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Assessment of Beam Modeling Methods for Rotor Blade Applications

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Abstract—There is no lack of composite beam theories. Quite to the contrary, there might be too many of them. Different approaches, notation, etc., are used by the authors of those theories, so it is not always straightforward to compare the assumptions made and to assess the quantitative consequences of those assumptions. Moreover, there is a serious lack of experimental results and benchmark problems. As a result, one finds that most theories perform about equally well on the few extant benchmark problems. This can obscure differences among theories and simultaneously create the false expectation that a specific theory will perform as well in *all* cases. The goal of this paper is to attempt to objectively assess theories within a common framework. The validity and relative importance of various assumptions that are present in the literature are discussed. It is hoped that this will be a first step toward the clearly desirable situation in which an engineer can safely and easily choose a composite beam theory based on the type of application and specific needs for fidelity.

Keywords—Beam, Composite, Cross-sectional analysis.

1. INTRODUCTION

Composite beam theories find extensive application in modeling rotor blades. Most commonly, rotor blades are modeled as thin-walled box-beams. However, the popularity of more sophisticated cross-sectional modeling is increasing, ranging from analytical theories for one- or two-celled thin-walled beams to finite element cross-sectional analyses which will model any cross section. Here we focus on blade modeling, since the geometry of flexbeams is somewhat simpler than that of blades; and such sophistication is usually not required for modeling flexbeams.

Once the geometric idealization is made, a separate logical step in beam modeling consists of selecting the type of beam theory to be employed. Any beam theory is associated with introduction of variables which depend only on the coordinate along the beam axis. For general deformation at least four such 1-D variables have to be introduced: extensional (u_1'), torsional (θ'),

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and two flexural variables (u_2'' and u_3'') corresponding to bending deformation along two orthogonal directions. The corresponding 1-D governing equations are uncoupled for isotropic beams with doubly symmetric cross sections and are given by the Euler-Bernoulli Theory for extension and bending and the St. Venant Theory for torsion. To extend this theory to composite beams, one must allow the governing equations to become coupled due to the appearance of off-diagonal terms in the cross-sectional stiffness matrix. This 4×4 stiffness matrix S characterizes elastic properties of the beam. Then, the strain energy per unit length is expressed in terms of the four 1-D strain measures as

$$2U_{\text{classical}} = \alpha^{\mathrm{T}} S \alpha, \tag{1}$$

where $\alpha^{\mathrm{T}} = \{u_1', u_2'', u_3'', \theta'\}$. Various methods have been proposed in the literature to improve the fidelity of such a theory, which below is referred to as "classical". Two important examples of such refinements are the theories due to Timoshenko [1] and Vlasov [2]. The latter is known to be more important for open cross sections [3]. In both cases, additional 1-D variables must be considered which are rotations of the cross sections and twist rate, respectively; the size of the cross-sectional stiffness matrix becomes 6×6 and 5×5, respectively, or 7×7 if both effects are included [4]. Apparent successes of these refinements have led to their extensive use for beams, even in cases where improvements over the classical theory are questionable at best. Moreover, these apparent successes have led some investigators to unnecessarily introduce still more 1-D variables. That was especially true for box-beams, where formulations having stiffness matrices as large as 9×9 can be found [5,6]. While these refinements inevitably result in more complicated models with more 1-D variables as well as more complicated boundary conditions, the benefits of those refinements are yet to be generally accepted. In other words, in many cases there seems to be no proof that the additional complexity of such theories is warranted. Even quite elaborate models such as [7] are based on assumptions that have to be verified for various material properties before their results can be totally trusted.

A seemingly different approach was taken in [8–10], where so-called "closed form fundamental beam solutions" are provided. Those solutions differ from traditional beam theories, because the warping used therein depends on the boundary conditions. Still other researchers have argued that beam modeling should be abandoned for box-beams altogether and more computationally intensive plate analysis needs to be employed [11].

In this paper, using asymptotically correct stiffnesses derived from [3,12], the results of classical beam theory are compared with various published refinements. It is the goal of this paper to critically examine these and other recent developments. The relative validity and importance of various assumptions that are made by different authors is discussed and illustrated.

2. ANALYSIS OF THEORIES

The present discussion is restricted to prismatic beams where 3-D constitutive as well as strain-displacement relationships can be considered linear. To facilitate the following discussion, let us introduce the following system of coordinates: a Cartesian one with the $x_1 \equiv x$ axis directed along the beam, and $x_2 \equiv y$ and $x_3 \equiv z$ axes defining the beam cross section; and a curvilinear one (for the case of thin- and thick-walled sections) with s and ξ being the contour and through-the-thickness coordinates, respectively. Here $\xi = 0$ corresponds to the shell midsurface.

While the traditionally classical 1-D variables are often introduced using intuitive considerations, this choice of variables for statics and low-frequency dynamics can be shown to be "natural" by applying rigorous mathematical procedures to 3-D elasticity without any *ad hoc* assumptions [13,14]. In particular, it follows that if the classical stiffness coefficients are calculated correctly, then the elastic behavior of beams is predicted well for slender beams, i.e., $a/\ell \ll 1$, where a is a characteristic dimension of the cross section and ℓ is a longitudinal dimension related to the wavelength of deformation. On the other hand, as the beam becomes less and less slender, so that $a/\ell \approx 1$, beam theory is practically useless. However, a quantitative connection between the slenderness ratio a/ℓ, and the accuracy of classical beam theory strongly depends on material properties and the cross-sectional geometry.

