A Geometrically Nonlinear Shear Deformation Theory for Composite Shells

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A geometrically nonlinear shear deformation theory has been developed for elastic shells to accommodate a constitutive model suitable for composite shells when modeled as a two-dimensional continuum. A complete set of kinematical and intrinsic equilibrium equations are derived for shells undergoing large displacements and rotations but with small, two-dimensional, generalized strains. The large rotation is represented by the general finite rotation of a frame embedded in the undeformed configuration, of which one axis is along the normal line. The unit vector along the normal line of the undeformed reference surface is not in general normal to the deformed reference surface because of transverse shear. It is shown that the rotation of the frame about the normal line is not zero and that it can be expressed in terms of other global deformation variables. Based on a generalized constitutive model obtained from an asymptotic dimensional reduction from the three-dimensional energy, and in the form of a Reissner-Mindlin type theory, a set of intrinsic equilibrium equations and boundary conditions follow. It is shown that only five equilibrium equations can be derived in this manner because the component of virtual rotation about the normal is not independent. It is shown, however, that these equilibrium equations contain terms that cannot be obtained without the use of all three components of the finite rotation vector. [DOI: 10.1115/1.1640364]

Introduction

For an elastic three-dimensional continuum, there are two types of nonlinearity: geometrical and physical. A theory is geometrically nonlinear if the kinematical (strain-displacement) relations are nonlinear but the constitutive (stress-strain) relations are linear. This kind of theory allows large displacements and rotations with the restriction that strain must be small. A physically (or materially) nonlinear theory is necessary for biological, rubber-like or inflatable structures where the strain cannot be considered small, and a nonlinear constitutive law is needed to relate the stress and strain. Although this classification seems obvious and clear for a structure modeled as a three-dimensional continuum, it becomes somewhat ambiguous to model dimensionally reducible structures—structures that have one or two dimensions much smaller than the other(s) such as beams, plates, and shells, [1]—using reduced one-dimensional or two-dimensional models. A nonlinear constitutive law for the reduced structural model can in some circumstances be obtained from the reduction of a geometrically nonlinear three-dimensional theory. For example, in the so-called Wagner or trapeze effect, [2–5], the effective torsional rigidity is increased due to axial force. This physically nonlinear one-dimensional model stems from a purely geometrically nonlinear theory at the three-dimensional level. On the other hand, the present paper focuses on a geometrically nonlinear analysis at the three-dimensional level which becomes a geometrically nonlinear analysis at the two-dimensional as well. That is, the two-dimensional generalized strain-displacement relations are nonlinear while the two-dimensional generalized stress-strain relations turn out to be linear.

A shell is a three-dimensional body with a relatively small thickness and a smooth reference surface. The feature of the small thickness attracts researchers to simplify their analyses by reducing the original three-dimensional problem to a two-dimensional problem by taking advantage of the thinness. By comparison with the original three-dimensional problem, an exact shell theory does not exist. Dimensional reduction is an inherently approximate process. Shell theory is a very old subject, since the vibration of a bell was attempted by Euler even before elasticity theory was well established, [6]. Even so, shell theory still receives a lot of attention from modern researchers because it is used so extensively in so many engineering applications. Moreover, many shells are now made with advanced materials that have only recently become available.

Generally speaking, shell theories can be classified according to *direct, derived*, and *mixed* approaches. The direct approach, which originated with the Cosserat brothers, [7], models a shell directly as a two-dimensional ‘‘orientated’’ continuum. Naghdi [8] provided an extensive review of this kind of approach. Although the direct approach is elegant and able to account for transverse and normal strains and rotations associated with couple stresses, it nowhere connects with the fact that a shell is a three-dimensional body and thus completely isolates itself from three-dimensional continuum mechanics. This could be the main reason that this approach has not been much appreciated in the engineering community. One of the complaints of these approaches that they are difficult for numerical implementation has been answered by Simo and his co-workers by providing an efficient formulation ‘‘free from mathematical complexities and suitable for large scale computation,’’ [9,10]. And more recently a similar theory was developed by Ibrahimbegovic [11] to include drilling rotations so that not-so-smooth shell structures can be analyzed conveniently. However, the main complaint remains that these approaches lack a meaningful way to find the constitutive models ‘‘which can only be experienced and formulated properly in our three-dimensional real world,’’ [12]. Reissner [13] developed a very general nonlin-

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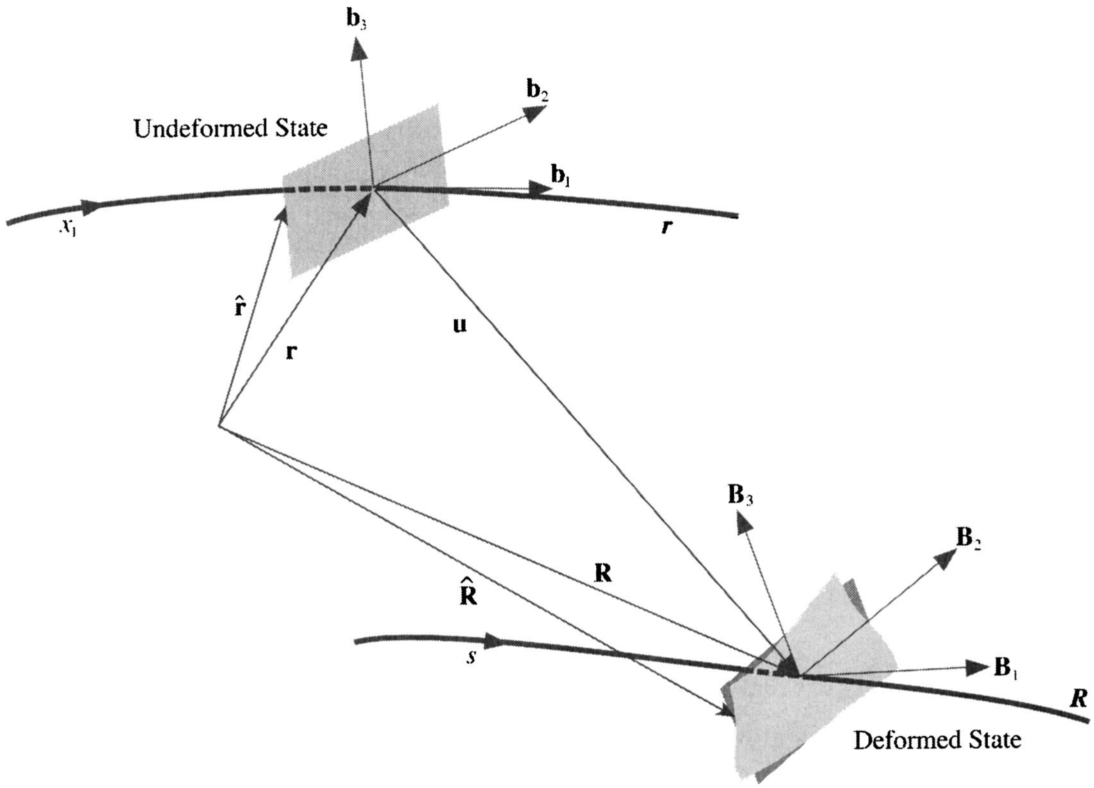

Fig. 1 Schematic of shell deformation

ear shell theory introducing 12 generalized strains by considering the dynamics of stress resultants and couples on the reference surface as the basis. He gracefully avoided the awkwardness of finding a proper constitutive model by pointing out two possible means to establish them. It is recommended in [13] that one could either design experiments to determine the constitutive constants without explicit reference to the three-dimensional nature of the structure or derive an appropriate two-dimensional model from the given knowledge of the constitutive relations for the real three-dimensional model of the structure.

