Generalized Timoshenko Theory of the Variational Asymptotic Beam Sectional Analysis

Wenbin Yu* and Dewey H. Hodges†

Georgia Institute of Technology, Atlanta, Georgia 30332-0150

Abstract

The refined theory for small-strain extension, twist, bending, and shearing of composite beams that is embedded in the computer program VABS (Variational Asymptotic Beam Sectional Analysis) has the same structure as Timoshenko's original theory for isotropic beams, but it has none of the restrictive assumptions of the original theory. An overview of this theory, referred to as a generalized Timoshenko theory, is presented so that readers can appreciate its general and rigorous framework. Certain theoretical details missing from previous developments are supplied, such as the proof of a kinematical identity and the expression of the recovery theory in terms of sectional stress resultants. As an analytical validation of the theory it is demonstrated that the VABS generalized Timoshenko theory reproduces the elasticity solution for the flexure problem of an isotropic prism. Additional numerical results are presented in support of the long-term validation effort, focusing especially on calculation of sectional stiffnesses (including shear correction factors) and shear center location, making use of the VABS model for composite beam analysis (including buckling and vibration), and recovering three-dimensional field variables over an interior cross section. The accuracy of the VABS generalized Timoshenko theory is demonstrated, and some of its practical advantages over three-dimensional finite element analysis are exhibited.

Introduction

Due to the special geometric feature of beams, in which one dimension is much larger than the other two, beam modeling has been regarded as an elasticity problem for centuries, starting with Galileo's inquiry, Ref. 1. In the more than three hundred years that followed, investigators tried to simplify the analysis by taking advantage of the geometric features to model beams as one-dimensional (1-D) problems. However, to obtain an accurate beam

*Post Doctoral Fellow, School of Aerospace Engineering. Member, AIAA. Presently, Assistant Professor, Department of Mechanical and Aerospace Engineering, Utah State University, Logan, Utah.

†Professor, School of Aerospace Engineering. Fellow, AIAA. Member, AHS, ASCE, and ASME.

representation for the physically three-dimensional (3-D) structure, one has to find a way to reproduce the 3-D energy stored in the structure as accurately as possible. That is, one must find a way to take into account the eliminated two-dimensional (2-D) cross-sectional coordinates. The famous Euler-Bernoulli hypothesis was introduced in order to meet this requirement. Therein it is assumed that a normal cross-sectional plane remains plane and normal to the reference line when the beam deforms. The theory resulting from this assumption can account for extension and bending in two directions. In order to account for torsion, the cross section cannot in general be constrained to remain plane. However, its shape and size in its own plane are assumed to be preserved during torsion, and the cross section can warp out of its plane freely. The simplest beam theories for isotropic beams account for extension, torsion and bending in two directions based on these assumptions and, in spite of the incorporation of Saint-Venant torsion, are sometimes called Euler-Bernoulli beam theories.

Refined theories are required for higher accuracy when either the beam structure is not slender or when the wavelength of the beam deformation is shorter than its length (such as when it is vibrating in a mode higher than the first). For isotropic beams the next logical step beyond Euler-Bernoulli beam theory is Timoshenko beam theory, in which there are six fundamental global deformations (bending and transverse shear in two directions, extension, and twist). The assumption of Euler-Bernoulli theory, that a cross section of the undeformed beam normal to the reference line remains normal during deformation, is relaxed. Rather, a cross section of the undeformed beam normal to the reference line will be, in general, oblique to the reference line of the deformed beam because of transverse shearing. To ensure that the reduced 1-D strain energy is equivalent to the original 3-D model over a broader range of slenderness ratios, shear correction factors are often introduced to modify the shear stiffness for isotropic beams. These factors typically reduce the transverse shear stiffness of a cross section relative to what it would have been were there no out-of-plane cross-sectional warping due to transverse shear. For long-wavelength deformation of slender, isotropic beams, it suffices to use Euler-Bernoulli theory. When the beam structure is not slender or when the wavelength of the beam deformation is shorter than its length, Timoshenko theory offers some improvement for various applications.

However, for beams made with generally anisotropic materials, the imposition of such *ad hoc* kinematic assumptions can introduce significant errors. To accurately capture such behavior when a theory governing extension, torsion, and bending in two directions is generalized to include composite material effects, one must abandon the restrictive assumptions of Euler-Bernoulli theory and include all possible deformation. In the limit of small h/l, where h is the cross-sectional characteristic dimension and l is the wavelength of the deformation along the beam reference line, the resulting theory governs fully coupled extension, twist, and bending. Thus, instead of four fundamental stiffnesses there could be as many as 10 (a fully populated 4×4 symmetric matrix). Such a generalized Euler-Bernoulli theory is typically termed "classical beam theory" and can predict well the static and low-frequency dynamic behavior of slender, composite beams, Ref. 2. The stiffness model of classical theory can be refined to take initial twist and curvature into account, Ref. 3, without changing the types of deformation or the number of stiffness constants. Classical theory only takes 3-D transverse shearing into account for the calculation of extension, twist, and bending stiffness constants.

Although for many cases of static and low-frequency dynamic deformation of beams it is shown in Ref. 4 that an asymptotically correct classical model is at least as accurate as refined theories based on *ad hoc* assumptions, classical theory has its limitations. It is not only possible but desirable to generalize Timoshenko theory for composite beams, so that a transverse shearing beam variable and short-wavelength phenomena can be included in the model. A significant literature review is beyond the scope of this paper. However, recent research has resulted in significant advances in this field; see the extensive literature reviews in Refs. 4–7.

One of the recent developments pointed out in the more recent review papers is the computer program VABS, a finite element based cross-sectional analysis for composite beam-like structures originally developed by Hodges and his co-workers, Refs. 2, 8–11. VABS takes the Variational Asymptotic Method (VAM), Ref. 12, as the mathematical foundation to decouple a general 3-D nonlinear anisotropic elasticity problem into a linear, 2-D, cross-sectional analysis and a nonlinear, 1-D, beam analysis. The cross-sectional analysis calculates the 3-D warping functions asymptotically and finds the constitutive model for the 1-D nonlinear beam analysis. After one obtains the global deformation from the 1-D beam analysis, the original 3-D fields (displacements, stresses, and strains) can be recovered using the already-calculated 3-D warping functions. VABS was first mentioned in Ref. 2. Its development over the past decade is described in Refs. 8–11, 13–16.

VABS can perform a classical analysis for beams with initial twist and curvature with arbitrary reference cross sections. VABS is also capable of capturing the trapeze and Vlasov effects, which are useful for specific beam applications. VABS is able to calculate the 1-D sectional stiffness matrix with transverse shear refinement for any initially twisted and curved, inhomogeneous, anisotropic beam with arbitrary geometry and material properties. Finally, VABS can recover asymptotically correct 3-D displacement, stress and strain fields within a modeled cross section. To the best of the authors' knowledge, there does not exist any other published treatment of nonlinear composite beam modeling with such generality and versatility. It should be emphasized, however, that the recovery operations within cross sections that are near beam boundaries, concentrated loads, or sudden changes in the cross-sectional geometry along the span are not accurate. Indeed, in these areas one has truly 3-D behavior, and the structure is not behaving as a beam.