A direct way to assess the importance of nonclassical effects is by studying the dispersion curves for a given cross section [14]. This study provides quantitative estimates for the decay rate of any neglected modes of deformation for any specified beam. "Slow" decay indicates deep penetration of these deformations into the interior of the beam, while "fast" decay indicates localization of those effects at the ends of the beams. (Here "slow" and "fast" are relative to the slenderness ratio of the beam.) Consequently, in the case of slow decay rates, such as one finds with an I-beam, nonclassical deformations are important for the global elastic behavior of the beam. The inadequacy of classical theory is clearly exhibited; in particular, the well-known importance of the Vlasov effect can be established using this type of analysis. In the latter case, however (e.g., box-beams), such analysis shows clearly that nonclassical deformations can be justifiably neglected. Thus, we expect box-beams, whether thin-walled or thick-walled, to be adequately modeled using classical theory. Indeed, based on this type of analysis [14], we argue that something is amiss when refinements of classical beam theory are claimed necessary in order to accurately predict box-beam behavior.

The popularity of refined theories is fed by continual claims in the literature that classical theory is inadequate, even for thin-walled box-beams. Unfortunately, these claims are often based on results obtained from an incorrect 4×4 classical stiffness matrix. This has the effect of giving the illusion that refined theories make a big difference in the right direction and facilitate a favorable comparison for benchmark problems. Obtaining the correct 4×4 cross-sectional stiffness matrix is then vital to establish the validity of such claims.

A correct analytical procedure for doing this appears to have first been outlined for thin-walled beams in [15]. However, a closed-form solution was provided as an example only for special "pseudoisotropic" materials. The introduction of the variational-asymptotic method in context of anisotropic beams in [13] allowed the treatment of this problem from a different perspective: beam theory can be obtained from 3-D elasticity without making any *ad hoc* assumptions using the small parameter $a/\ell \ll 1$. For a cross section with arbitrary geometry and material distribution, the problem is reduced to a system of 2-D equations over the cross-sectional plane. Development of a numerical solution of this problem is presented in [12], providing a means to calculate the matrix of cross-sectional stiffness constants for any kind of cross section whatsoever, including thin- and thick-walled box beams. Results consistent with VABS can also be obtained using NABSA [16]. Even though both approaches use a 2-D finite element method for cross-sectional analysis, the amount of calculation is very modest and is easily compensated for by the gained simplicity of the resulting beam theory. By taking advantage of the existence of an additional small parameter [3] provides asymptotically correct closed-form expressions for the stiffnesses of thin-walled beams.

In what follows, we will use results from [3,12] to depict the behavior of consistent classical formulations (i.e., with 4×4 cross-sectional stiffness matrices) and compare these with results from "refined" or "higher-order" approaches (having larger cross-sectional stiffness matrices). The intent is to show that classical theory is sufficiently accurate for all the cases we present, so that refinements are unnecessary.

First, let us turn to the assumption of vanishing stresses in the plane of the cross section, the so-called Bernoulli hypothesis or uniaxial stress hypothesis. Since this is a very popular assumption [7–10], commonly invoked explicitly for solid sections and implicitly for thin-walled ones, let us consider it in more detail.

2.1. Uniaxial Stress

Unlike the case of thin plates and shells, where the analogous assumption that $\sigma_{zz} = 0$ makes perfect sense, it is easy to verify that a uniaxial stress field $\sigma_{yy} = \sigma_{zz} = \sigma_{yz} = 0$ for a beam is incompatible unless the material is isotropic. While the asymptotically correct displacement field leads to a coupled problem for warping in and out of the cross-sectional plane, only the out-of-plane warping need be considered when the stress field is uniaxial. This resulting simplicity, with in-plane warping effectively eliminated from the problem, versus the complexity of the more rigorous approach, might explain why the assumption is so popular, despite the incompatibility. While it is possible that cross-sectional constants based on this assumption differ only slightly from their asymptotically correct counterparts for many cases, a comprehensive investigation of this assumption is long overdue. Until then, a false sense of security might be somewhat dangerous. It is reasonable to expect that for certain configurations the discrepancy might become significant. The situation is thus analogous to the neglect of hoop and shell bending stress measures for thin-walled, closed sections [3]. One such configuration is considered below in the "thin-walled beams" section.

It is interesting to note that in the theory developed by Kim and White [7], in which this assumption is invoked, the resulting 6×6 cross-sectional stiffness matrix (the extra 2 rows and columns are for transverse shear since the theory is Timoshenko-like) is not symmetric. Let us recall that an alternative way for accounting for transverse shear effects due to Levinson is used in the paper. The original elegant derivation in [17] was conducted for isotropic rectangular sections, and even the author himself mentions that extension of the method to other geometries is far from straightforward. Thus, without some justification for the way in which this theory was extended to composite box-beams in [7], one has reasons to question its rigor. The "uniaxial stress" assumption combined with adjusted material constants effectively transfers the effect of in-plane warping into the out-of-plane warping. Even if the calculation of the beam stiffness is carried out correctly from that point, the out-of-plane warping is not necessarily close to the result from 3-D elasticity. Rather, it is an "equivalent" warping when 3-D effects are condensed to 1-D form. The inconsistency of the displacement field modeled in this way versus that from 3-D elasticity is most likely the reason for the beam stiffness matrix turning out to be nonsymmetric in [7].