Derived approaches reduce the original three-dimensional elasticity problem into a two-dimensional problem to be solved over the reference surface. Such reductions are usually carried out in one of two ways. The most common approach is to assume a priori the distribution of three-dimensional quantities through the thickness and then to construct a two-dimensional strain energy per unit area by integrating the three-dimensional energy per unit volume through the thickness. Remarkably, classical (also known as Kirchhoff-Love type theory), first-order shear deformation (also known as Reissner-Mindlin type theory), higher-order, and layer-wise shell theories all fall into this category, including the theories proposed by Reddy [14], for example. Another approach is to apply an asymptotic method to expand all quantities into an asymptotic series of the thickness coordinate, so that a sequence of two-dimensional problems can be solved according to the different orders.

The mixed approach is used in [15] based on the argument that all the three-dimensional elasticity equations except the constitutive relations are independent of the material properties, such as the kinematical relations, equilibrium of momentum and forces. The constitutive law must be determined experimentally, and hence it is avoidable that it is approximate. Libai and Simmonds [15] obtain exact shell equations for the balance of momentum, heat flow and an entropy inequality from the three-dimensional continuum mechanics via integration through the thickness. An analogous two-dimensional constitutive law is postulated due to the fact that even three-dimensional constitutive laws are inexact.

There is a sense in which the present approach can also be considered as mixed. The two-dimensional constitutive model is obtained by the variational asymptotic method (VAM), [16], such that the two-dimensional energy is as close to an asymptotic approximation of the original three-dimensional energy as possible, [17]. The process of constructing the constitutive model defines the reference surface and the kinematics of this surface are geometrically exact formulated in an intrinsic format. The two-dimensional equilibrium equations are obtained from the two-dimensional energy with the knowledge of the variations of the generalized strains. The only approximate part of our two-dimensional shell theory is the constitutive law which is not postulated but is mathematically obtained by VAM.

Shell Kinematics

The equations of two-dimensional shell theory are written over the domain of the reference surface, on which every point can be represented by a position vector $\mathbf{r}$ in the undeformed configuration and $\mathbf{R}$ in the deformed configuration (see Fig. 1) with respect to a fixed point O in the space. A set of two curvilinear coordinates, x_α, are required to located a point on the reference surface. The coordinates are so-called *convected* coordinates such that every point of the configuration has the same coordinates during the deformation. (Here and throughout the paper Latin indices assume 1, 2, 3; and Greek indices assume values 1 and 2. Dummy indices are summed over their range except where explicitly indicated.) Without loss of generality, x_α are chosen to be the lines of curvatures of the surface to simplify the formulation. For the purpose of representing finite rotations, an orthonormal triad $\mathbf{b}_i$ is introduced for the initial configuration, such that

$$\mathbf{b}_\alpha = \mathbf{a}_\alpha / A_\alpha \qquad \mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 \tag{1}$$

where $\mathbf{a}_\alpha$ is the set of base vectors associated with x_α and A_α are the Lamé parameters, defined as

$$\mathbf{a}_\alpha=\mathbf{r}_{,\alpha} \quad A_\alpha=\sqrt{\mathbf{a}_\alpha\cdot\mathbf{a}_\alpha}. \tag{2}$$

From the differential geometry of the surface and following [13] and [18], one can express the derivatives of $\mathbf{b}_i$ as

$$\mathbf{b}_{i,\alpha}=A_\alpha\mathbf{k}_\alpha\times\mathbf{b}_i \tag{3}$$

where $\mathbf{k}_\alpha$ is the curvature vector measured in $\mathbf{b}_i$ with the components

$$k_\alpha=\lfloor -k_{\alpha2} \quad k_{\alpha1} \quad k_{\alpha3}\rfloor^T \tag{4}$$

in which $k_{\alpha\beta}$ refers to out-of-plane curvatures. We note that $k_{12}=k_{21}=0$ because the coordinates are the lines of curvatures. The geodesic curvatures $k_{\alpha3}$ can be expressed in terms of the Lamé parameters as

$$k_{13}=-\frac{A_{1,2}}{A_1A_2} \quad k_{23}=\frac{A_{2,1}}{A_1A_2}. \tag{5}$$

When the shell deforms, the particle that had position vector $\mathbf{r}$ in the undeformed state now has position vector $\mathbf{R}$ in the deformed shell. The triad $\mathbf{b}_i$ rotates to be $\mathbf{B}_i$. The rotation relating these two triads can be arbitrarily large and represented in the form of a matrix of direction cosines $C(x_\alpha)$ so that

$$\mathbf{B}_i=C_{ij}\mathbf{b}_j \quad C_{ij}=\mathbf{B}_i\cdot\mathbf{b}_j. \tag{6}$$

A definition of the two-dimensional generalized strain measures is needed for the purpose of formulating this problem in an intrinsic form. Following [13] and [18], they can be defined as

$$\mathbf{R}_{,\alpha}=A_\alpha(\mathbf{B}_\alpha+\epsilon_{\alpha\beta}\mathbf{B}_\beta+2\gamma_{\alpha3}\mathbf{B}_3) \tag{7}$$

and

$$\mathbf{B}_{i,\alpha}=A_\alpha(-K_{\alpha2}\mathbf{B}_1+K_{\alpha1}\mathbf{B}_2+K_{\alpha3}\mathbf{B}_3)\times\mathbf{B}_i \tag{8}$$

where $\epsilon_{\alpha\beta}$ are the two-dimensional in-plane strains, and K_{ij} are the curvatures of the deformed surface, which are the summation of curvatures of undeformed geometry k_{ij} and curvatures introduced by the deformation κ_{ij}, and $\gamma_{\alpha3}$ are the transverse strains because $\mathbf{B}_3$ is not constrained to be normal to the reference surface after deformation. Please note that the two-dimensional generalized strain measures are defined by Eqs. (7) and (8) in an intrinsic fashion, the symmetry of the inplane strain measures such that $\epsilon_{12}=\epsilon_{21}$ does not hold automatically. Nevertheless, one is free to set $\epsilon_{12}=\epsilon_{21}$, i.e.,

$$\frac{\mathbf{B}_1\cdot\mathbf{R}_{,2}}{A_2}=\frac{\mathbf{B}_2\cdot\mathbf{R}_{,1}}{A_1} \tag{9}$$

which is a constraint used in [17] to make the three-dimensional formulation unique.

At this point sufficient preliminary information has been obtained to develop a geometrically nonlinear shell theory.

Compatibility Equations

It is well known that a rigid body in three-dimensional space has only six degrees-of-freedom. Thus, the kinematics of an element of the deformed shell reference surface can be expressed in terms of *at most* six independent quantities: three measures of displacement, say $\mathbf{u}\cdot\mathbf{b}_i$, and three measures of the rotation of $\mathbf{B}_i$ (since the global rotation tensor $\mathbf{C}$, which brings $\mathbf{b}_i$ into $\mathbf{B}_i$, can be expressed in terms of three independent quantities). However, we have the 11 two-dimensional strain measures ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{\alpha3}$, $\kappa_{\alpha\beta}$, and $\kappa_{\alpha3}$ as defined in Eqs. (7) and (8). Thus, they are not independent; there are some compatibility equations among these eleven quantities. In [19] and [13] appropriate compatibility equations are derived by first enforcing the equalities

$$\mathbf{R}_{,12}=\mathbf{R}_{,21} \tag{10}$$

and

$$\mathbf{B}_{i,12}=\mathbf{B}_{i,21}. \tag{11}$$

These two vector equations lead to six independent compatibility equations equivalent to a form of those found in [13]. These equations are rewritten here for convenience in the present notation. First, from the $\mathbf{B}_3$ components of Eq. (10), we obtain

$$(1+\epsilon_{22})\kappa_{12}-(1+\epsilon_{11})\kappa_{21}=\frac{(A_2 2\gamma_{23})_{,1}}{A_1A_2}-\frac{(A_1 2\gamma_{13})_{,2}}{A_1A_2}+\epsilon_{12}(K_{22}-K_{11}). \tag{12}$$