To emphasize that *ad hoc* assumptions such as those used in the original Timoshenko theory are not invoked in development of the VABS composite beam theory that accounts for bending and transverse shear in two directions, extension, and twist, it is referred to as a *generalized Timoshenko theory*. For composite beams, instead of six fundamental stiffnesses, there could be as many as 21 in a fully populated 6×6 symmetric matrix. The purpose of this paper is to explain, validate and assess this theory embedded in VABS.

We first present an overview of the VABS generalized Timoshenko theory along with a few theoretical details missing from previous treatments of this topic such as Refs. 10, 11. Then, the relation between VABS and elasticity theory is briefly reported as an analytical validation of the theory, building on Ref. 17. Finally, several examples are presented as numerical validations in which VABS results are compared with those from the commercial finite element analysis (FEA) package ANSYS and other approaches in the literature where available and appropriate.

The Generalized Timoshenko Theory of VABS

Construction of Strain Energy Density

The first step of developing the generalized Timoshenko beam theory of VABS is to find a strain energy asymptotically correct up to the second order of h/l and h/R, where h is the characteristic size of the section, l the characteristic wavelength of deformation along the beam axial coordinate and R the characteristic radius of initial curvatures and twist of the beam. We conclude that a complete second-order strain energy is sufficient for the purpose of constructing a generalized Timoshenko model because it is generally accepted that the transverse shear strain measures are one order less than classical beam strain measures (extension, torsion and bending in two directions). By formulating the beam kinematics exactly in an intrinsic fashion, one obtains the 3-D strain field in terms of beam strain measures and arbitrary warping functions. The 3-D warping functions are solved by VAM asymptotically. Finally, a strain energy asymptotically approximating the 3-D energy up to the second order can be achieved. All the derivation of this procedure is presented in Ref. 10. Here only the resulting asymptotically correct strain energy is presented, given as

$$2U^* = \varepsilon^T A \varepsilon + 2\varepsilon^T B \varepsilon' + \varepsilon'^T C \varepsilon' + 2\varepsilon^T D \varepsilon'' \tag{1}$$

where A, B, C, D are matrices that carry information on both the geometry and material of the cross section; $\varepsilon = \lfloor \overline{\gamma}_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3 \rfloor^T$ are the strain measures defined in the classical beam theory; $(\ )'$ means the partial derivative with respect to the beam axial coordinate x_1; and x_α are the local Cartesian coordinates for the cross section. (Here and throughout the paper, Greek indices assume values 2 and 3 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated.)

Although the strain energy expressed in Eq. (1) is asymptotically correct, it is difficult to use in practical analyses because it contains derivatives of the classical strain measures, which requires boundary conditions that are more complicated than necessary. Timoshenko beam theory, commonly used in engineering practice, is free from such drawbacks. Therefore, the second step is to fit the obtained asymptotically correct strain energy, Eq. (1), into a generalized Timoshenko model of the form

$$2U = \epsilon^T X \epsilon + 2\epsilon^T F \gamma_s + \gamma_s^T G \gamma_s \tag{2}$$

where $\epsilon = \lfloor\ \gamma_{11}\ \ \beta_1\ \ \beta_2\ \ \beta_3\ \rfloor^T$, the classical strain measures (defined in the framework of a generalized Timoshenko model), and $\gamma_s = \lfloor 2\gamma_{12}\ \ 2\gamma_{13} \rfloor^T$ transverse shear strains. Generally, it is impossible to achieve this transformation while keeping the resulting model asymptotically correct. Although it is noted that Berdichevsky and Starosel'skii, Ref. 18, used changes of variable to achieve a model of the form of Eq. (2), the 1-D transverse shear strain measures are not equivalent to those commonly used. Hence, because the model of Ref. 18 differs in this way, direct use of this model could produce misleading results if one does not take this subtlety into account, Ref. 17. However, one should be able to recover 3-D fields based on that work that are equivalent to those obtained in the VABS generalized Timoshenko theory.

To ensure that the generalized Timoshenko model represents the original asymptotically correct model as accurately as possible, the best one can do is to make use of all the known information between these two models. First, we need to find a relation to express the beam strain measures (ϵ and γ_s) defined in the generalized Timoshenko model in terms of the strain measures (ε) defined in the asymptotically correct model. In fact, the strain measures of the asymptotically correct and generalized Timoshenko models are associated with two different triads, $\mathbf{T}_i$ and $\mathbf{B}_i$, respectively. As sketched in Fig. 1, they are related according by the following equation:

$$\begin{Bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{Bmatrix} = C^{BT} \begin{Bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \mathbf{T}_3 \end{Bmatrix} \tag{3}$$

where

$$C^{BT} = \begin{bmatrix} 1 & -2\gamma_{12} & -2\gamma_{13} \\ 2\gamma_{12} & 1 & 0 \\ 2\gamma_{13} & 0 & 1 \end{bmatrix} \tag{4}$$

Based on the definition of 1-D "force-strain" measures, Ref. 19, we have the identity

$$\mathbf{R}' = (1+\gamma_{11})\mathbf{B}_1 + 2\gamma_{1\alpha}\mathbf{B}_\alpha = (1+\overline{\gamma}_{11})\mathbf{T}_1 \tag{5}$$

where $\overline{\gamma}_{11}$ is the extensional strain associated with $\mathbf{T}_i$ and γ_{11} and $2\gamma_{1\alpha}$ are the "force-strains" associated with $\mathbf{B}_i$. Dot multiplying this equality with $\mathbf{B}_1$, making use of Eq. (3), and assuming the strain components are small, one obtains

$$\overline{\gamma}_{11} = \gamma_{11} \tag{6}$$

According to Ref. 19, the "moment-strain" measures can be related by

$$K_B = \frac{\left(I - \frac{1}{2}\widetilde{\alpha}\right)\alpha'}{1 + \frac{1}{4}\alpha^T\alpha} + C^{BT}K_T \tag{7}$$

with I denoting the 3×3 identity matrix and

$$\begin{aligned} K_B &= \beta + k \\ K_T &= \kappa + k \end{aligned} \tag{8}$$

where β and κ are the column matrices representing the "moment-strain" measures associated with bases $\mathbf{B}_i$ and $\mathbf{T}_i$, respectively; the column matrix k contains the initial twist and curvatures measured in basis $\mathbf{b}_i$; and the column matrix $\alpha = \lfloor 0 \quad 2\gamma_{13} \quad -2\gamma_{12} \rfloor^T$ denotes the Rodrigues parameters corresponding to the direction cosine matrix in Eq. (4) for small strain. By virtue of the restriction to small strain, the generalized Timoshenko constitutive model being sought is linear. Thus, one can rewrite Eq. (7) as

$$\kappa = \beta - \alpha' + k - C^{BT}k \tag{9}$$

which can be written explicitly as

$$\begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -2\gamma_{13}' \\ 2\gamma_{12}' \end{Bmatrix} + \begin{Bmatrix} 2\gamma_{1\alpha}k_\alpha \\ -2\gamma_{12}k_1 \\ -2\gamma_{13}k_1 \end{Bmatrix} \tag{10}$$

