The Transfer Matrix Technique
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Prof. Dewey H. Hodges

When the modal bending moment and shear force distributions are required, the transfer matrix approach is very powerful and very efficient. It is based on a first-order mixed formulation including displacement, rotation, and stress resultants. As an example of its application, consider a rotating beam whose motion is governed by the ordinary differential equation

$$(EIv'')'' - (Tv')' - \omega^2 mv = 0 \tag{1}$$

Here EI is the bending stiffness, T is the tension force given by

$$T = \Omega^2 \int_x^\ell mx^* dx^* \tag{2}$$

Ω is the angular speed of the beam rotation, m is the mass per unit length, ω is the frequency of free vibration, x is the axial coordinate, $(\ )'$ denotes differentiation with respect to x, and ℓ is the length of the undeformed beam. We now write these equations in first order form with the deflection, slope, moment, and shear as the state variables:

$$\begin{aligned} v' &= \beta \\ \beta' &= \frac{M}{EI} \\ M' &= -V + T\beta \\ V' &= -\omega^2 mv \end{aligned} \tag{3}$$

This can be expressed in matrix form as

$$\begin{Bmatrix} v \\ \beta \\ M \\ V \end{Bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{EI} & 0 \\ 0 & T & 0 & -1 \\ -m\omega^2 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v \\ \beta \\ M \\ V \end{Bmatrix} \tag{4}$$

or $z' = Az$. By defining the transfer matrix τ such that

$$z(x) = \tau(x) z(\ell) \tag{5}$$

for example, one can verify that τ satisfies the equation

$$\tau' = A\tau \tag{6}$$

with starting value $\tau(\ell) = I$ where I is the 4×4 identity matrix. One can integrate this equation numerically (starting at the tip) and iterate to find those values of frequency that satisfy the equations and boundary conditions. For a cantilever beam, $v(0) = \beta(0) = M(\ell) = V(\ell) = 0$. Thus,

$$\left\{ \begin{array}{c} v(0) \\ \beta(0) \end{array} \right\} = \begin{bmatrix} \tau_{11}(0) & \tau_{12}(0) \\ \tau_{21}(0) & \tau_{22}(0) \end{bmatrix} \left\{ \begin{array}{c} v(\ell) \\ \beta(\ell) \end{array} \right\} \tag{7}$$

Standard integration schemes can be used to calculate $\tau(0)$ for any value of ω^2. Although the eigenvalue ω is unknown, we know that for the left hand side of Eq. (7) to vanish, the determinant Δ of the matrix of coefficients, which may be now regarded as a function of ω^2, must vanish yielding

$$\Delta(\omega^2) = \tau_{11}(0)\tau_{22}(0) - \tau_{12}(0)\tau_{21}(0) = 0 \tag{8}$$

A variety of single-equation root solvers will find as many values of ω^2 as desired.

Once ω^2 is known, substitution into Eq. (4) will yield the mode shapes for v, β, M, and V. The starting values (at the tip) of M and V are zero. Since the determinant is zero, once $v(\ell)$ is specified (say unity), then $\beta(\ell)$ is determined from Eq. (7) to be

$$\beta(\ell) = \frac{-\tau_{21}(0)v(\ell)}{\tau_{22}(0)} = \frac{-\tau_{11}(0)v(\ell)}{\tau_{12}(0)} \tag{9}$$

The accuracy of the results depends on the accuracy of the numerical integration scheme used to solve Eq. (6) and the tolerance in the root finding algorithm used to solve Eq. (8).