Next, from the $\mathbf{B}_\alpha$ components of Eq. (10) we obtain two equations for $\alpha=1$ and 2, respectively, as

$$\begin{aligned}(1+\epsilon_{22})K_{13}-\epsilon_{12}K_{23}&=\frac{(A_2\epsilon_{12})_{,1}}{A_1A_2}-\frac{[A_1(1+\epsilon_{11})]_{,2}}{A_1A_2}-2\gamma_{13}\kappa_{21}+2\gamma_{23}K_{11}\\(1+\epsilon_{11})K_{23}-\epsilon_{12}K_{13}&=\frac{[A_2(1+\epsilon_{22})]_{,1}}{A_1A_2}-\frac{(A_1\epsilon_{12})_{,2}}{A_1A_2}-2\gamma_{13}K_{22}+2\gamma_{23}\kappa_{12}.\end{aligned} \tag{13}$$

Finally, from the three components of Eq. (11) we have nine identities. However, there are only three independent equations, given by

$$\begin{aligned}&\frac{(A_1K_{11})_{,2}}{A_1A_2}-\frac{(A_2\kappa_{21})_{,1}}{A_1A_2}+K_{13}K_{22}-\kappa_{12}K_{23}=0\\&\frac{(A_1\kappa_{12})_{,2}}{A_1A_2}-\frac{(A_2K_{22})_{,1}}{A_1A_2}+K_{23}K_{11}-\kappa_{21}K_{13}=0\\&\frac{(A_2K_{23})_{,1}}{A_1A_2}-\frac{(A_1K_{13})_{,2}}{A_1A_2}+K_{11}K_{22}-\kappa_{12}\kappa_{21}=0.\end{aligned} \tag{14}$$

There are now 11 quantities which are related by six compatibility equations. This means that these strain measures can be determined in terms of *only five* independent quantities—*not six*.

In the process of dimensional reduction of [17] to find an accurate constitutive model for composite shells, the authors encountered the question whether one should include κ_{21} and κ_{12} as two different generalized strain measures. This was determined by the following argument. Let us denote a new twist measure $2\omega=\kappa_{12}+\kappa_{21}$. From Eq. (12) the difference between κ_{21} and κ_{12} can be obtained as

$$\frac{\kappa_{12}-\kappa_{21}}{2}=\frac{\dfrac{(A_2 2\gamma_{23})_{,1}-(A_1 2\gamma_{13})_{,2}}{A_1A_2}+\epsilon_{12}(K_{22}-K_{11})+\omega(\epsilon_{11}-\epsilon_{22})}{(2+\epsilon_{11}+\epsilon_{22})}. \tag{15}$$

This difference is clearly $O(\varepsilon h/\ell^2)$ or $O(\varepsilon/R)$ disregarding the nonlinear terms (ε is the order of generalized strains, h is the thickness of the shell, l is the wavelength of in-plane deformation and R is the characteristic radius of shell). One can show that it contributes terms that are $O(\varepsilon h^2/\ell^2 h/R)$ or $O(\varepsilon h^2/R^2)$ to the three-dimensional strains. Clearly, such terms will not be counted in a physically linear theory with only correction up to the order of h/R and $(h/l)^2$.

Equations (13) can be solved for the in-plane curvatures κ_{13} and κ_{23}, and Eq. (15) can be used to express κ_{12} and κ_{21} in terms of ω. Now, using these expressions, one can rewrite the *three* Eqs. (14) entirely in terms of the *eight* strain measures ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$, $2\gamma_{23}$, κ_{11}, 2ω, and κ_{22}. This confirms that *only five independent measures of displacement and rotation are necessary to define these strain measures* as we will demonstrate conclusively below by deriving such measures.

Global Displacement and Rotation Variables

There is no unique choice for the global deformation variables. For this reason, the importance (not to mention the beauty) of an intrinsic formulation is widely appreciated. On the other hand, for the purpose of understanding the displacement field more fully, for practical computational algorithms, and for easy derivation of virtual strain-displacement relations, it is expedient to introduce a suitable set of displacement measures.

The displacement measures we choose are derived by expressing $\mathbf{R}$ in terms of $\mathbf{r}$ plus a displacement vector so that

$$\mathbf{R}(x_1,x_2)=\mathbf{r}(x_1,x_2)+u_i\mathbf{b}_i \tag{16}$$

Differentiating both sides of Eq. (16) with respect to x_α, and making use of Eq. (7), one can obtain the identity

$$\mathbf{B}_\alpha+\epsilon_{\alpha\beta}\mathbf{B}_\beta+2\gamma_{\alpha 3}\mathbf{B}_3=\mathbf{b}_\alpha+u_{i;\alpha}\mathbf{b}_i+u_i\mathbf{k}_\alpha\times\mathbf{b}_i \tag{17}$$

where $(\ )_{;\alpha}=1/A_\alpha\partial(\ )/\partial\alpha$. The above formula allows the determination of the strain measures $\epsilon_{\alpha\beta}$ and $2\gamma_{\alpha 3}$ in terms of C, u_i and the derivatives of u_i. Introducing column matrices $u=\lfloor u_1\ u_2\ u_3\rfloor^T$, $e_1=\lfloor 1\ 0\ 0\rfloor^T$, $e_2=\lfloor 0\ 1\ 0\rfloor^T$, $\gamma_1=\lfloor \epsilon_{11}\ \epsilon_{12}\ 2\gamma_{13}\rfloor^T$, and $\gamma_2=\lfloor \epsilon_{21}\ \epsilon_{22}\ 2\gamma_{23}\rfloor^T$, we can obtain the following identity in matrix form:

$$e_\alpha+\gamma_\alpha=C(e_\alpha+u_{;\alpha}+\widetilde{k_\alpha}u) \tag{18}$$

where C is the matrix of direction cosines from Eq. (6), k_α is defined in Eq. (4), and $\widetilde{(\ )}_{ij}=-e_{ijk}(\ )_k$.

Rodrigues parameters, [20], can be used as rotation measures to allow a compact expression of C. These are derived based on Euler's theorem, which shows that any rotation can be represented as a rotation of magnitude Θ about a line parallel to a unit vector $\mathbf{e}$. Defining the Rodrigues parameters $\rho_i=2\mathbf{e}\cdot\mathbf{b}_i\tan(\Theta/2)$ and arranging these in a column matrix $\rho=\lfloor\rho_1\ \rho_2\ \rho_3\rfloor^T$, the matrix C can simply be written as

$$C=\frac{\left(1-\dfrac{\rho^T\rho}{4}\right)I-\tilde{\rho}+\dfrac{\rho\rho^T}{2}}{1+\dfrac{\rho^T\rho}{4}} \tag{19}$$