It should be noted that in spite of the linearity of the constitutive model, which is a byproduct of the restriction to small strain, the 1-D beam analysis (including both kinematical equations and equations of motion) is geometrically exact; see, for example, the work of Ref. 19. Combining Eqs. (6) and (10), one obtains a kinematical identity between these two sets of strain measures, given by

$$\varepsilon = \epsilon + Q\,\gamma_s' + P\,\gamma_s \tag{11}$$

with

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 0 \\ k_2 & k_3 \\ -k_1 & 0 \\ 0 & -k_1 \end{bmatrix} \tag{12}$$

Making use of Eq. (11), one can express the asymptotically correct strain energy, Eq. (1), in terms of the generalized Timoshenko beam strain measures as

$$\begin{aligned} 2U^* =& \epsilon^T A\epsilon + 2\epsilon^T AQ\gamma_s' + 2\epsilon^T AP\gamma_s + \\ & 2\epsilon^T B\epsilon' + \epsilon'^T C\epsilon' + 2\epsilon^T D\epsilon'' \end{aligned} \tag{13}$$

To fit Eq. (13) into the generalized Timoshenko form as in Eq. (2), we must express the derivatives of strain measures in terms of the strain measures themselves. Fortunately, the 1-D equilibrium equations based on the generalized Timoshenko model provide a convenient way to relate the derivatives of strain measures with the strain measures themselves. Taking advantage of such relations, the unknowns in Eq. (2) can be calculated and a generalized Timoshenko constitutive model can be obtained. The details of this process are given in Ref. 10.

Recovery Relations

There are at least two applications of the generalized Timoshenko model. First, the main application of the constitutive model obtained from VABS is to use it as input for 1-D beam analysis. Although VABS casts the strain energy into a form that has types of deformation similar to those of the original Timoshenko beam theory, it does not make the limiting kinematical assumptions of that theory. In fact, VABS considers all possible 3-D deformation but still creates a seamless connection to traditional beam theories so that the 1-D beam analysis remains essentially the same. The additional 3-D information, which is eliminated in the construction of the 1-D beam analysis, is included by introducing arbitrary warping functions and is retained to a degree sufficient to give accurate stress, strain, and displacement recovery. Note that any general 1-D beam solver can directly make use of the

VABS generalized Timoshenko constitutive model to carry out buckling, dynamic, static and aeroelastic analyses as long as the 1-D beam solver uses 1-D strain measures that are equivalent to the standard ones that are used in the VABS development; see, for example, Ref. 19.

A second application is to use this model to calculate the shear center location for a cross section with arbitrary geometry and material. The reader can refer to Refs. 10, 11 for the details of using the VABS generalized Timoshenko model to locate the shear center.

There are several criteria for evaluation of composite beam theories and modeling approaches. Although it is necessary for a methodology to provide accurate results for the various types of beam global behavior (*i.e.*, static deflections, natural frequencies, mode shapes, nonlinear transient behavior, buckling loads, etc.), this is not sufficient. Indeed, it is misleading to focus only on the 1-D behavior, per se, because an insufficiently detailed study of published results may lead one to believe that differences among the various published composite beam theories are insignificant. Actually, the adequacy of a composite beam modeling approach should be measured based on how well it predicts 3-D behavior of the original 3-D structure. Therefore, a full set of recovery relations should be provided to complete the modeling. By recovery relations we mean expressions for the 3-D displacements, strains and stresses in terms of 1-D beam quantities and the local cross-sectional coordinates, x_α.

Such relations are provided in the VABS generalized Timoshenko model. The recovery relations of the 3-D strains and stresses in terms of the 1-D generalized strains and their derivatives are reported briefly in Ref. 11. However, it is difficult to obtain the strain derivatives if the 1-D analysis is solved by a 1-D finite element method using lower-order shape functions. To be consistent with the procedure used to construct the generalized Timoshenko theory, the final version of recovery theory in VABS for 3-D displacements, strains and stresses will now expressed in terms of sectional stress resultants and the applied and inertial loads using the 1-D equations of motion.

For modeling an initially curved and twisted beam, the warping that is asymptotically correct up to the order of h/R and h/l can be expressed as

$$w(x_i) = (V_0 + V_{1R})\varepsilon + V_{1S}\varepsilon' \tag{14}$$

where $w(x_i)$ are the 3-D warping functions, V_0, V_{1R}, and V_{1S} are the asymptotically correct warping functions for classical modeling, the correction due to initial curvatures/twist, and the refined warping of the order of h/l, respectively.

The recovered 3-D displacement field of the generalized Timoshenko model can be expressed as

$$U_i(x_1, x_2, x_3) = u_i(x_1) + x_\alpha[C_{\alpha i}(x_1) - \delta_{\alpha i}] + w_i(x_1, x_2, x_3) \tag{15}$$

where U_i are the 3-D displacements, u_i are the 1-D beam displacements, $C_{\alpha i}$ are components of the direction cosine matrix representing the rotation of beam triads during deformation, and $\delta_{\alpha i}$ is the kronecker symbol. Strictly speaking, although the warping field has been calculated only to first order, the strain energy density is asymptotically correct through the second order in the small parameters. With the first-order warping the 3-D fields can only be recovered through the first order. To recover the 3-D fields up to the second order requires calculation of the second-order warping field, which means additional complexity and computation. Here the 3-D results will be recovered based on the first-order warping and all the other information we have. Numerical examples show that such recovery relations yield accurate results without introducing additional computational cost.

The 3-D strain field can be recovered as follows:

$$\begin{aligned}\Gamma = {} & [(\Gamma_h + \Gamma_R)(V_0 + V_{1R}) + \Gamma_\epsilon]\,\varepsilon \\ & + [(\Gamma_h + \Gamma_R)V_{1S} + \Gamma_l(V_0 + V_{1R})]\,\varepsilon' \\ & + \Gamma_l V_{1S} \varepsilon''\end{aligned} \tag{16}$$

where the 3-D field

$$\Gamma = \lfloor\ \Gamma_{11}\ \ 2\Gamma_{12}\ \ 2\Gamma_{13}\ \ \Gamma_{22}\ \ 2\Gamma_{23}\ \ \Gamma_{33}\ \rfloor^T \tag{17}$$

All the operators in Eq. (16) can be found in Ref. 10. As mentioned previously, it is more useful to write the recovery relations in terms of stress resultants because one can obtain those quantities and their derivatives through the 1-D equilibrium equations. Denoting S as the 6×6 stiffness matrix for the generalized Timoshenko beam model, one can obtain the generalized strain measures in terms of sectional stress resultants as

$$\epsilon_t = S^{-1} F \tag{18}$$

where $\epsilon_t = \lfloor\ \gamma_{11}\ \ 2\gamma_{12}\ \ 2\gamma_{13}\ \ \beta_1\ \ \beta_2\ \ \beta_3\ \rfloor^T$ are the generalized 1-D strain measures obtained from a generalized Timoshenko model, and $F = \lfloor\ F_1\ \ F_2\ \ F_3\ \ M_1\ \ M_2\ \ M_3\ \rfloor^T$ the cross-sectional stress and moment resultants. To find the derivatives of stress resultants, the 1-D nonlinear equilibrium equations can be arranged as

$$F' = -RF - f = \begin{bmatrix} \widetilde{K_B} & \widetilde{0} \\ \widetilde{e_1} + \widetilde{\gamma} & \widetilde{K_B} \end{bmatrix} F - f \tag{19}$$

with

$$\widetilde{(\ )}_{ij} = -e_{ijk}(\ )_k \tag{20}$$

$$\gamma = \lfloor \gamma_{11} \ \ 2\gamma_{12} \ \ 2\gamma_{13} \rfloor^T \tag{21}$$

$$e_1 = \lfloor 1 \ \ 0 \ \ 0 \rfloor^T \tag{22}$$

$$f = \lfloor f_1 \ \ f_2 \ \ f_3 \ \ m_1 \ \ m_2 \ \ m_3 \rfloor^T \tag{23}$$

where e_{ijk} are the components of the permutation tensor in the Cartesian system, and f are the known distributed 1-D generalized applied and inertial loads. Note that the generalized strains in Eq. (19) can be obtained by Eq. (18). It is clear that one can differentiate Eq. (19) on both sides to get higher derivatives as

$$\begin{aligned} F'' &= (R^2 - F')F + Rf - f' \\ F''' &= (-R^3 + RR' + 2R'R - R'')F + (-R^2 + 2R')f + Rf' - f'' \end{aligned} \tag{24}$$