2.2. Solid-Cross-Section Beams

Solid rectangular sections are not widely used in the rotorcraft industry, with one notable exception being flex-beams (and even there open sections are more prevalent). Still, some research in this area might be useful to the rotorcraft community, because it could be a basis for the development of a beam analysis for different cross sections, such as thick-walled box-beams [7] (see the discussion below). From this point of view, it is interesting to look at assumptions made in [8–10]. Similar to its treatment in Euler-Bernoulli Theory, the displacement field is assumed to consist of a combination of four "rigid" motions of the cross sections, which correspond to beam (1-D) variables related to extension, torsion and bending in two directions, as well as an unknown out-of-plane (3-D) warping Ψ. The uniaxial stress hypothesis (see the discussed above) is used. Thus, the constitutive equations are reduced to

$$\begin{Bmatrix} \sigma_{xx} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & 0 & C_{16} \\ 0 & C_{55} & 0 \\ C_{11} & 0 & C_{16} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{2}$$

and a problem for the warping Ψ (as a function of beam variables) can be posed. If the solution for warping is superimposed over the "rigid" cross-sectional displacement field, with the resulting strain energy density consequently integrated over the cross section, one obtains an Euler-Bernoulli-like theory with a 4×4 stiffness matrix. Then ψ corresponds to at St. Venant torsional warping function, which is, of course, the same as for the uncoupled case (i.e., when $C_{16} = 0$). Rather than considering the four 1-D strain measures as independent, which would allow one to later satisfy arbitrary beam boundary conditions, constraints are posed on those

1-D strain measures in [8]. This allows for the warping to satisfy appropriate equations for given boundary conditions. Both the constraints on the 1-D strain measures and resulting warping become dependent on the boundary conditions. For example, in the case of a cantilever beam with a torque applied at the tip,

$$\phi_{,x} = \bar{\phi}_{,x} - \frac{C_{16}}{2\bar{C}_{16}} w_{,xx},$$

where $\bar{\phi}_{,x}$ is a twist rate for an uncoupled lay-up ($C_{16} = 0$). From the viewpoint of beam equations, this implies that

$$S_{23} = S_{22}\frac{C_{16}}{2\bar{C}_{16}}.$$

Here S_{22} is the torsional stiffness and S_{23} is the coupling term between bending and torsion. It should be noted that the words "exact analytical solution" used in [8–10] are somewhat misleading since the solutions presented there are clearly not exact solutions of 3-D elasticity.

For comparison, the configuration in Table 1 is used. As one can see in Figure 1 for the homogeneous solid rectangular cross section, the correlation with VABS is excellent (the plot is for a square section, but other aspect ratios were checked as well). Thus, we can conclude that for this particular type of cross section, the approach in [8] leads to a solution that is practically equivalent to the asymptotically correct one. In particular, it implies that the uniaxial stress assumption performs satisfactorily for homogeneous, solid cross sections. However, extension of this methodology to nonhomogeneous beams is not straightforward, and there is clearly no reason to expect that the uniaxial stress hypothesis will lead to accurate results in the general case, as demonstrated below for box-beams.

Table 1. Properties of homogeneous, solid-cross-section beams used in study.

Outer Dimensions Height $b = 0.1\,\text{m}$	Width $a = 0.1\,\text{m}$
Layup	$[\theta]_N$
Material Properties	$E_l = 130\,\text{GPa}$
$E_t = 12.0\,\text{GPa}$	$G_{lt} = 6.0\,\text{GPa}$
$G_{tn} = 4.0\,\text{GPa}$	$\nu_{lt} = 0.3$, $\nu_{tn} = 0.5$

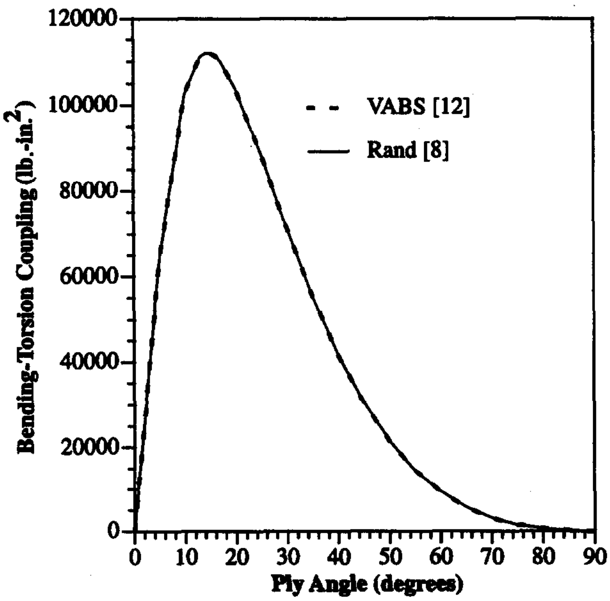

Figure 1. Torsional-bending coupling stiffness for a homogeneous square solid cross section.

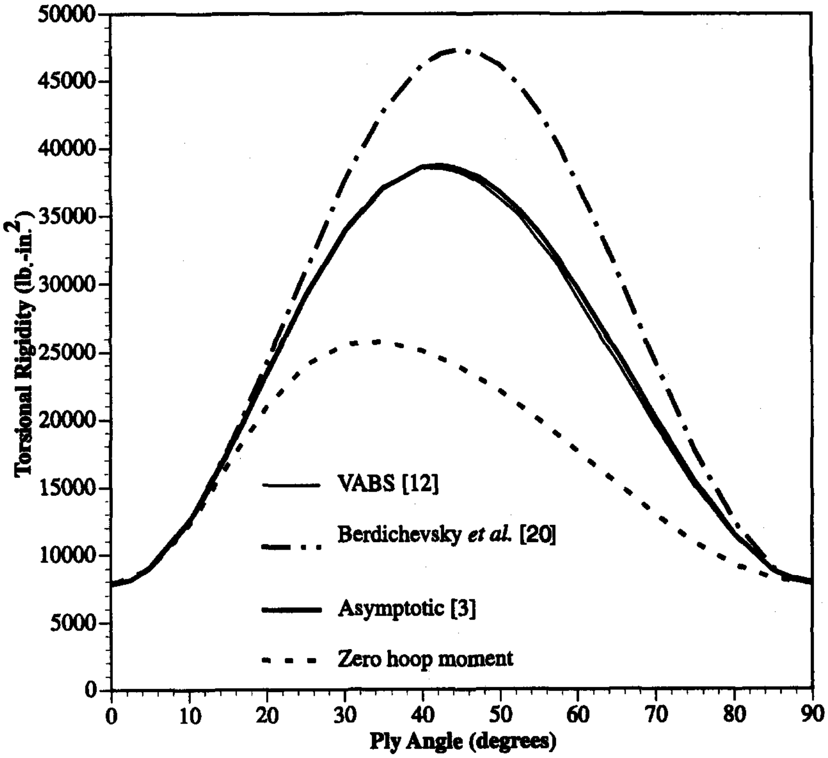

Figure 2. Torsional rigidity of box-beam.