Let us also denote the direction cosines of $\mathbf{B}_3$ by

$$C_{3i}=\delta_{3i}+\theta_i \tag{20}$$

Hodges [21] has shown that, given the third row of C, the Rodrigues parameters can be uniquely expressed in terms of θ_i as

$$\rho_1=\frac{\rho_3\theta_1-2\theta_2}{2+\theta_3}$$

$$\rho_2=\frac{\rho_3\theta_2+2\theta_1}{2+\theta_3} \tag{21}$$

$$\rho_3=2\tan\left(\frac{\phi_3}{2}\right)$$

where ρ_3 can be understood as a change of variables to simplify later parts of the derivation. Later on we will discuss the meaning of ϕ_3 for a special case. Finally, it is noted that the three rotational parameters θ_i are not independent but instead satisfy the constraint

$$\theta_1^2+\theta_2^2+(1+\theta_3)^2=1. \tag{22}$$

When Eq. (21) is substituted into Eq. (19), the resulting elements of C can be expressed as functions of θ_i and ϕ_3

$$C_{11}=\frac{(2+\theta_3-\theta_1^2)\cos\phi_3-\theta_1\theta_2\sin\phi_3}{2+\theta_3}$$

$$C_{12}=\frac{(2+\theta_3-\theta_2^2)\sin\phi_3-\theta_1\theta_2\cos\phi_3}{2+\theta_3}$$

$$C_{13}=-\theta_1\cos\phi_3-\theta_2\sin\phi_3$$

$$C_{21}=\frac{-(2+\theta_3-\theta_1^2)\sin\phi_3-\theta_1\theta_2\cos\phi_3}{2+\theta_3}$$

$$C_{22}=\frac{(2+\theta_3-\theta_2^2)\cos\phi_3+\theta_1\theta_2\sin\phi_3}{2+\theta_3} \tag{23}$$

$$C_{23}=\theta_1\sin\phi_3-\theta_2\cos\phi_3$$

$$C_{31}=\theta_1$$

$$C_{32}=\theta_2$$

$$C_{33}=1+\theta_3.$$

This representation reduces to those of [22] when considering small, finite rotations. There is an apparent singularity in the present scheme when $\theta_3=-2$ (i.e., when the shell deforms in such a way that $\mathbf{B}_3$ is pointed in the opposite direction of $\mathbf{b}_3$). This should pose no practical problem, however, since $\theta_1=\theta_2=0$ for that condition, and none of the kinematical relations become infinite in the limit as $\theta_3\rightarrow-2$.

When these expressions for the direction cosines are substituted into Eq. (18), explicit expressions for the strain measures can be found as

$$\begin{aligned}\epsilon_{11}=&\left[\frac{(2+\theta_3-\theta_1^2)(1+u_{1;1}-k_{13}u_2+k_{11}u_3)-\theta_1\theta_2(u_{2;1}+k_{13}u_1)}{2+\theta_3}+\theta_1(k_{11}u_1-u_{3;1})\right]\cos\phi_3\\&+\left[\frac{(2+\theta_3-\theta_2^2)(u_{2;1}+k_{13}u_1)-\theta_1\theta_2(1+u_{1;1}-k_{13}u_2+k_{11}u_3)}{2+\theta_3}+\theta_2(k_{11}u_1-u_{3;1})\right]\sin\phi_3-1\end{aligned}$$

$$\begin{aligned}\epsilon_{22}=&\left[\frac{(2+\theta_3-\theta_2^2)(1+u_{2;2}+k_{23}u_1+k_{22}u_3)-\theta_1\theta_2(u_{1;2}-k_{23}u_2)}{2+\theta_3}-\theta_2(u_{3;2}-k_{22}u_2)\right]\cos\phi_3\\&+\left[\frac{(2+\theta_3-\theta_1^2)(k_{23}u_2-u_{1;2})+\theta_1\theta_2(1+u_{2;2}+k_{23}u_1+k_{22}u_3)}{2+\theta_3}+\theta_1(u_{3;2}-k_{22}u_2)\right]\sin\phi_3-1\end{aligned}$$

$$\begin{aligned}\epsilon_{12}=&\left[\frac{(2+\theta_3-\theta_2^2)(u_{2;1}+k_{13}u_1)-\theta_1\theta_2(1+u_{1;1}-k_{13}u_2+k_{11}u_3)}{2+\theta_3}+\theta_2(k_{11}u_1-u_{3;1})\right]\cos\phi_3\\&-\left[\frac{(2+\theta_3-\theta_1^2)(1+u_{1;1}-k_{13}u_2+k_{11}u_3)-\theta_1\theta_2(u_{2;1}+k_{13}u_1)}{2+\theta_3}+\theta_1(k_{11}u_1-u_{3;1})\right]\sin\phi_3\end{aligned}$$

$$\epsilon_{21}=\left[\frac{(2+\theta_3-\theta_1^2)(u_{1;2}-k_{23}u_2)+\theta_1\theta_2(1+u_{2;2}+k_{23}u_1+k_{22}u_3)}{2+\theta_3}+\theta_1(u_{3;2}-k_{22}u_2)\right]\cos\phi_3$$
$$+\left[\frac{(2+\theta_3-\theta_2^2)(1+u_{2;2}+k_{23}u_1+k_{22}u_3)-\theta_1\theta_2(u_{1;2}-k_{23}u_2)}{2+\theta_3}-\theta_2(u_{3;2}-k_{22}u_2)\right]\sin\phi_3$$
$$2\gamma_{13}=\theta_1(1+u_{1;1}-k_{13}u_2+k_{11}u_3)+\theta_2(u_{2;1}+k_{13}u_1)+(1+\theta_3)(u_{3;1}-k_{11}u_1)$$
$$2\gamma_{23}=\theta_1(u_{1;2}-k_{23}u_2)+\theta_2(1+u_{2;2}+k_{23}u_1+k_{22}u_3)+(1+\theta_3)(u_{3;2}-k_{22}u_2). \tag{24}$$

These expressions explicitly depend on $\sin\phi_3$ and $\cos\phi_3$. It is evident that one can choose ϕ_3 so that $\epsilon_{12}=\epsilon_{21}$, yielding

$$\tan\phi_3=\frac{n_1+\theta_2^2(u_{2;1}+k_{13}u_1)-\theta_1^2(u_{1;2}-k_{23}u_2)+\theta_1\theta_2[u_{1;1}-u_{2;2}+(k_{11}-k_{22})u_3-k_{23}u_1-k_{13}u_2]}{n_2+\theta_1^2(u_{1;1}+k_{11}u_3-k_{13}u_2)+\theta_2^2(u_{2;2}+k_{22}u_3+k_{23}u_1)+\theta_1\theta_2(u_{1;2}+u_{2;1}-k_{23}u_2+k_{13}u_1)} \tag{25}$$

where

$$n_1=(2+\theta_3)[u_{1;2}-u_{2;1}-\theta_1(u_{3;2}-k_{22}u_2)+\theta_2(u_{3;1}-k_{11}u_1)-k_{13}u_1-k_{23}u_2] \tag{26}$$
$$n_2=(2+\theta_3)[\theta_1(u_{3;1}-k_{11}u_1)+\theta_2(u_{3;2}-k_{22}u_2)-u_{1;1}-u_{2;2}-2-\theta_3-k_{23}u_1-(k_{22}+k_{11})u_3+k_{13}u_2]$$

It is now clear that once the functions u_1, u_2, u_3, θ_1 and θ_2 are known, the entire deformation is determined. Because of this, one should expect that a variational formulation would yield only five equilibrium equations–*not six*.

For small displacement and small strain, one can obtain ϕ_3 as

$$\phi_3=\frac{(A_2u_2)_{,1}-(A_1u_1)_{,2}}{2A_1A_2} \tag{27}$$

which is half the angle of rotation about $\mathbf{B}_3$, the same as obtained in [23].

Although one can now find exact expressions for ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$ and $2\gamma_{23}$ which are independent of ϕ_3, such expressions are rather lengthy and are not given here. Alternatively, one could leave ϕ_3 in the equations and regard Eq. (25) as a constraint. This would allow the construction of a shell finite element which would be compatible with beam elements which have three rotational degrees-of-freedom at the nodes.