Having F', F'' and F''', one can obtain ϵ_t', ϵ_t'' and ϵ_t''' from Eq. (18). Substituting these values into Eq. (16) and using Eq. (11), one can express the 3-D strain field in terms of the known quantities F, f, f' and f'' from the 1-D beam analysis. Finally, the 3-D stress field can be obtained using the 3-D constitutive law. The results obtained from use of these recovery relations are identically the same as from those written in terms of 1-D strains and their derivatives found in Eq. (16), Ref. 11.

Validation of the VABS Generalized Timoshenko Theory

The aforementioned generalized Timoshenko theory for initially curved and/or twisted composite beams has been implemented in the computer program VABS. To the authors' knowledge, there are no other published treatments of composite beam modeling that have its level of consistency, rigor, and generality. VABS is in use internationally among researchers who are involved in the design and analysis of composite beam-like structures. Nevertheless, more validation is always helpful to demonstrate further the accuracy and versatility of this theory.

Validation of VABS can be undertaken in two different ways. One is analytical and the other numerical. Several examples are given in this section to support the continuing validation of the generalized Timoshenko theory of VABS. The examples include isotropic and anisotropic cases. The comparison is made against 3-D elasticity theory, 3-D FEA and other published results whenever possible and appropriate.

Analytical Validation

Analytical validation of VABS is accomplished by establishing the connection between VABS and 3-D elasticity theory. The first such example appeared in Ref. 11 in which the variational asymptotic cross-sectional analysis of an elliptic isotropic prism with transverse shear refinement is solved by using Ritz method. Later, a one-to-one correspondence between the 3-D elasticity theory and the VABS generalized Timoshenko theory was found, Ref. 17. There, the general formulation from VABS is specialized to treat isotropic, prismatic beams. The governing differential equations and associated boundary conditions of the VABS classical model and generalized Timoshenko model are exactly the same as those of the Saint-Venant problem and flexure problem in 3-D elasticity theory, respectively. This discovery demonstrates that VABS yields exact results for shear correction factors and shear center location for isotropic prismatic beams. Therefore, any existing model that does not produce the same results as VABS does for these kinds of beams should be considered inferior. The fact that VABS reproduces the results of elasticity theory clearly confirms the fact that VABS avoids the difficulties of dealing with 3-D elasticity while at the same time obtains results that agree very well with exact solutions.

Although it may not be possible to analytically validate the general theory of VABS for anisotropic beams in the same way, it is a natural deduction based on the derivation of Ref. 17 that the results for generally anisotropic beams should be the same as those calculated by methods based on 3-D anisotropic elasticity theory, such as 3-D FEA. Indeed, as 3-D FEA allows one to go beyond the limitations of 3-D elasticity, VABS may also be considered as a means for going beyond those limits when considering the cross-sectional analysis of beam-like structures. When coupled with a suitable 1-D beam solver, 3-D stresses on the interior of a beam are obtained for a computational cost that is two to three orders of magnitude less than that of 3-D FEA.

Numerical Validation Examples

Another means of validation is to compare results from VABS with those published and with 3-D FEA; this sort of validation can be called numerical validation. Most of the numerical examples presented in previous publications on VABS are validations of this kind, Refs. 9–11, 14–16. In Ref. 11 some VABS results are assessed against the 3-D FEA package ABAQUS.

In this section, several numerical examples related will be investigated to demonstrate the accuracy and advantages of the VABS generalized Timoshenko theory. The problems related with generalized Timoshenko modeling of composite beams can be classified into the following three groups:

- obtaining the matrix of cross-sectional stiffness constants for general beams, particularly the transverse shear stiffnesses or shear correction factors for isotropic sections;
- locating the shear center, especially for structures that are anisotropic and not thin-walled;
- recovering 3-D results for the purpose of detailed analysis.

Stiffness Model and Shear Correction Factors

One of the main outcomes of a generalized Timoshenko model is a 6×6 stiffness model including the transverse shear stiffnesses for a cross section. Particularly, for isotropic sections, shear correction factors are often used to obtain the equivalent shear stiffness to be used for a Timoshenko beam analysis. These factors are defined as

$$c_\alpha = \frac{S_\alpha}{GA} \tag{25}$$

where c_α is the shear correction factor in the x_α direction, S_α the equivalent shear stiffness in the x_α direction, G the shear modulus and A the sectional area.

Taking for example a rectangular section with a length of $2a$ in the x_2 direction and $2b$ in the x_3 direction, we know that the factor 5/6 is normally used in engineering practice. However, this number is only correct for the x_2 direction when $b << a$ and/or the Poisson's ratio $\nu = 0$. The exact factor can be obtained by the VABS generalized Timoshenko theory as

$$c_2^{-1} = \frac{6}{5} + \left(\frac{\nu}{1+\nu}\right)^2 \left[\frac{1}{5\rho^4} - \frac{18}{\rho^5\pi^5}\sum_{m=1}^{\infty}\frac{\tanh(m\pi\rho)}{m^5}\right] \tag{26}$$

with $\rho = a/b$. As mentioned in Ref. 17, Eq. (26) is exactly the same as that which comes from the flexure problem of elasticity theory. The same result was also obtained independently by Refs. 20,21. There are other values proposed in the literature, Refs. 18,22,23, but these are actually approximations to the exact value and not as accurate as what VABS calculates.

Eq. (26) is obtained by mathematically solving the governing differential equations of the VABS generalized Timoshenko theory. It will be interesting to investigate the convergence of the numerical result from the computer program VABS. Letting $2a = 1$ in., $2b = 2$ in., $G = 10^9$ psi and $\nu = 0.3$, then the exact value of c_2 is 0.784442. Seven different meshes: 1×2, 2×4, 3×6, 4×8, 5×10, 6×12 and 10×20, where the first number is the number of elements along x_2 and the second is the number of elements along x_3, are created for this purpose. The section is meshed by ANSYS *shell93* elements and this mesh is imported to VABS using special-purpose macros created for interfacing VABS and ANSYS. The results for the relative error of c_2 with respect to the exact result are plotted versus the number of elements along

x_2 on a log-log scale in Fig. 2. One can observe from the figure that the numerical results from VABS converge monotonically to the exact result as one refines the mesh. Even if the mesh is only comprised of 8 elements, the difference between the VABS numerical result and the exact solution is less than 1%, which clearly demonstrates the accuracy of VABS in calculating the shear correction factors for rectangular sections.