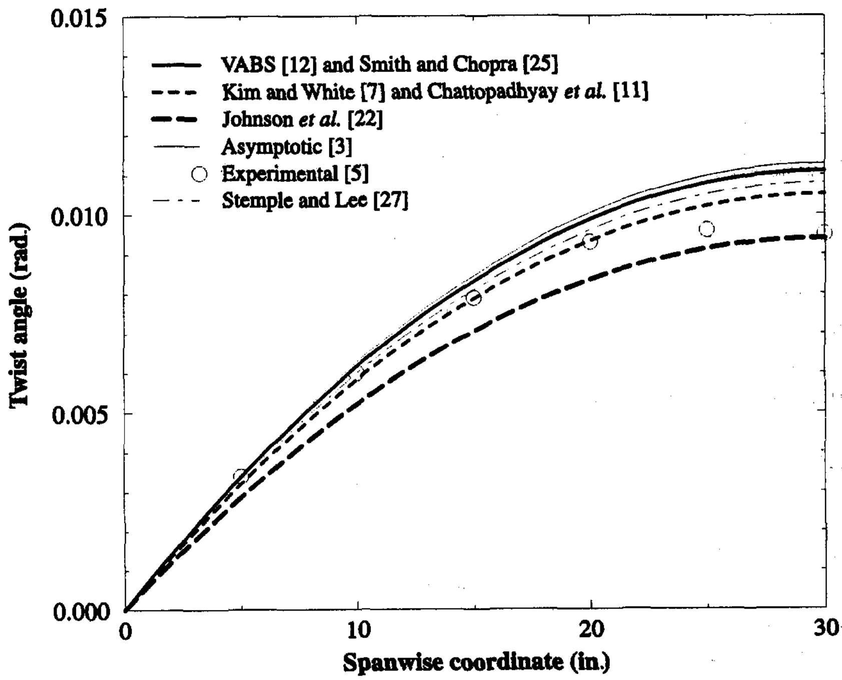

Figure 3. Twist of a CAS thin-walled beam $\theta = 15°$ due to a tip shear force (1 lb.).

2.3. Open Sections

Experimental data for open sections practically boils down to I-beams studied in [18]. Such scarcity of experimental data, similarly to the closed-section case, leads to a peculiar situation when, depending on one's point of view, either most of the existing theories correlate reasonably

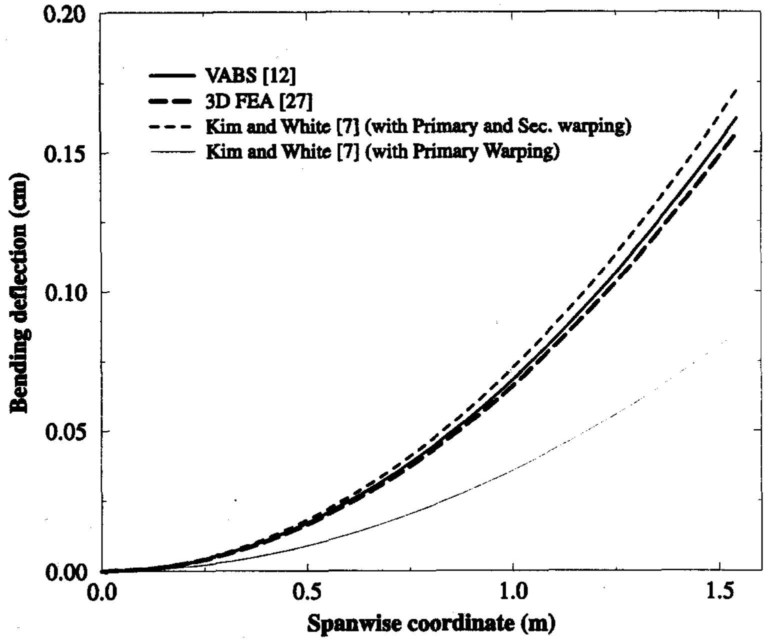

Figure 4. Vertical deflection for a thick-walled CAS box-beam due to a 1.13 KN-m tip twisting moment.

well with these data, or none correlates well enough. As a typical example let us consider a lamination angle of 15°, results for which are shown in Figures 2 and 3, where distribution of a bending slope and a twist are depicted, respectively. In both cases, a unit shear load is applied at the free tip. As one can see, a simple asymptotically correct theory with 5×5 stiffness matrix performs at least as well as other, more complicated theories. On the other hand, an experimental data for a twist distribution due to a tip torque (see Figure 4) is not predicted well by any theory, if one takes for granted the statement made in [18] that the "warping was restrained" at the free end, i.e., kinematic condition $\theta' = 0$ is used. In order to avoid widespread confusion, it has to be remembered that sometimes the term "restrained warping" is used just to distinguish the Vlasov Theory, which takes into consideration changes in the rate of twist, i.e., θ'' from a St. Venant Theory, where only θ' is included in the analysis. The latter theory is then referred to as "free warping theory". It can be seen from Figure 4 that experimental data fall somewhere between the curves produced using such a "restrained end" (kinematic) boundary conditions and a "free end" (natural) one with zero bimoment at the free end.