Expressions for the curvatures can be found in terms of C as

$$\widetilde{K_\alpha}=-C_{;\alpha}C^T+C\widetilde{k_\alpha}C^T \tag{28}$$

where

$$K_\alpha=\lfloor -k_{\alpha2}\ \ k_{\alpha1}\ \ k_{\alpha3}\rfloor^T+\lfloor -\kappa_{\alpha2}\ \ \kappa_{\alpha1}\ \ \kappa_{\alpha3}\rfloor^T \tag{29}$$

Following [24], the curvature vector can also be found using Rodrigues parameters

$$K_\alpha=\frac{I-\frac{\tilde{\rho}}{2}}{1+\frac{\rho^T\rho}{4}}\rho_{;\alpha}+Ck_\alpha. \tag{30}$$

Using the form of C from Eqs. (23), the curvatures become

$$\kappa_{\alpha1}=\theta_{1;\alpha}\cos\phi_3+\theta_{2;\alpha}\sin\phi_3-\frac{\theta_{3;\alpha}(\theta_1\cos\phi_3+\theta_2\sin\phi_3)}{2+\theta_3}+\hat{k}_{\alpha1}-k_{\alpha1}$$
$$\kappa_{\alpha2}=-\theta_{1;\alpha}\sin\phi_3+\theta_{2;\alpha}\cos\phi_3+\frac{\theta_{3;\alpha}(\theta_1\sin\phi_3-\theta_2\cos\phi_3)}{2+\theta_3}+\hat{k}_{\alpha2}-k_{\alpha2} \tag{31}$$
$$\kappa_{\alpha3}=\phi_{3;\alpha}+\frac{\theta_{1;\alpha}\theta_2-\theta_1\theta_{2;\alpha}}{2+\theta_3}+\hat{k}_{\alpha3}$$

where

$$\hat{k}_{\alpha1}=\left(k_{\alpha3}-\frac{k_{\alpha2}\theta_1}{2+\theta_3}+\frac{k_{\alpha1}\theta_2}{2+\theta_3}\right)(\theta_1\sin\phi_3-\theta_2\cos\phi_3)+k_{\alpha2}\sin\phi_3+k_{\alpha1}\cos\phi_3$$
$$\hat{k}_{\alpha2}=\left(k_{\alpha3}-\frac{k_{\alpha2}\theta_1}{2+\theta_3}+\frac{k_{\alpha1}\theta_2}{2+\theta_3}\right)(\theta_1\cos\phi_3+\theta_2\sin\phi_3)+k_{\alpha2}\cos\phi_3-k_{\alpha1}\sin\phi_3 \tag{32}$$
$$\hat{k}_{\alpha3}=-k_{\alpha2}\theta_1+k_{\alpha1}\theta_2+k_{\alpha3}\theta_3.$$

As before, ϕ_3 can be eliminated from these expressions, so that all six curvatures can be expressed in terms of five independent quantities. Note that $\kappa_{\alpha3}$ are not independent two-dimensional generalized strains. They will, however, appear in the equilibrium equations because of their appearance in the virtual strain-displacement relations derived later.

Two-Dimensional Constitutive Law

To complete the analysis for an elastic shell, a two-dimensional constitutive law is required to relate two-dimensional generalized stresses and strains. As mentioned before the constitutive law can not be exact, however, one should try to avoid introducing any unnecessary approximation in addition to the already-approximate three-dimensional constitutive relations.

Among many approaches that have been proposed to deal with dimensional reduction, the approach in [17] stands out for its accuracy and simplicity. In that work, a simple Reissner-Mindlin type energy model is constructed that is as close as possible to being asymptotically correct. Moreover, the original three-dimensional results can be recovered accurately. The resulting model can be expressed as

$$2\Pi=\epsilon^TA\epsilon+\gamma^TG\gamma+2\epsilon^TF \tag{33}$$

where $\epsilon=\lfloor\epsilon_{11}\ 2\epsilon_{12}\ \epsilon_{22}\ \kappa_{11}\ \kappa_{12}+\kappa_{21}\ \kappa_{22}\rfloor^T$ and $\gamma=\lfloor 2\gamma_{13}\ 2\gamma_{23}\rfloor^T$. It is noticed that there is only one in-plane shear strain ϵ_{12} in Eq. (33). This is possible only after one uses the constraints in Eq. (9). Moreover, the strain energy is independent of $\kappa_{\alpha3}$ so that the rotation about the normal only appears algebraically, making it possible for it to be eliminated.

This simple constitutive model is rigorously reduced from the original three-dimensional model for multilayer shells, each layer of which is made with an anisotropic material with monoclinic symmetry. The variational asymptotic method [16] is used to guarantee the resulting two-dimensional shell model to yield the best approximation to the energy stored in the original three-dimensional structure by discarding all the insignificant contribution to the energy higher than the order of $(h/l)^2$ and h/R. The stiffness matrices A and G obtained through this process carry all the material and geometry information through the thickness (see Eqs. (63) and (73) in Ref. [17] for detailed expressions). The term containing the column matrix F is produced by body forces in the shell structure and tractions on the top and bottom surfaces, and it is very important for the recovery of the original three-dimensional results. Interested readers can refer to Ref. [17] for details of constructing the model in Eq. (33) for multilayered composite shells.

Having obtained the two-dimensional constitutive law from three-dimensional elasticity, one can derive all the other relations over the reference surface of the shell, a two-dimensional continuum.

Virtual Strain-Displacement Relations

In order to derive intrinsic equilibrium equations from the two-dimensional energy, it is necessary to express the variations of generalized strain measures in terms of virtual displacements and virtual rotations.

The variation of the energy expressed in Eq. (33) can be written as

$$\delta\Pi=\frac{\partial\Pi}{\partial\epsilon_{11}}\delta\epsilon_{11}+\frac{\partial\Pi}{\partial\epsilon_{12}}\delta\epsilon_{12}+\frac{\partial\Pi}{\partial\epsilon_{22}}\delta\epsilon_{22}+\frac{\partial\Pi}{\partial\gamma_{13}}\delta\gamma_{13}+\frac{\partial\Pi}{\partial\gamma_{23}}\delta\gamma_{23}+\frac{\partial\Pi}{\partial\kappa_{11}}\delta\kappa_{11}+\frac{\partial\Pi}{\partial\omega}\delta\omega+\frac{\partial\Pi}{\partial\kappa_{22}}\delta\kappa_{22}. \tag{34}$$

It is now obvious that one must express $\delta\epsilon_{11},\ldots,\delta\kappa_{22}$, in terms of virtual displacements and rotations in order to obtain the final Euler-Lagrange equations of the energy functional in their intrinsic form. Following Ref. [24], we introduce measures of virtual displacement and rotation that are "compatible" with the intrinsic strain measures. For the virtual displacement, we note the form of Eq. (18) and choose

$$\overline{\delta q}=C\,\delta u. \tag{35}$$

Similarly, for the virtual rotation, we note the form of Eq. (28) and write

$$\widetilde{\overline{\delta\psi}}=-\delta CC^T \tag{36}$$

where $\overline{\delta\psi}$ is a column matrix arranged similarly as the curvature column matrix in Eq. (4) $\overline{\delta\psi}=\lfloor-\overline{\delta\psi_2}\ \overline{\delta\psi_1}\ \overline{\delta\psi_3}\rfloor^T$. The bars indicate that these quantities are not necessarily the variations of functions. Using these relations it is clear that

$$\delta u=C^T\overline{\delta q} \tag{37}$$

and

$$\delta C=-\widetilde{\overline{\delta\psi}}C. \tag{38}$$

Let us begin with the generalized strain-displacement relationship, Eq. (18). A particular in-plane strain element can be written as

$$\epsilon_{\alpha\beta}=e_\beta^T[C(e_\alpha+u_{;\alpha}+\widetilde{k_\alpha}u)-e_\alpha]. \tag{39}$$

Taking a straightforward variation, one obtains

$$\delta\epsilon_{\alpha\beta}=e_\beta^T[\delta C(e_\alpha+u_{;\alpha}+\widetilde{k_\alpha}u)+C(\delta u_{;\alpha}+\widetilde{k_\alpha}\delta u)]. \tag{40}$$

The right-hand side contains $u_{;\alpha}$ and $\delta u_{;\alpha}$, which must be eliminated in order to obtain variations of the strain that are independent of displacements. These are needed to derive intrinsic equilibrium equations.

