494×10^4	0.76211033×10^4	7.5
S_{33}	0.30464257×10^5	0.30600072×10^5	0.5
S_{44}	0.28661977×10^7	0.28661977×10^7	0

provided in Ref. [16] was

$$S_{12} = \frac{Ec^3h}{12}\left(1-\left(\frac{h}{c}\right)^2\right)k_1 \tag{57}$$

Although both results converge to the same value as h/c goes to zero, the result of Ref. [16] introduces an error of the order $(h/c)^2$ in comparison to that of Ref. [15], which is inevitable when one attempts to take advantage of the smallness of wall thickness in an ad hoc manner that is not asymptotically correct. On the other hand, the present theory, which is still a thin-walled beam theory, can reproduce the asymptotically correct results obtained without invoking the thin-walled assumption.

The second example is a single-layer strip made from an anisotropic material with properties given by

$$\begin{aligned} &E_{11}=25\times10^6\ \text{psi} \qquad E_{22}=E_{33}=10\times10^6\ \text{psi} \qquad G_{12}=G_{13}=5\times10^6\ \text{psi} \\ &G_{23}=2\times10^6\ \text{psi} \qquad \nu_{12}=\nu_{13}=\nu_{23}=0.25 \end{aligned} \tag{58}$$

The ply orientation is $-15°$. Even if there is no initial twist, this strip exhibits bending–twist coupling. We are going to compare our result with VABS which has been comprehensively validated against experimental results, 3D elasticity solutions, and 3D finite element solutions [11,17–20]. For the purpose of comparing the results with VABS, we assume $c=2$ in., $h=0.2$ in., and $k_1=0.1$/in. The results were tabulated in Table 1. One can observe that the present theory, which takes advantage of the thin-walled structure, agrees reasonably well with VABS, which is also based on the variational asymptotic method but is capable of treating cross-sections of arbitrary geometry. The present theory gives the exact solution for the stiffness in the stiff directions such as extension and bending around the normal of the strip. However, for stiffness in the soft directions, particularly the torsional stiffness and bending–twist coupling, the present theory introduces some errors because of the fact we have taken advantage of the smallness of h/c during the reduction from 3D to 2D.

When the strip is initially twisted, extension–twisting coupling will appear in the stiffness model. The present theory yields an excellent prediction for the extension–twisting coupling. The present theory also predicts a very weak extension–bending coupling which is three orders of magnitude less than the extension–twisting coupling. Although VABS also predicts such a small term, it is not meaningful to judge the accuracy

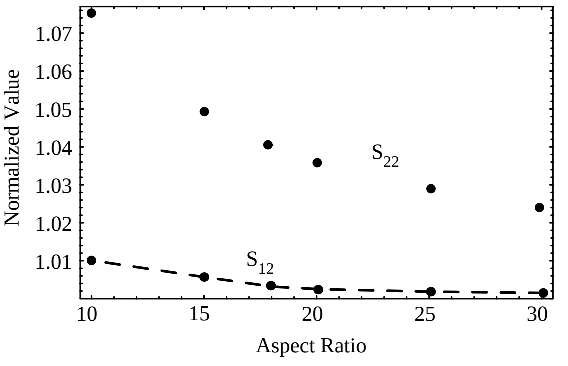

Fig. 2. Convergence study of thin-walled beam theory.

based on this difference. It is not clear to the authors at this stage what would be the source of this term, although the values are reported for completeness.

It may seem that the errors for S_{22} and S_{23} are too large to be acceptable. It is comforting to find out that even for an untwisted isotropic strip with the same aspect ratio ($c/h = 10$), the exact torsional stiffness ($0.312Gch^3$) is about 6.7% different from the result $Gch^3/3$ that is based on thin-walled assumptions. Therefore, the amount of error for composite beams as shown in Table 1 is not unreasonable. Theories such as the present one, which take advantage of the smallness of thickness, should converge to the exact solution when the aspect ratio h/c goes to zero. For this purpose, initially twisted composite strips with different aspect ratios are studied; the convergence trends of S_{12}, S_{22} are plotted in Fig. 2, where the dashed line-dot plot for S_{12} and solid line-dot for S_{22}. The values of the present theory are normalized by VABS results. It is found that S_{23} always has the same ratio as S_{22}, which can be explained by the fact that both values are proportional to ch^3. S_{12}

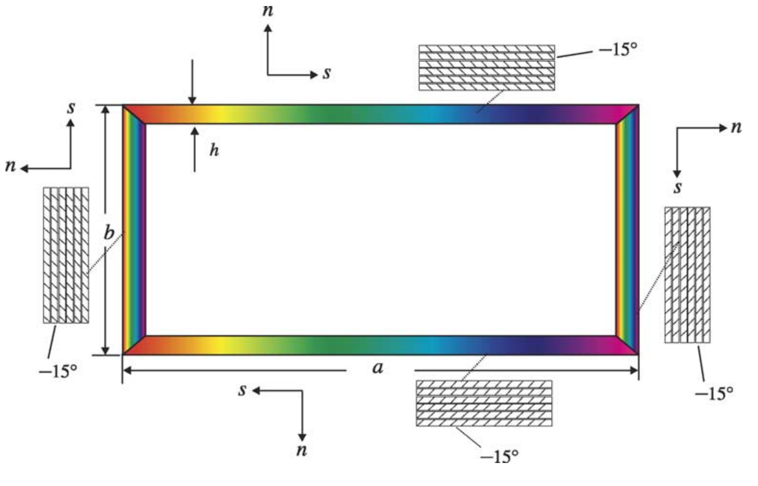

Fig. 3. Schematic of cross-section of a thin-walled box-beam.

converges to the VABS result faster than S_{22} because part of it is contributed by the bending stiffness in the stiff direction according to Eq. (56). The torsional stiffness is very sensitive to the assumptions one uses to develop the theory. Even when the aspect ratio is 30, the difference of the torsional stiffness S_{22} has a value around 2.3%. Nevertheless, as one can observe from the plot, the values from the present theory indeed converge to the more accurate values produced by VABS.