2.4. Thin-Walled Beams with Closed Sections

We now turn to thin-walled beams, which are reasonable approximations used to represent rotor blades in preliminary design. Thin-walled beams have attracted a lot of research due to the feasibility of analytical solutions. The box-beam configurations used in the examples, all of which have an effective length of 30 in., are described in Table 3. Applying the variational-asymptotic procedure to thin-walled cross sections, where another small parameter exists, namely $h/a \ll 1$ (where h is a wall thickness), allows one to start with shell theory rather than 3-D elasticity. This procedure was used first in [20] to obtain analytical solutions for closed sections using only membrane strain measures, and was expanded further to account for multicell configurations with active materials [21]. Recently, asymptotically correct formulae for a general anisotropic thin-walled beams were provided in [3]. Let us recall that there are three membrane shell strain measures $\gamma^{\mathrm{T}} = \{\gamma_{xx}, \gamma_{ss}, 2\gamma_{xs}\}$ and three bending/twisting shell strain measures $\rho^{\mathrm{T}} = \{\rho_{xx}, \rho_{ss}, \rho_{xs}\}$. (The latter are asymptotically equivalent to the three curvatures $\gamma^{\mathrm{T}} = \{\kappa_{xx}, \kappa_{ss}, \kappa_{xs}\}$.) For a generally anisotropic shell these six strain shell measures are connected with stress resultants

Table 2. Properties of an I-beam used in study.

Outer Dimensions Wall Thickness $h = 0.04$	Flange Width $b = 1.0$ in. Web Height $a = 0.5$ in
Layup	
Web	$[0/90]_4$
Right Upper Flange	$90/0/90/0_2/90/\theta_2$
Left Upper Flange	$[0/90]_3/\theta_2$
Left Lower Flange	$[0/90]_3/[-\theta]_2$
Right Lower Flange	$[90/0/90/0_2/90/[-\theta]_2]$
Material Properties	$E_l = 20.59 \times 10^6$ psi
$E_t = 1.42 \times 10^6$ psi	$G_{lt} = 8.7 \times 10^5$ psi
$G_{tn} = 6.96 \times 10^5$ psi	$\nu_{lt} = \nu_{tn} = 0.42$

Table 3. Properties of thin-walled box beams used in study.

Outer Dimensions Wall Thickness $h = 0.03$	Width $a = 0.953$ in Height $b = 0.53$ in
Layup	
Right and Upper Wall CAS1	$[\theta_3/-\theta_3]$
Left and Lower Wall CAS1	$[-\theta_3/\theta_3]$
Right Wall CAS2	$[\theta/-\theta]_3$
Left Wall CAS2	$[-\theta/\theta]_3$
Upper Wall CAS2	$[\theta]_6$
Lower Wall CAS2	$[-\theta]_6$
All Walls CUS1	$[\theta]_6$
Material Properties	$E_l = 20.59 \times 10^6$ psi
$E_t = 1.42 \times 10^6$ psi	$G_{lt} = 8.7 \times 10^5$ psi
$G_{tn} = 6.96 \times 10^5$ psi	$\nu_{lt} = \nu_{tn} = 0.42$

$N^{\mathrm{T}} = \{N_{xx}, N_{ss}, N_{xs}\}$ and moments $M^{\mathrm{T}} = \{M_{xx}, M_{ss}, M_{xs}\}$ by a fully populated 6×6 stiffness matrix.

It is the relative importance of these six strain measures that allows one to sort out the various assumptions made for different types of thin-walled beam cross sections. Indiscriminate retention of all of these measures in an analysis without regard for their relative importance inevitably leads to an overly complex analysis. Moreover, inappropriately neglecting a measure can introduce inaccuracies into the analysis.

As an example of the former, consider [5] where retaining κ_{xx} led to the introduction of derivatives of beam transverse shear strain measures as independent 1-D variables—leading to a 9×9 cross-sectional stiffness matrix. (A later analysis, [19], modified this approach so that the matrix is reduced to 7×7 while still retaining the derivatives of the transverse shear strain measures.) However, the asymptotic approach, [3], clearly shows that retention of κ_{xx} is inconsistent with shell theory, where terms of order h/a are neglected with respect to unity. In thin-walled beam analysis, the torsional stiffness of open cross sections stems from κ_{xs} of the underlying shell theory. On the other hand, torsional stiffness of closed sections arises from γ_{xs} while the portion from κ_{xs} (sometimes called "secondary warping") ought to be neglected for exactly the same reasons as stated above for κ_{xx}. This in turn implies the difference in relative importance of a higher-order Vlasov's term (which stems from γ_{xx}). It becomes important for open cross sections due to the presence of the inverse of the small parameter h/a; clearly, this is not the case for closed sections.

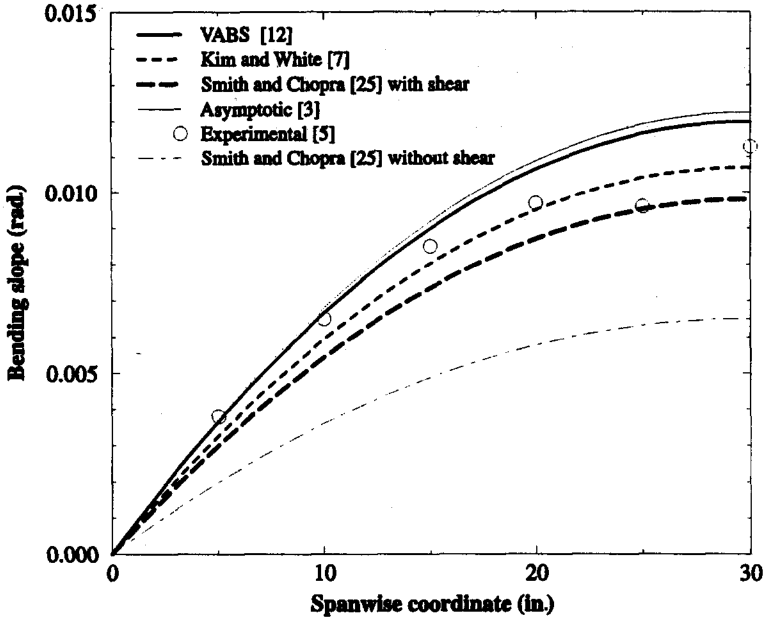

Figure 5. Bending slope of a CUS box-beam due to a tip shear force (1 lb.).