Premultiplying both sides of Eq. (18) by C^T, making use of Eq. (36), and finally using a property of the tilde operator that, for arbitrary column matrices Y and Z, $\widetilde{Y}Z=-\widetilde{Z}Y$, one can make the first term in brackets on the righthand side independent of $u_{;\alpha}$. After all this, one obtains

$$\begin{aligned}\delta C(e_\alpha+u_{;\alpha}+\widetilde{k_\alpha}u)&=\delta CC^T(e_\alpha+\gamma_\alpha)=-\widetilde{\overline{\delta\psi}}(e_\alpha+\gamma_\alpha)\\&=(\widetilde{e_\alpha}+\widetilde{\gamma_\alpha})\overline{\delta\psi}.\end{aligned} \tag{41}$$

An expression for the second term in brackets on the right-hand side of Eq. (40) can now be obtained by differentiating Eq. (37) with respect to x_α and premultiplying by C. This yields

$$C(\delta u_{;\alpha}+\widetilde{k_\alpha}\delta u)=C(C^T\overline{\delta q})_{;\alpha}+C\widetilde{k_\alpha}\delta u=\overline{\delta q}_{;\alpha}+\widetilde{K_\alpha}\overline{\delta q}. \tag{42}$$

Substituting Eqs. (41) and (42) into Eq. (40), one obtains an intrinsic expression for the variation of the in-plane strain components as

$$\delta\epsilon_{\alpha\beta}=e_\beta^T[\overline{\delta q}_{;\alpha}+\widetilde{K_\alpha}\overline{\delta q}+(\widetilde{e_\alpha}+\widetilde{\gamma_\alpha})\overline{\delta\psi}] \tag{43}$$

where $e_\beta^T\widetilde{e_\alpha}$ vanishes when $\alpha=\beta$. This matrix equation can be written explicitly as four scalar equations:

$$\begin{aligned}\delta\epsilon_{11}&=\overline{\delta q}_{1;1}-K_{13}\overline{\delta q}_2+K_{11}\overline{\delta q}_3-2\gamma_{13}\overline{\delta\psi}_1+\epsilon_{12}\overline{\delta\psi}_3\\\delta\epsilon_{12}&=\overline{\delta q}_{2;1}+K_{13}\overline{\delta q}_1+\kappa_{12}\overline{\delta q}_3-2\gamma_{13}\overline{\delta\psi}_2-(1+\epsilon_{11})\overline{\delta\psi}_3\\\delta\epsilon_{21}&=\overline{\delta q}_{1;2}-K_{23}\overline{\delta q}_2+\kappa_{21}\overline{\delta q}_3-2\gamma_{23}\overline{\delta\psi}_1+(1+\epsilon_{22})\overline{\delta\psi}_3\\\delta\epsilon_{22}&=\overline{\delta q}_{2;2}+K_{23}\overline{\delta q}_1+K_{22}\overline{\delta q}_3-2\gamma_{23}\overline{\delta\psi}_2-\epsilon_{12}\overline{\delta\psi}_3.\end{aligned} \tag{44}$$

The variations $\delta\epsilon_{12}$ and $\delta\epsilon_{21}$ should be equal due to Eq. (9); hence, one can solve for the virtual rotation component about $\mathbf{B}_3$ as

$$\overline{\delta\psi}_3=\frac{\overline{\delta q}_{2;1}-\overline{\delta q}_{1;2}+K_{13}\overline{\delta q}_1+K_{23}\overline{\delta q}_2+(\kappa_{12}-\kappa_{21})\overline{\delta q}_3-2\gamma_{13}\overline{\delta\psi}_2+2\gamma_{23}\overline{\delta\psi}_1}{2+\epsilon_{11}+\epsilon_{22}}. \tag{45}$$

It is now possible to write the variations of all strain measures in terms of three virtual displacement and two virtual rotation components as

$$\begin{aligned}\delta\epsilon_{11}&=\overline{\delta q}_{1;1}-K_{13}\overline{\delta q}_2+K_{11}\overline{\delta q}_3-2\gamma_{13}\overline{\delta\psi}_1+\epsilon_{12}\overline{\delta\psi}_3\\\delta\epsilon_{22}&=\overline{\delta q}_{2;2}+K_{23}\overline{\delta q}_1+K_{22}\overline{\delta q}_3-2\gamma_{23}\overline{\delta\psi}_2-\epsilon_{12}\overline{\delta\psi}_3\\2\delta\epsilon_{12}&=\overline{\delta q}_{2;1}+\overline{\delta q}_{1;2}+K_{13}\overline{\delta q}_1-K_{23}\overline{\delta q}_2+2\omega\overline{\delta q}_3\\&\quad-2\gamma_{13}\overline{\delta\psi}_2-2\gamma_{23}\overline{\delta\psi}_1+(\epsilon_{22}-\epsilon_{11})\overline{\delta\psi}_3\end{aligned} \tag{46}$$

with $\overline{\delta\psi}_3$ taken from Eq. (45).

Let us now consider the transverse shear strains

$$2\gamma_{\alpha3}=e_3^T[C(e_\alpha+u_{;\alpha}+\widetilde{k_\alpha}u)-e_\alpha]. \tag{47}$$

Following a procedure similar to the above, one can obtain the virtual strain-displacement equation for transverse shear strains as

$$2\,\delta\gamma_{\alpha 3} = e_3^T[\overline{\delta q}_{;\alpha} + \widetilde{K_\alpha}\overline{\delta q} + (\widetilde{e_\alpha} + \widetilde{\gamma_\alpha})\overline{\delta\psi}]. \tag{48}$$

Explicit expressions for the variations of the shear strain components are now easily written as

$$2\,\delta\gamma_{\alpha 3} = \overline{\delta q}_{3;\alpha} + \overline{\delta\psi}_\alpha + \epsilon_{\alpha\beta}\overline{\delta\psi}_\beta - K_{\alpha\beta}\overline{\delta q}_\beta . \tag{49}$$

Finally, variations of the curvatures are found. First, taking the straightforward variation of Eq. (28), one obtains

$$\delta\widetilde{\kappa_\alpha} = -\frac{\delta C_{,\alpha}C^T}{A_\alpha} - \frac{C_{,\alpha}\delta C^T}{A_\alpha} + \delta C\widetilde{k_\alpha}C^T + C\widetilde{k_\alpha}\delta C^T. \tag{50}$$

In order to eliminate $\delta C_{,\alpha}$, we differentiate Eq. (36) with respect to x_α

$$\widetilde{\overline{\delta\psi}}_{,\alpha} = -\delta C_{,\alpha}C^T - \delta C C^T_{,\alpha} . \tag{51}$$

In order to eliminate δC, we can use Eq. (38). Then, Eq. (50) becomes

$$\delta\widetilde{\kappa_\alpha} = \widetilde{\overline{\delta\psi}}_{;\alpha} + \widetilde{K_\alpha}\widetilde{\overline{\delta\psi}} - \widetilde{\overline{\delta\psi}}\widetilde{K_\alpha}. \tag{52}$$

Using another tilde identity ($\widetilde{\widetilde{Y}Z} = \widetilde{Y}\widetilde{Z} - \widetilde{Z}\widetilde{Y}$) one can find the virtual strain-displacement relation as

$$\delta\kappa_\alpha = \overline{\delta\psi}_{;\alpha} + \widetilde{K_\alpha}\,\overline{\delta\psi}. \tag{53}$$

In explicit form

$$\delta\kappa_{11} = \frac{\overline{\delta\psi}_{1,1}}{A_1} - K_{13}\overline{\delta\psi}_2 + \kappa_{12}\overline{\delta\psi}_3$$

$$\delta\kappa_{22} = \frac{\overline{\delta\psi}_{2,2}}{A_2} + K_{23}\overline{\delta\psi}_1 - \kappa_{21}\overline{\delta\psi}_3 \tag{54}$$

$$2\,\delta\omega = \frac{\overline{\delta\psi}_{1,2}}{A_2} + \frac{\overline{\delta\psi}_{2,1}}{A_1} + K_{13}\overline{\delta\psi}_1 - K_{23}\overline{\delta\psi}_2 + (K_{22} - K_{11})\,\overline{\delta\psi}_3$$

where $\overline{\delta\psi}_3$ can again be eliminated by using Eq. (45).