To demonstrate that VABS can calculate the shear correction factors correctly for arbitrary sections, an irregular section as sketched and meshed in Fig. 3 is studied. VABS results are compared with those of Ref. 24 and ANSYS beam capability (*beam 188/189*) in Table 1. VABS results agree with those of agree with those from Ref. 24 to within 0.2%, while ANSYS results are off by as much as 1.3%. Note that the results from ANSYS are independent of Poisson's ratio and thus can be only considered as approximations to the exact solution when $\nu = 0$. Although it is said in the ANSYS manual that the cross-sectional capability is valid for composite sections, the word "composite" just means that one can build up several isotropic sections into an arbitrary section. It is *not* applicable to sections made with anisotropic materials. It should be noted that the work of Ref. 24 is devoted to calculate the shear correction factors only for arbitrary isotropic sections, which is much less versatile than VABS because this capability is just a small subset of the VABS functionalities.

Usually, when the six global deformations are uncoupled, it is sufficient to obtain the shear correction factors instead of calculating the 6×6 stiffness matrix. However, more often than not, the global deformations are coupled, and the full stiffness matrix is needed. Especially for composite beams, the stiffness matrix is generally coupled as the examples in Refs. 10, 11 have demonstrated. Here a more complex composite beam with a cross section described in Fig. 4 is modeled by VABS. This cross section is comprised of two straight strips and two half circles, each of which is made with two laminated layers. The material properties are listed in Table 2, and the dimensions are given in Fig. 4. The cross-sectional properties are listed in Table 3, from which one can observe extension-twist and shear-bending couplings. The results labelled as "SVBT" are produced by a computer program based on Ref. 25, a generalized application of the Saint-Venant approach. Therein, the behavior of all quantities versus the axial coordinate is represented by polynomials of zeroth and first degree, while the cross-sectional variations are handled by means of finite element approximation. Although SVBT is not based on asymptotic methods, the results from it and from VABS agree very well for most cases compared to date. The generalized Saint-Venant methodology, however, is limited to linear problems and lacks rigorous connections to treatments of end effects such as are modeled by Vlasov theory.

Based on the above in combination with previously published demonstrations, we conclude that one can confidently use the generalized Timoshenko stiffness model of VABS to carry out structural analysis for a wide variety of beam cross-sectional geometries and materials.

Locating the Shear Center

Another important outcome of generalized Timoshenko modeling is the shear center location. For thin-walled sections this is straightforward, but for arbitrary sections there is not much published information. Although it is not a common practice to carry out generalized Timoshenko modeling to find the shear center, it is indeed very simple to accurately locate the shear center once an accurate 6×6 stiffness model such as that from VABS is obtained.

The first example is an isotropic open channel-like section with unequal edges and equal thickness (see Fig. 5) studied in Ref. 26. VABS results are compared against those of Ref. 26 and ANSYS. As one can observe from Table 4, the three sets of results agree within 1.7% for c_2 and to three significant figures for c_3.

Another example studied in Ref. 26 is a closed section with three cells (see Fig. 6 for geometry and mesh). Again the results from VABS, Ref. 26 and ANSYS are compared in Table 5 and agreement to within 0.5% is observed.

While most research concerning the shear center is focused on isotropic, thin-walled sections, VABS can also locate the shear center for generally anisotropic cross sections, whether or not they are thin-walled. On this subject not much work can be found in the literature.

Beam Analysis with the VABS Generalized Timoshenko Model

Composite beam modeling does not stop at the stage of obtaining the cross-sectional properties such as beam stiffness matrix (including shear correction factors), and shear center location, etc. The ultimate goal of beam modeling is to use the obtained sectional properties to carry out the 1-D beam analysis to predict the global behavior, such as deflections, buckling load, natural frequencies and so on. Here, we assume the cross section in Fig. 4 is cut from a cantilever beam with length $L = 20$ in. The sectional mass properties can be obtained by VABS as:

$$\begin{aligned} m = &\ 7.42513 \times 10^{-2} \text{ lb/in} \\ m_{11} = &\ 7.10661 \times 10^{-2} \text{ lb.in} \\ m_{22} = &\ 9.88336 \times 10^{-3} \text{ lb.in} \\ m_{33} = &\ 6.11827 \times 10^{-2} \text{ lb.in} \end{aligned} \tag{27}$$

where m is the mass per unit span and m_{ii} is the mass moments of inertia per unit span about coordinate x_i. We use the VABS stiffness model, Table 3, and sectional mass matrix, Eq. (27), along with a 1-D nonlinear beam solver (here we use DYMORE, Ref. 27) to carry out a 1-D dynamic analysis to obtain the natural frequencies and a 1-D buckling analysis to obtain the critical Euler buckling load (compression). The natural frequencies for the first three bending modes are listed in Table 6, where $b_{i\alpha}$ denotes the ith bending mode in the direction of x_α. Model 1 is the generalized Timoshenko model given in Table 3 and Model 2 is the corresponding classical model. It should be noted that the 4×4 matrix of classical stiffness constants can be found by inverting the 6×6 cross-sectional stiffness matrix from the generalized Timoshenko model, removing the two columns associated with transverse shear, and then inverting the resulting 4×4 matrix. One can observe from Table 6 that there is a significant difference between the natural frequencies obtained from the generalized Timoshenko and classical models. Especially for the higher bending modes, it is clear that results from the classical model are useless. This shows that to obtain accurate flapping and lagging frequencies for a rotor blade, one should use a generalized Timoshenko model. We note that there is no difference between the two models for torsional frequencies.

A static stability problem is also studied for this composite beam. The tip deflection is excited by a very small tip shear force. As it is obvious from the plots, when the compression load is approaching the critical load, there is an abrupt change of the tip deflection. There is a small difference (3.5%) between the results obtained from classical and generalized Timoshenko models (see Fig. 7).

Recovering 3-D Results

Another capability of VABS is that it can recover the asymptotically correct distributions of the 3-D fields (displacements, strains, and stresses) over the cross section after obtaining the 1-D global behavior, which is very important for analyzing the failure of some critical areas. For demonstration purposes, we consider a composite beam of length 5 inches with a rectangular cross section, dimensions of which are b=0.25 inches, and h=1 inch; see Fig. 8. The material properties are given in Table 7, and the layup is a repeating quasi-isotropic pattern with 80 layers.

A unit shear force was applied in the x_3 direction at one end, and the other end was constrained to have zero displacement at every node. This means that the VABS model requires an input of F_3=1 lb., and M_2=2.5 lb.-in. for the purpose of recovering the stress components on the cross section at the mid-span of the beam ($x_1 = 2.5$ in.); specifically, we look at the stress along the line $x_2 = 0$ for various x_3 locations. For the ANSYS run, the total number of elements was 25,600, and about one hour of computer time was required. For the VABS run, the number of elements was 640, and the computer time was less than two seconds. As can be seen for stress components τ_{13} and τ_{12} in Figs. 9 and 10, respectively, the results from VABS and ANSYS are so close to one another that it's difficult to distinguish them from the plots.