Most researchers neglected all bending strain measures for closed sections [20,22]. This would seem to be logical since κ_{xs} and κ_{xx} are small. Indeed, one can show that for circumferentially uniform stiffness (CUS) beams, the approach of [20] yields asymptotically correct results. However, as shown in [3], for certain layups, such as those with circumferentially asymmetric stiffness (CAS) construction, κ_{ss} cannot be neglected without introducing significant errors. To illustrate this, consider the thin-walled box-beam CAS1, the properties of which are given in Table 3. In the thin-walled approximation, the coupling is negligible which results in a diagonal 4×4 cross-sectional stiffness matrix. Note the upper curve in Figure 5, taken from [20] which severely *over* predicts the torsional rigidity.

It is worthwhile to remark that, for thin-walled beams, the uniaxial stress hypothesis (see the discussion above) is analogous to setting $N_{ss} = M_{ss} = 0$. This also has serious consequences for thin-walled beams. Results for the torsional rigidity are plotted in Figure 5. These results demonstrate the qualitative consequences of this assumption, which are not negligible at all. Indeed, the torsional rigidity is severely *under* predicted when this assumption is invoked for the box-beam CAS1.

In [8], a similar methodology as described above for solid cross sections is developed for thin-walled beams with CUS construction. Results obtained for extensional-torsional stiffnesses by using the method of [8] are asymptotically correct. Extension of this method to other types of beams, such as those with CAS construction, is not straightforward. Moreover, cases which are nonhomogeneous through the thickness need to be considered with particular caution (see the discussion on uniaxial stress assumption).

2.5. Transverse Shear

Another important point that needs to be made concerning results obtained for box-beams is associated with the need of having explicit transverse shear measures in the 1-D theory. Several cases in the literature where this claim is made resulted from an inconsistent handling of the shear effects. VABS includes transverse shear effects in the 3-D model but without explicit transverse shear measures in the 1-D model. The good correlation between VABS and the results of [7] indicates that the explicit transverse shear effects in the 1-D model are not very important in the cases studied. As was shown in [12,13,23,24], and for beams made of anisotropic materials transverse shear is not zero within the framework of classical beam theory (i.e., a consistent generalization of Euler-Bernoulli Theory to the anisotropic case). Rather, in general, the two transverse shear measures can be found as functions of the four 1-D variables of classical theory [24]. It is only for structural configurations which have zero bending-shear coupling (such as prismatic beams made of homogeneous, isotropic material) that the transverse shear measures vanish. Thus, a proper judgment of the importance of Timoshenko's correction should be based on the comparison of such a consistent implementation of classical theory for an anisotropic beam (e.g., VABS [12]) with the results of a theory where transverse shear variables are considered as independent 1-D strain measures. Needless to say, comparing results to an inconsistent implementation of Euler-Bernoulli Theory (i.e., one in which the transverse shear measures set to zero from the outset) may lead one to misleading conclusions on the importance of transverse shear [23,24].

As a first example, consider the case of a CUS configuration, exhibiting extension-twist coupling. The cross-sectional properties are given under CUS1 in Table 3, with θ chosen to be 15° for the results presented. Previous studies (for example [25]), attribute the discrepancy in the results solely to the presence of the transverse shear measures in the 1-D model. However, according to the results in [23,24], accurate predictions can be obtained without the explicit use of transverse shear variables as long as the classical stiffnesses are correctly calculated. This can be seen in Figure 6 where the two asymptotic analyses [3,12], neither of which has transverse shear in its 1-D model, correctly predict the bending stiffness of this beam.

As another example related to transverse shear, consider a box-beam with configuration CAS2, Table 3, which exhibits bending-twist coupling. Typical examples of 1-D results for this beam with $\theta = 45°$ are shown in Figure 7. It shows the twist distribution due to a tip vertical shear force of 1 lb. Differences between the results from the plate analysis of [11] and the beam analysis of [25], shown in Figure 7, were attributed by the authors of [11] to transverse shear effects not being correctly treated in [25]. That this claim would be made in the first place is somewhat puzzling, because [25] does include this effect in its treatment. It is a moot point, however, for the following reason. It can be seen that both asymptotic approaches [3,12], correlate better with experimental data than the plate analysis employed in [11, Figure 12]. But in fact, neither of the 1-D asymptotic models have explicit 1-D transverse shear variables in them. Obviously then, the conclusion of [11] cannot be correct.

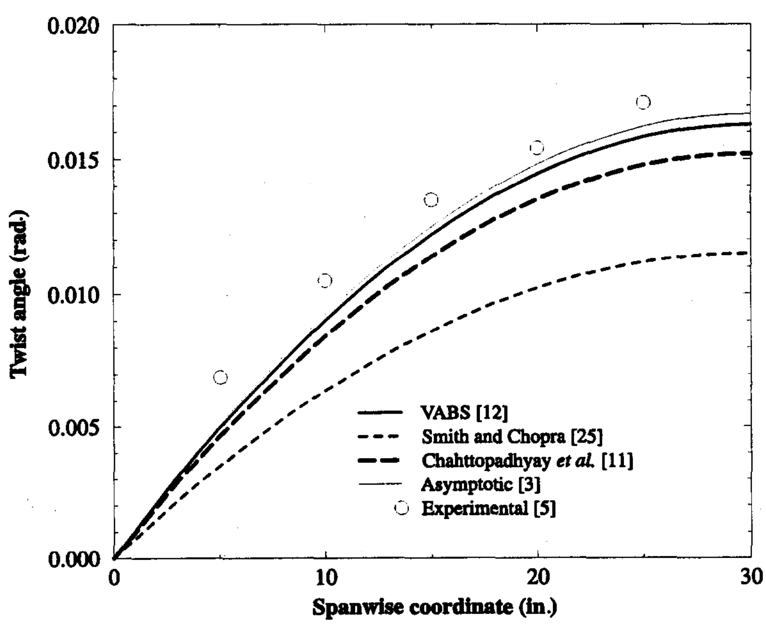

Figure 6. Twist of a CAS thin-walled beam, $\theta = 45°$ due to a tip shear force (1 lb.).