Intrinsic Equilibrium Equations

In this section, we will make use of the virtual strain-displacement relations in the variation of the internal strain energy in order to derive the intrinsic equilibrium equations. Here we define the generalized forces as

$$\frac{\partial\Pi}{\partial\epsilon_{11}} = N_{11} \quad \frac{\partial\Pi}{\partial\epsilon_{22}} = N_{22} \quad \frac{1}{2}\frac{\partial\Pi}{\partial\epsilon_{12}} = N_{12}$$

$$\frac{\partial\Pi}{\partial\kappa_{11}} = M_{11} \quad \frac{\partial\Pi}{\partial\kappa_{22}} = M_{22} \quad \frac{1}{2}\frac{\partial\Pi}{\partial\omega} = M_{12} \tag{55}$$

$$\frac{1}{2}\frac{\partial\Pi}{\partial\gamma_{13}} = Q_1 \quad \frac{1}{2}\frac{\partial\Pi}{\partial\gamma_{23}} = Q_2 .$$

To use the principle of virtual work to derive the equilibrium equations, one needs to know the applied loads. In addition to the applied loads used in the modeling process, $\tau_i\mathbf{B}_i$ at the top surface, $\beta_i\mathbf{B}_i$ at the bottom surface and body force $\phi_i\mathbf{B}_i$ [17], one can also specify appropriate combinations of displacements, rotations (geometrical boundary conditions), running forces and moments (natural boundary conditions) along the boundary around the reference surface. It is trivial to apply the geometrical boundary conditions. Although it is possible in most cases that natural boundary conditions can be derived from Newton's law, the procedure is tedious and not easily applied here because the physical meanings for some of the generalized forces are not clear. Thus, natural boundary conditions are best derived from the principle of virtual work.

Suppose on boundary Γ (see Fig. 2), we specify a force resultant $\hat{N}_{\nu\nu}$ and moment resultant $\hat{M}_{\nu\nu}$ along the outward normal of

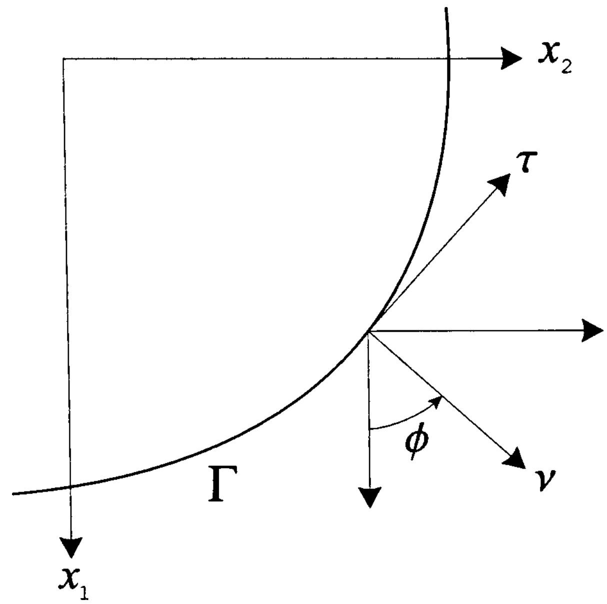

Fig. 2 Schematic of an arbitrary boundary

the boundary curve tangent to the reference surface ν, $\hat{N}_{\nu\tau}$ and $\hat{M}_{\nu\tau}$ along the tangent of the boundary curve τ, $\hat{N}_{\nu 3}$ along the normal of the reference surface. Then the principle of virtual work (strictly speaking, the principle of virtual displacements) can be stated as:

$$\int\int_s(\delta\Pi - \overline{\delta q}_i f_i - \overline{\delta\psi}_\alpha m_\alpha)A_1A_2\,\mathrm{d}x_1\mathrm{d}x_2 - \int_\Gamma(\hat{N}_{\nu\nu}\overline{\delta q}_\nu + \hat{N}_{\nu\tau}\overline{\delta q}_\tau + \hat{N}_{\nu 3}\overline{\delta q}_3 + \hat{M}_{\nu\nu}\overline{\delta\psi}_\nu + \hat{M}_{\nu\tau}\overline{\delta\psi}_\tau)\mathrm{d}\Gamma = 0 \tag{56}$$

where f_i and m_α are taken directly from [17].

It is now possible to obtain intrinsic equilibrium equations and consistent edge conditions by use of the principle of virtual work and the virtual strain-displacement relations derived in the previous section. The equilibrium equations are

$$\frac{(A_2N_{11})_{,1}}{A_1A_2} + \frac{[A_1(N_{12}+\mathcal{N})]_{,2}}{A_1A_2} - K_{13}(N_{12}-\mathcal{N}) - K_{23}N_{22} + Q_1K_{11} + Q_2\kappa_{21} + f_1 = 0$$

$$\frac{(A_1N_{22})_{,2}}{A_1A_2} + \frac{[A_2(N_{12}-\mathcal{N})]_{,1}}{A_1A_2} + K_{23}(N_{12}+\mathcal{N}) + K_{13}N_{11} + Q_1\kappa_{12} + Q_2K_{22} + f_2 = 0$$

$$\frac{(A_2Q_1)_{,1}}{A_1A_2} + \frac{(A_1Q_2)_{,2}}{A_1A_2} - K_{11}N_{11} - K_{22}N_{22} - 2\,\omega N_{12} + (\kappa_{12} - \kappa_{21})\mathcal{N} + f_3 = 0 \tag{57}$$

$$\frac{(A_2M_{11})_{,1}}{A_1A_2} + \frac{(A_1M_{12})_{,2}}{A_1A_2} - Q_1(1+\epsilon_{11}) - Q_2\epsilon_{12} + 2\,\gamma_{13}N_{11} + 2\,\gamma_{23}(N_{12}+\mathcal{N}) - M_{12}K_{13} - M_{22}K_{23} + m_1 = 0$$

$$\frac{(A_2M_{12})_{,1}}{A_1A_2} + \frac{(A_1M_{22})_{,2}}{A_1A_2} - Q_2(1+\epsilon_{22}) - Q_1\epsilon_{12} + 2\,\gamma_{13}(N_{12}-\mathcal{N}) + 2\,\gamma_{23}N_{22} + M_{11}K_{13} + M_{12}K_{23} + m_2 = 0$$

where

$$\mathcal{N}=\frac{(N_{22}-N_{11})\epsilon_{12}+N_{12}(\epsilon_{11}-\epsilon_{22})+M_{22}\kappa_{21}-M_{11}\kappa_{12}+M_{12}(K_{11}-K_{22})}{2+\epsilon_{11}+\epsilon_{22}}. \tag{58}$$