The third example is a thin-walled isotropic box-beam with length a, width b, and thickness h (see Fig. 3 for dimensions). The box-beam is made of isotropic material with Young's modulus E and Poission's ratio ν. Following a similar procedure as for the isotropic strip, analytical formulas can be obtained for the stiffnesses as

$$
\begin{aligned}
S_{11} &= 2(a+b)Eh \\
S_{12} &= \frac{Ehk_1(a^4+4ba^3-3a^2(2b^2+h^2)+a(4b^3-6bh^2)+b^4-3b^2h^2)}{6(a+b)} \\
S_{22} &= \frac{Eh(a^2(3b^2+h^2)+2abh^2+b^2h^2)}{3(a+b)(1+\nu)} \\
S_{33} &= \frac{Eh(b^3+3ab^2+ah^2)}{6} \qquad S_{44} = \frac{Eh(a^3+3a^2b+bh^2)}{6}
\end{aligned}
\tag{59}
$$

One can verify that the extension–twisting coupling satisfies the exact relation in Eq. (56). Note that S_{11}, S_{33}, and S_{44} are the same as the exact solutions. The difference between the traditional thin-walled theory (assuming h/a and h/b are zero) and the present theory is of the order of $(h/c)^2$, with c denoting the larger value of a and b.

The last example is a pretwisted thin-walled box-beam with length $a=0.923$ in., width $b=0.5$ in., thickness $h=0.03$ in., and initial twist $k_1=0.1$/in. This box-beam is made of an anisotropic material with the same properties as those for the composite strip, which are given by Eq. (58). The six layers have the same ply angles of $-15°$. The results are shown in Table 2. Again excellent agreement is found between VABS and the present theory. Except for the torsional stiffness, the differences of other values are less than 1%. It is noticed that without including D_1 in Eq. (25) an error of about 20% is introduced for S_{12} in comparison to VABS results. Hence, a systematic dimensional reduction should be carried out by consistently including all the first-order correction due to k_1 in both steps of reduction from 3D to 2D and from 2D to 1D. Direct use of stiffness matrices from classical

Table 2
Stiffness for pretwisted composite box-beam

S	VABS	Present	Difference (%)
S_{11}	0.19600322×10^7	0.19594756×10^7	0.028
S_{12}	0.91355458×10^5	0.91558325×10^5	0.22
S_{22}	0.52561275×10^5	0.516076716×10^5	1.8
S_{33}	0.90793318×10^5	0.90459553×10^5	0.37
S_{44}	0.23141874×10^6	0.23051801×10^6	0.39

shell theory without geometric correction, as suggested in Refs. [9,21], will introduce some differences to the final thin-walled beam models with initial twist. However, for the isotropic case, it is verified that differences between the present theory and those of Refs. [9,21] are of the order of $(h/c)^2$, which is allowable in thin-walled beam theories (i.e. theories that take advantage of the smallness of h/c). Indeed, it is encouraging to find out that results from the present theory satisfy the analytical formula in Eq. (56), obtained using the VAM for arbitrary beams. In spite of the discontinuity in the slope of the contour, the box-beam case clearly demonstrates the usage and accuracy of the present theory for thin-walled beams with closed sections.

6. Conclusions

A general framework to model initially twisted, thin-walled, composite beams has been developed to consistently capture the first-order correction to the beam stiffness from initial twist. The unique geometry of a thin-walled beam ($h/c \ll 1$ and $c/l \ll 1$) makes it possible to analytically reduce the original 3D representation to a shell model and then further to a beam model. It is shown that to consistently calculate the first-order correction from initial twist one has to obtain this correction during both dimensional reductions, which is against the usual practice in the literature that the pretwisted thin-walled beams are modeled directly from a classical shell theory without geometric corrections.

Simple strip and box-beam examples have been studied to demonstrate the usage and accuracy of the present theory. Results show that the present theory provides a very accurate beam model to approximate the original 3D representation. The present theory can be used with confidence in the design and analysis of pretwisted thin-walled composite beams. A good compromise between accuracy and efficiency has been achieved for modeling thin-walled beams so that tradeoffs and analysis can be performed efficiently and without significant loss of accuracy. It is worthy to point out that the present theory applies to all types of thin-walled beams including strips and closed or open sections without using different theory for different types as traditional techniques demand.

The analytical formulas developed here are implemented using Mathematica™, a symbolic manipulator. The present theory provides an alternative to VABS for structure engineers who want a quick way to calculate the section properties of composite thin-walled beams without tedious modeling effort and computation cost of the finite element method. Any single-celled, thin-walled cross-section can be modeled using the present theory. Additional work is required to extend the present theory to include multiple cells or to introduce refinements to the 1D model such as shear deformation or the Vlasov effect, both of which are already treated by VABS

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