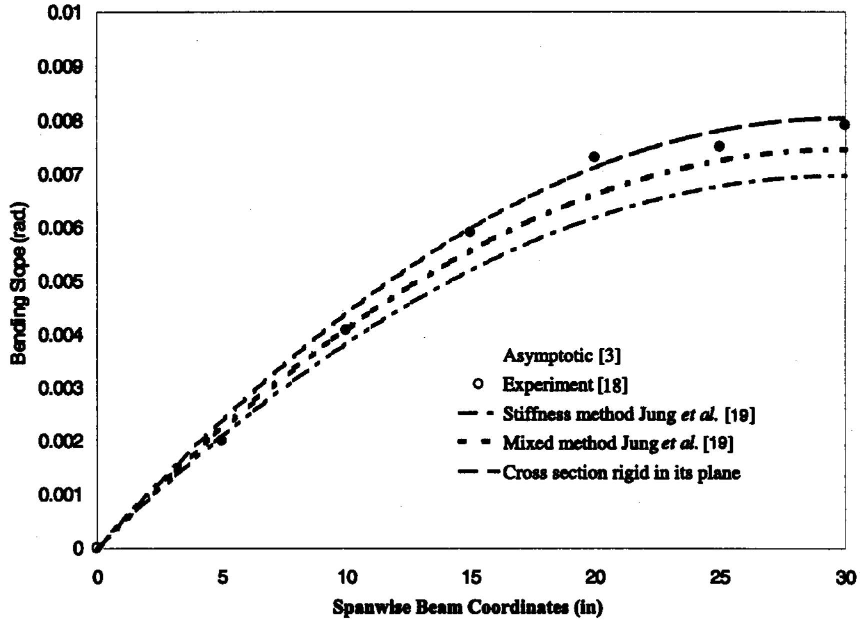

Figure 7. Bending slope of an I-beam $\theta = 15°$ due to a tip shear force (1 lb.).

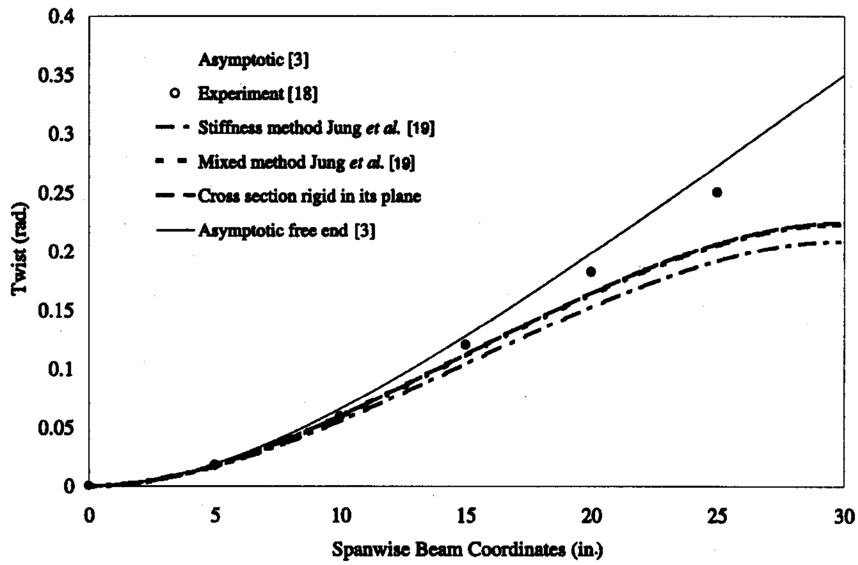

Figure 8. Twist of an I-beam $\theta = 15°$ due to a tip torque (1 lb.).

Consider again a box-beam with configuration CAS2 but this time with lamination angle of 15°, results for which are shown in Figure 8. Due to uncertainty in the 3-D material constants and in the experimental data, and effects neglected from the 3-D theory, one could not conclude that a theory which happened to coincide with experiment within half a percent for one case is any better than one which predicts within 3%. This figure, along with Figure 7, confirms the point made earlier, that many of the theories predict essentially the same behavior for the test cases in the literature. A notable exception to this statement is certainly Figure 5, an uncoupled beam which should be considered as a good test for beam theory.

2.6. Thick-Walled Beams

Of course, the above discussion on relative importance of shell strain measures does not hold for thick-walled beams; but to the best of the authors' knowledge, a rigorous *analytical* treatment