The associated natural boundary conditions on Γ are

$$\begin{aligned}
\hat{N}_{\nu\nu}&=\nu_1^2N_{11}+2\nu_1\nu_2N_{12}+\nu_2^2N_{22}\\
\hat{N}_{\nu\tau}&=\nu_1\nu_2(N_{22}-N_{11})+(\nu_1^2-\nu_2^2)N_{12}-\mathcal{N}\\
\hat{N}_{\nu\nu}&=\nu_1^2N_{11}+2\nu_1\nu_2N_{12}+\nu_2^2N_{22}\\
\hat{N}_{\nu3}&=\nu_1Q_1+\nu_2Q_2\\
\hat{M}_{\nu\nu}&=\nu_1^2M_{11}+2\nu_1\nu_2M_{12}+\nu_2^2M_{22}\\
\hat{M}_{\nu\tau}&=\nu_1\nu_2(M_{22}-M_{11})+(\nu_1^2-\nu_2^2)M_{12}
\end{aligned} \tag{59}$$

where $\nu_1=\cos\phi$, $\nu_2=\sin\phi$, and ϕ is the angle between the outward normal of the boundary and the x_1 direction as shown in Fig. 2. The terms containing $\mathcal{N}$ stem from consistent inclusion of the finite rotation from undeformed triad to deformed triad although the *nonzero* rotation about $\mathbf{B}_3$ is expressed in terms of other kinematical quantities. Similar terms are found in the shell equations derived by Berdichevsky [16] where only five equilibrium equations are derived.

In a mixed formulation, $\mathcal{N}$ can be shown to be the Lagrange multiplier that enforces Eq. (45). To further understand the nature of $\mathcal{N}$ one can undertake the following exercise: Setting $P_i=0$ and $\epsilon_{12}=\epsilon_{21}$ for the equilibrium equations given in [13], $(N_{21}-N_{12})/2$ can be solved from Reissner's sixth equilibrium equation. This shows that Reissner's $(N_{21}-N_{12})/2$ is the same as our $\mathcal{N}$, and Reissner's $(N_{21}-N_{12})/2$ is the same as our N_{12}. Finally, substitution of this sixth equation into the other five yields the five equilibrium equations given here in Eqs. (57). It is noted that Reissner's equilibrium equations are derived based on the basis of Newton's law of motion without consideration of either constitutive law or strain-displacement relations. However, the present derivation is purely displacement-based. The reproduction of those equilibrium equations by the present derivation illustrates that, as long as the formulation is geometrically exact, one can derive exact equilibrium equations.

A few investigators have noted an apparent conflict between the symmetry of the stress resultants and the satisfaction of moment equilibrium about the normal. In reality there is no conflict, but one must be careful. We have shown herein that the triad $\mathbf{B}_i$ can always be chosen so that $\epsilon_{12}=\epsilon_{21}$. If this relation is enforced strongly, there is only one in-plane shear stress resultant, N_{12}, that can be derived from the energy. In that case the physical quantity associated with the antisymmetric part of Reissner's in-plane stress resultants, while it is not available from the constitutive law, is nevertheless available as a reactive quantity from the moment equilibrium equation about the normal. However, it must be stressed that the moment equilibrium equation about the normal is not available from a conventional energy approach, in which the virtual displacements and rotations must be independent.

In a somewhat similar vein, not being able to obtain the antisymmetric part of the moment stress resultants from derivatives of the two-dimensional strain energy is a result of the approximate dimensional reduction process in which it was determined, based on asymptotic considerations and *geometrically* nonlinear three-dimensional elasticity, that the antisymmetric term $\kappa_{12}-\kappa_{21}$ does not appear as an independent generalized strain measure in the two-dimensional constitutive law with correction only to the order of h/R. However, if a more refined theory with respect to h/R is required, then $\kappa_{12}-\kappa_{21}$ would appear as a generalized strain in the two-dimensional constitutive law and a new generalized moment would be defined based on the constitutive law.

For practical computational schemes, equilibrium equations and boundary conditions need to use the constitutive law to relate with the generalized two-dimensional strains. Finally a set of kinematical equations is needed. Depending on how this part is done, the analysis can be completed in either of two fundamentally different ways: a purely intrinsic form, relying on compatibility equations, and a mixed form relying on explicit strain-displacement relations.

In the intrinsic form we have five equilibrium equations, Eqs. (57); six compatibility equations, Eqs. (12)–(14); and the eight constitutive equations—a total of 19 equations. The 19 unknowns are the eight stress resultants, N_{11}, N_{12}, N_{22}, Q_1, Q_2, M_{11}, M_{12}, and M_{22}; and the 11 strain measures ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$, $2\gamma_{23}$, κ_{11}, $2\omega_{12}$, and κ_{22}, along with κ_{13}, κ_{23}, and $\kappa_{12}-\kappa_{21}$. The last three strain measures appear in the equilibrium equations but not in the constitutive law.

In a mixed formulation one would use the same five equilibrium equations and eight constitutive equations. One would also need a set of strain-displacement relations among the 11 generalized strain measures ϵ_{11}, $2\epsilon_{12}$, ϵ_{22}, $2\gamma_{13}$, $2\gamma_{23}$, κ_{11}, 2ω, and κ_{22}, along with κ_{13}, κ_{23}, and $\kappa_{12}-\kappa_{21}$, and the five global displacement and rotational variables u_1, u_2, u_3, θ_1, and θ_2. One possible set of such equations is as follows: use five of Eqs. (24), using either ϵ_{12} or ϵ_{21}; use the six Eqs. (31). There are also the two other rotational variables θ_3 and ϕ_3, which are governed by Eqs. (22) and (25), respectively. This way there are 26 equations and 26 unknowns. This mixed formulation is capable of handling boundary conditions on two-dimensional stress resultants and displacement/rotation variables. At least in principle, one could recover a displacement formulation by eliminating all the unknowns except the displacement and rotation variables.

Equations (57) and (58) contain terms that could be disregarded because of the original assumption of small strain. We will not undertake this simplification here, because it is out of the scope of the present study to actually implement the two-dimensional nonlinear theory. Therefore, our equilibrium equations and kinematical equations are geometrically exact; all approximations stem from the dimensional reduction process used to obtain the two-dimensional constitutive law.

The present work is a direct extension of [18] to treat shells. If one sets $k_{ij}=0$ and $A_\alpha=1$, all the formulas developed here will reduce to those in [18], which indirectly verifies that derivation.

Conclusions

A nonlinear shear-deformable shell theory has been developed to be completely compatible with the modeling process in [17]. The compatibility equations, kinematical relations and equilibrium equations are derived for arbitrarily large displacements and rotations under the restriction that the strain must be small. The resulting formulas are compared with others in the literature. The following conclusions can be drawn from the present work:

1. The variational asymptotic method can be used to decouple the original three-dimensional elasticity problem of a shell into a one-dimensional, through-the-thickness analysis, [17], and a two-dimensional, shell analysis. The through-the-thickness analysis provides both an accurate two-dimensional constitutive law for the nonlinear shell theory and accurate through-the-thickness recovery relations for three-dimensional displacement, strain, and stress. This way, an intimate relation between the shell theory and three-dimensional elasticity is established.

2. A full finite rotation must be applied to fully specify the displacement field. However, since the strain energy on which the

formulation is based is independent of $\kappa_{\alpha 3}$, the rotation about the normal is not independent and can be expressed in terms of other quantities. Thus, it can be chosen so that the two-dimensional, in-plane shear strain measures are equal. This way all the strain measures can be expressed in terms of five independent quantities: three displacement and two rotation measures, and only one stress resultant for in-plane shear can be derived from the two-dimensional energy.

3. Only five equilibrium equations are obtainable in a displacement-based variational formulation. Moment equilibrium about the normal is satisfied implicitly. If one does not include the full finite rotation, but rather sets the rotation about the normal equal to zero, the correct equilibrium equations cannot be obtained. This should shed some light on the nature of ''drilling'' degrees of freedom.

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