The 3-D stress distributions clearly identify which areas within a cross section are experiencing extreme stresses. This capability of VABS enables the designer to make certain adjustments to avoid possible damage in the structure before it is actually built and tested, which means that the cost of a composite beam structure may be reduced. It should be emphasized, however, that the VABS analysis can only provide accurate stresses away from beam boundaries, concentrated loads, and sudden changes in the cross-sectional geometry along the span. In these areas one has truly 3-D behavior, and the structure does not behave as a beam.

Conclusions

We have provided an overview of the VABS generalized Timoshenko theory so that the generality and rigor of the framework is emphasized and exhibited to the reader. Certain theoretical derivations that were missing from earlier publications are presented herein. These include a proof of the kinematical identity and a presentation of the recovery theory in terms of sectional stress resultants. Thus, one now has, with previous publications, a complete formulation of the theory. Examples are presented herein to demonstrate that the generalized Timoshenko theory in VABS can reproduce the results of elasticity theory, accurately find the shear correction factors, and locate the shear center for beams made from anisotropic materials. It is also shown that the generalized Timoshenko model obtained from VABS can be used to carry out 1-D beam analyses such as buckling, vibration, etc. Significant differences are found between the results from the classical beam model and the generalized Timoshenko model. VABS can also be used to recover the 3-D distribution over the cross section with much less modeling and computational time relative to 3-D FEA.

In summary, the VABS generalized Timoshenko theory provides an accurate prediction of the behavior of the original 3-D beam structure because it asymptotically approximates the 3-D elasticity theory for the interior behavior of beams. It can predict the required results with much less labor time and computational time. Designers should be able to use this tool at both preliminary and detailed design stages to carry out needed tradeoffs more effectively, so that better and more cost-effective composite beam-like structures can be produced. Results from ongoing validation efforts will be presented in later publications for realistic helicopter blade sections.

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Table 1: Shear correction factors for an arbitrary section

ν	Resources	0	0.25	0.5
c_2/a	VABS	0.7404	0.7367	0.7306
	Ref. 24	0.7395	0.7355	0.7294
	ANSYS	0.7402	0.7402	0.7402
c_3/a	VABS	0.6780	0.6764	0.6736
	Ref. 24	0.6767	0.6753	0.6727
	ANSYS	0.6778	0.6778	0.6778

Table 2: Properties of the anisotropic pipe section

right wall:	$[45°/-45°]$
left wall:	$[45°/-45°]$
upper wall:	$[90°/0°]$
lower wall:	$[90°/0°]$
Material properties:	$E_l = 20.59 \times 10^6$ psi
$E_t = 1.42 \times 10^6$ psi	$G_{lt} = G_{tn} = 8.7 \times 10^5$ psi
$\nu_{lt} = \nu_{tn} = 0.42$	$\rho = 0.057$ lb/in^3

Table 3: The sectional stiffness matrix of the anisotropic pipe

Stiffness	VABS	SVBT	Rel. error (%)
S_{11} (lb.)	1.03890×10^7	1.03892×10^7	0.0019%
S_{22} (lb.)	7.84299×10^5	7.85310×10^5	0.13%
S_{33} (lb.)	3.29002×10^5	3.29279×10^5	0.084%
S_{14} (lb.-in.)	9.82878×10^4	9.84575×10^4	0.17%
S_{25} (lb.-in.)	-8.18782×10^3	-8.21805×10^3	0.37%
S_{36} (lb.-in.)	-5.18541×10^4	-5.20981×10^4	0.45%
S_{44} (lb.-in.^{2})	6.86973×10^5	6.87275×10^5	0.044%
S_{55} (lb.-in.^{2})	1.88236×10^6	1.88238×10^6	0.0011%
S_{66} (lb.-in.^{2})	5.38972×10^6	5.38987×10^6	0.0028%

Table 4: Shear center for the channel section

Resources	c_2 (in.)	c_3 (in.)
VABS	-0.176	0.186
Ref. 26	-0.179	0.186
ANSYS	-0.177	0.186

Table 5: Shear center for the 3-cell closed section

Resources	c_2 (in.)	c_3 (in.)
VABS	4.356	1.000
Ref. 26	4.337	1.000
ANSYS	4.356	1.001

Table 6: Natural frequencies (rad./sec.) from different models

Modes	Model 1	Model 2	Difference
b_{13}	8.417×10^2	8.693×10^2	3.3%
b_{12}	1.410×10^3	1.465×10^3	3.9%
b_{23}	4.488×10^3	5.423×10^3	20.8%
b_{22}	7.244×10^3	8.932×10^3	23.2%
b_{33}	1.054×10^4	1.507×10^4	43.0%
b_{32}	1.665×10^4	2.399×10^4	44.1%

Table 7: Material properties of anisotropic beam with rectangular cross section

Layup:	$[(-45/+45/0/90)10]_s$
Material properties:	$E_l = 20.59 \times 10^6$ psi
$E_t = 1.42 \times 10^6$ psi	$G_{lt} = G_{tn} = 8.7 \times 10^5$ psi
$\nu_{lt} = \nu_{tn} = 0.42$	$\rho = 0.057$ lb/in^3

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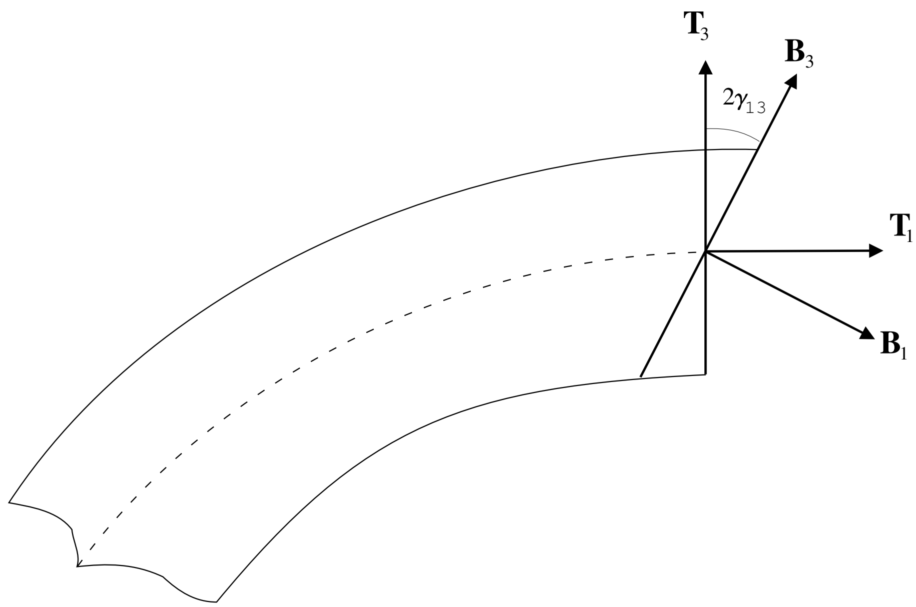
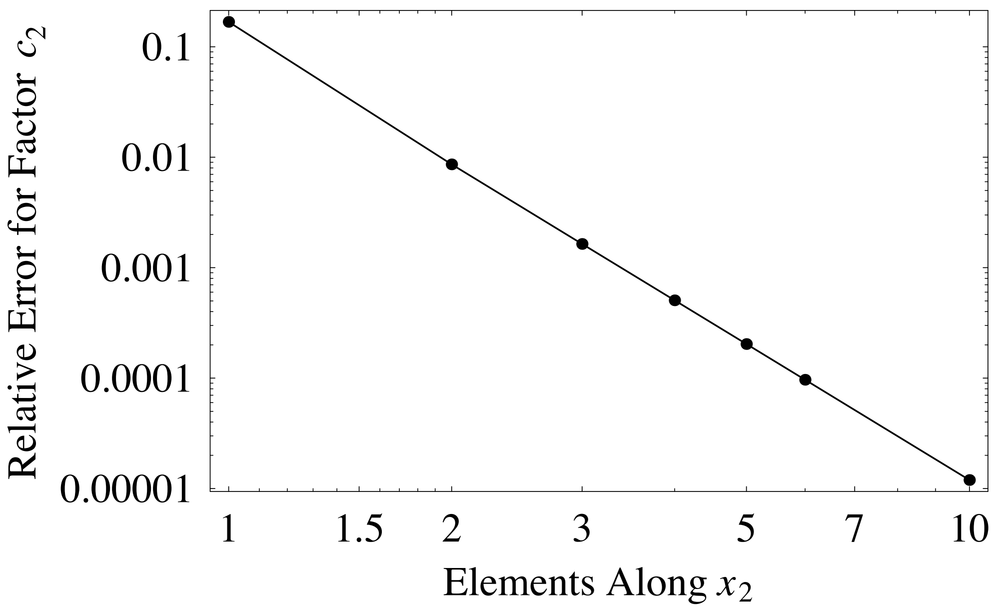