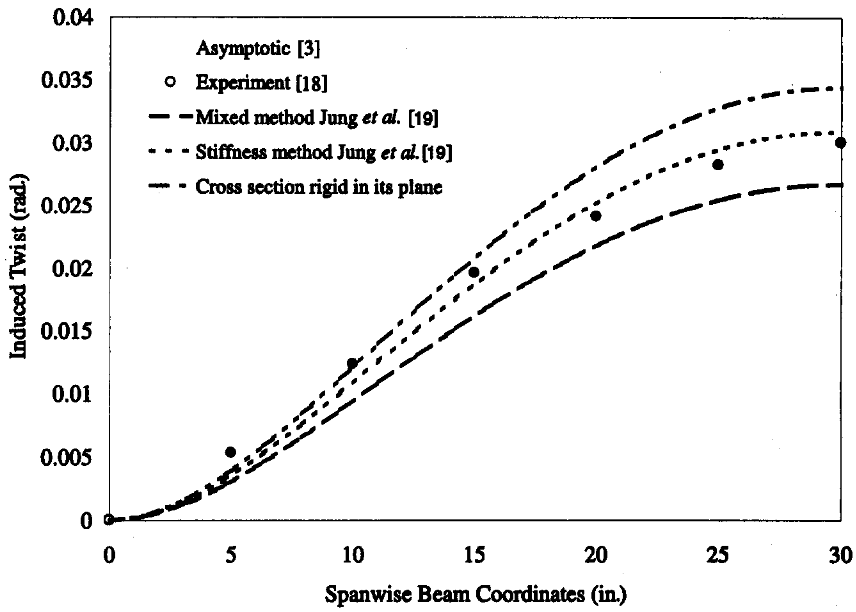

Figure 9. Twist of an I-beam $\theta = 15^\circ$ due to a tip shear force (1 lb.).

Table 4. Properties of thick-walled CAS beam used in study.

Outer Dimensions Wall Thickness $h = 15.24\,\text{mm}$	Width $a = 106.7\,\text{mm}$ Height $b = 50.8\,\text{mm}$
Upper Wall Layup CAS5 Lower Wall Layup CAS5 Right Wall Layup CAS5 Left Wall Layup CAS5	$[\theta_{20} / -\theta_{20}]_3$ $[-\theta_{20} / \theta_{20}]_3$ $[\theta]_{120}$ $[-\theta]_{120}$
Material Properties $E_t = 9.79\,\text{GPa}$ $G_{tn} = 4.83\,\text{GPa}$	$E_l = 141.96\,\text{GPa}$ $G_{lt} = 6.0\,\text{GPa}$ $\nu_{lt} = 0.24 \quad \nu_{tn} = 0.5$

of such beams has not been conducted. Such an analysis would have to involve a refined shell theory, at least including the shell transverse shear effects (implying at least a Reissner-like shell theory). In the meantime, various *ad hoc* theories are suggested in the literature. Probably the most elaborate one is suggested for thick-walled box-beams by Kim and White [7]. To obtain this theory, the uniaxial stress hypothesis is invoked (see the discussion above pertaining to this assumption). A factor 4/3 for quadratic distribution of the out-of-plane warping in [7] is obtained directly from the integration over the cross sectional coordinate. In addition to problems related to the incompatibility described above for nonisotropic materials, this result implies homogeneity of the material properties.

In spite of the fact that this model is a "refined" theory, with a 6×6 cross-sectional stiffness matrix, good correlation of the results reported in [7] is obtained with a 4×4 model from VABS is obtained. A typical example of these results is depicted in Figure 9, where a vertical bending deflection due to 1.13 KN-m tip torque is given. These results correspond to [7, Figure 29], for a thick-walled box-beam with bending-twist coupling (CAS5). The material (AS4/3501-6) properties and dimensions are given in Table 4 for a beam of length 1,524 mm.

It is worth mentioning at this point that Kim and White [7], do not include restrained warping effects. If the claims of [6,26] on the importance of restrained warping for thick-walled beams are to be believed, these would be cases that should be tested. However, in view of the dispersion analysis discussed above, this effect is not at all important for thin-walled box-beams, and it certainly should not be any more important for thick-walled beams than it is for thin-walled ones.

3. CONCLUSIONS AND RECOMMENDATIONS

Based on investigations of various beam theories for beams with a variety of geometry and material properties, the following has been concluded.

1. Most of the reported improvements in the cited references on refined beam theory for the modeling of box-beams are due to wrong stiffness coefficients for classical theory having been used as baselines for the comparison. When asymptotically correct stiffness constants are used for classical theory, the corresponding correlation with 3-D finite element and/or experiments is not inferior to any of the studied "refined" theories.
2. Claims that any significant contribution to the 1-D results is due to transverse shear, Vlasov's warping, or any other effects which are not accounted for by a classical theory, are not found plausible for any of the box-beams considered in the study. Of course, these effects become more important for short beams, but other end effects (neglected) can be expected to be at least as important.
3. These claims and associated motivation to introduce additional 1-D variables can be partially attributed to the misconception that a cross section remains rigid in its own plane in classical theory. In reality, an asymptotically correct classical theory will account for a consistent in-plane deformation which is proportional to classical 1-D strain measures. In a similar fashion, for general anisotropy, transverse shear strains that are proportional to classical 1-D strain measures are present as well.
4. There is a fundamental difference between open- and closed-cross-section beams which explains why the Vlasov correction is needed in the former case but not in the latter.
5. The "fundamental closed-form solutions" for solid rectangular cross sections for beams made of homogeneous orthotropic material [8], are practically equivalent to asymptotic beam solutions. This is also true for homogeneous thin-walled sections of CUS construction. Extension of this work to more general cases is not straightforward.

The study presented here suggests strongly that additional experimental data are needed to validate composite blade models. In particular, a set of benchmark problems needs to be designed which can be used to evaluate ranges of validity for various theories. Wall thickness, initial twist and curvature, and full elastic couplings of all types should be widely varied in this set. It is very easy to evaluate the importance of Vlasov's term and other end effects for thick-walled sections. Tests only have to be run for beams having the same cross-sectional properties but differing lengths. A far more difficult problem involves the validation of transverse shear stiffnesses in a 1-D model, since their effect in static analyses is quite small. It is possible that experiments involving dynamics would have to be performed. Because of the extreme difficulty involved in devising meaningful and useful experiments, other alternatives should be explored as well. One such alternative to experimental validation is 3-D finite element calculations. These cannot be taken lightly, since they would be very computationally intensive, requiring approximately 10^5 degrees of freedom for realistic rotor blade analysis.

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