Figure 1: Coordinate systems used for transverse shear formulation



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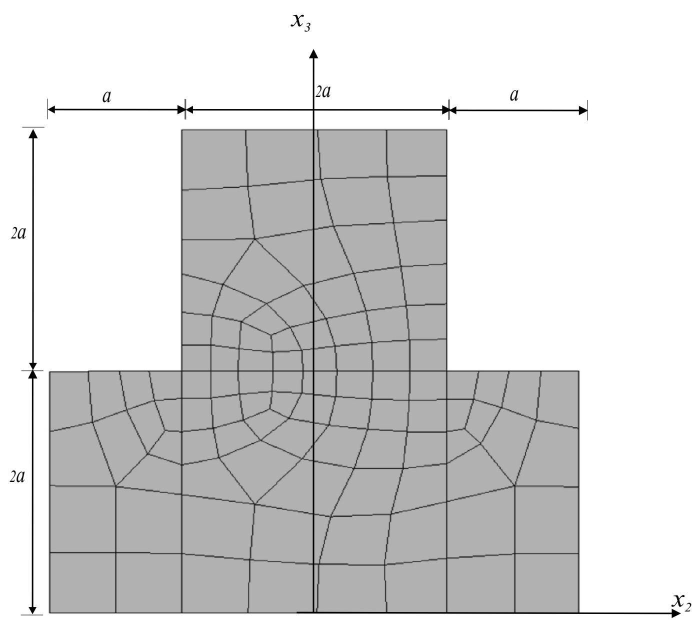

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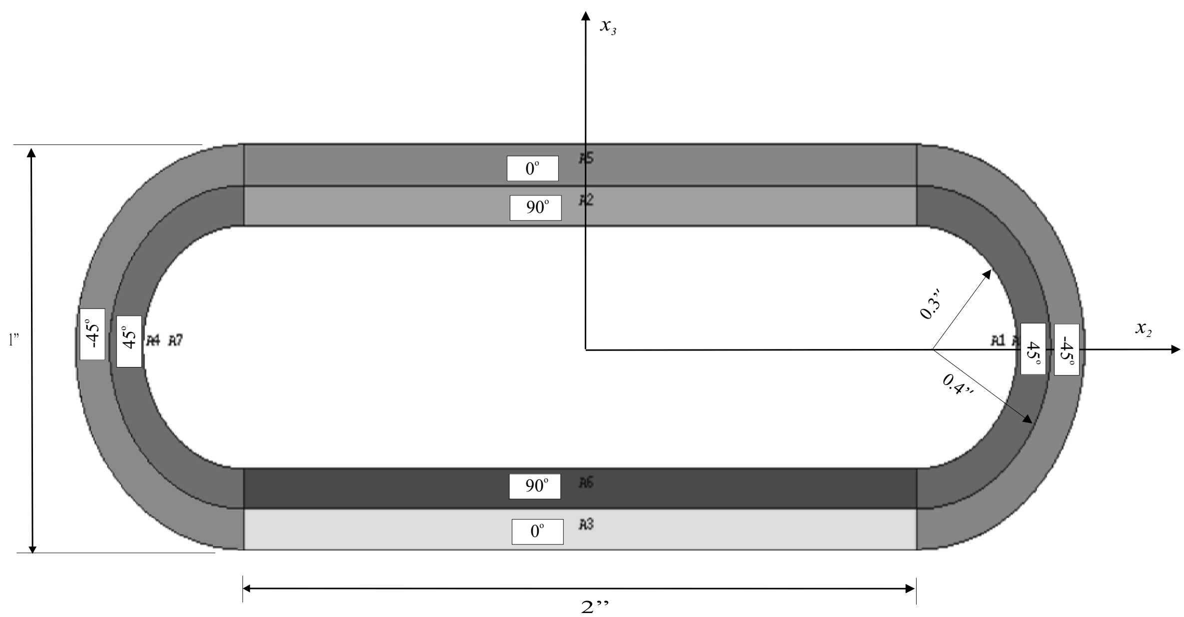

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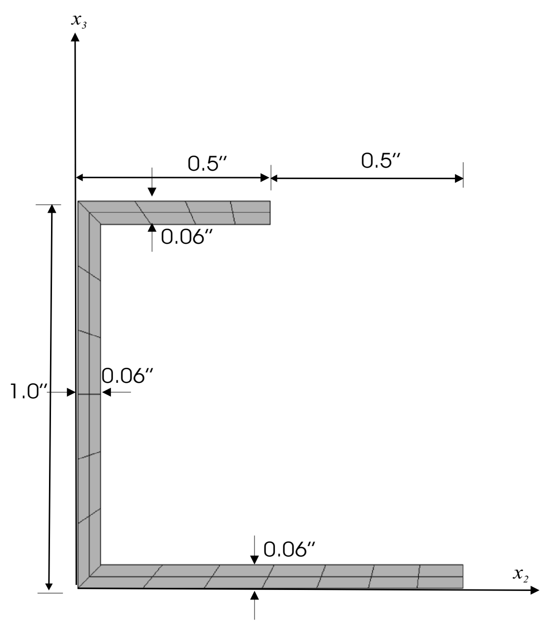

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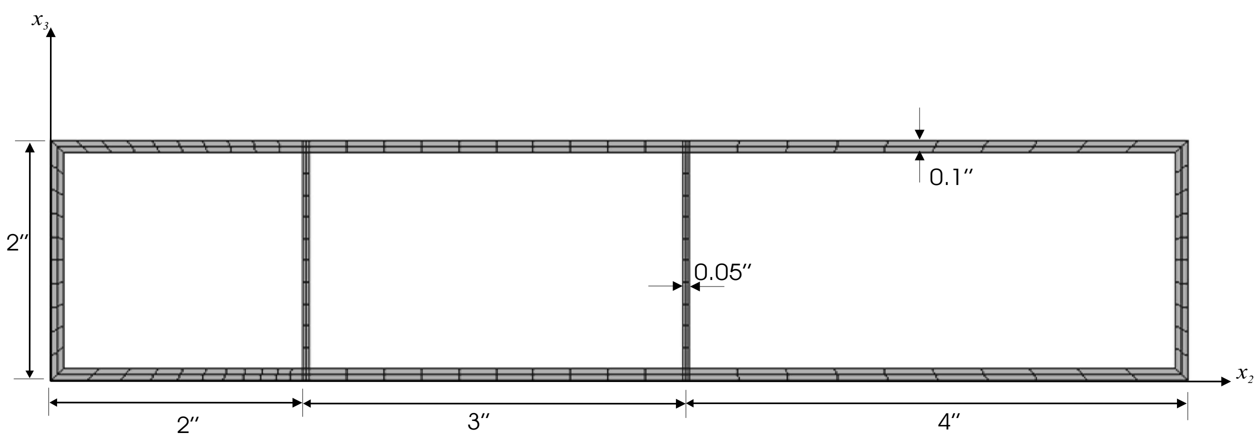

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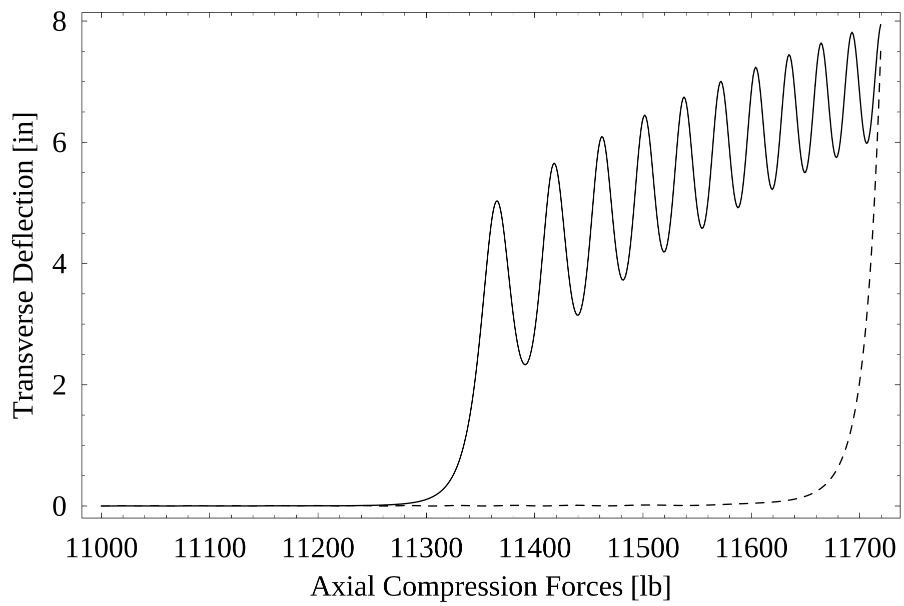

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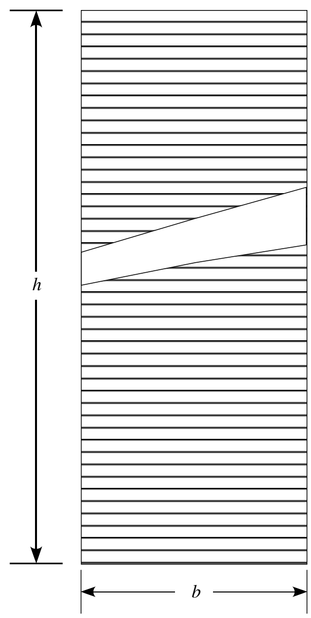

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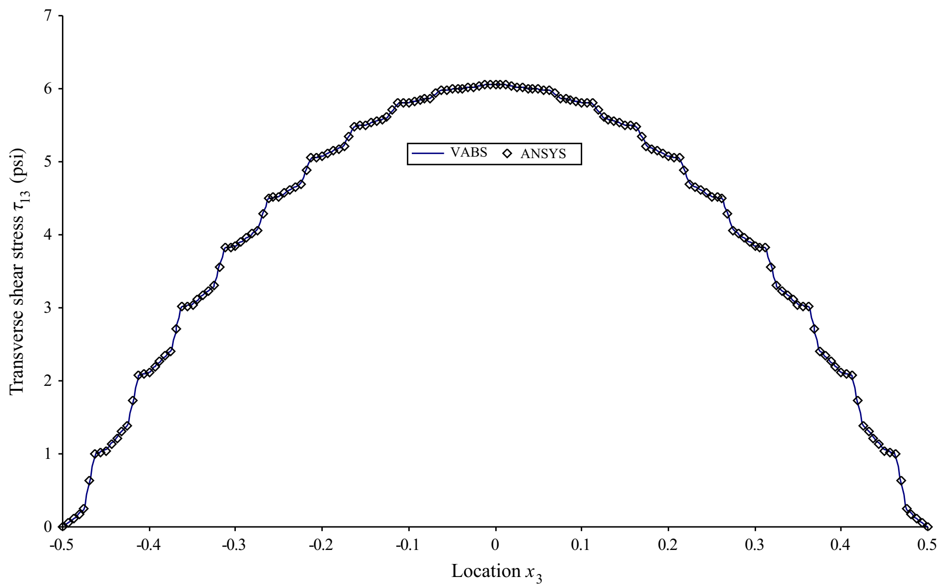
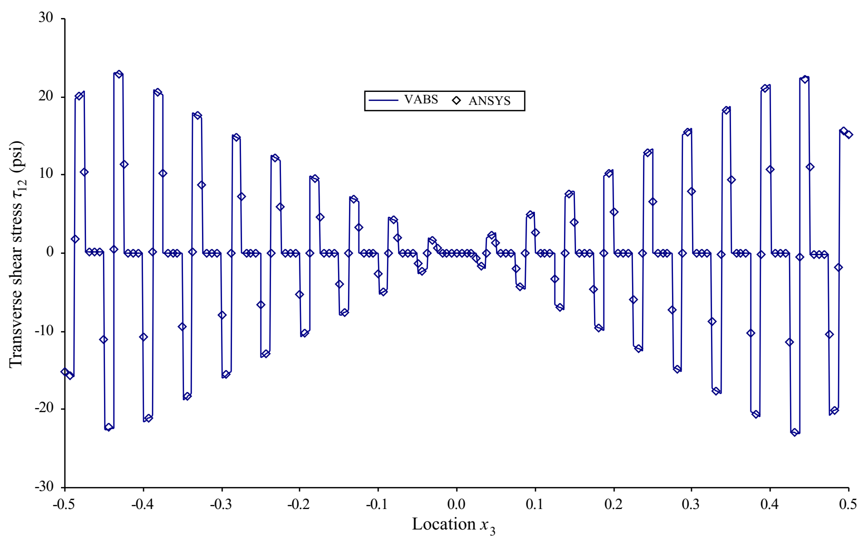


Figure 9: Stress component τ_{13} at mid-span and $x_2 = 0$



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