Special Cases and Theoretical Issues

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Revised June 17, 2008

There are certain issues of a theoretical nature that are not appreciated by most newcomers to the field of rotary-wing aircraft modeling. These issues are rather complex in some cases and can be best understood in the context of simplified examples. In addition there are rather important insights that can be gained through thorough examination of special cases. The purpose of this section is to examine some of these special cases and issues. First we will consider uncoupled equations for the main kinematical variables of flap, lead- lag, and torsion. Then the nature of nonlinear couplings among these variables will be explored.

Uncoupled Linear Equations

We have already explored the uncoupled flap bending equation in some detail. In this section we will summarize it and briefly compare with it the uncoupled lead-lag bending and torsion equations. The notation is primarily that of Hodges and Dowell (1974). For example, displacement components in the axial, lead-lag, and flap directions are denoted as u, v, and w while elastic torsion rotation is ϕ. The pretwist angle is θ.

Flap equation

The flap bending equation is given by

$$(EI_2 w'')'' - (Tw')' + m\ddot{w} = 0$$

where EI_2 is the bending stiffness about the x_2 axis, T is the axial force, and w is the transverse displacement normal to the plane of rotation. This equation was the subject of your third homework assignment in which you should have discovered that

1) the fundamental flap bending frequency increases as a function of rotor angular speed;
2) the fundamental flap bending frequency is greater than or equal to once per rotor revolution (for $EI_2 \geq 0$) (when the blade is outward directed with hub offset greater than or equal to zero);
3) the fundamental displacement mode shape tends to straighten out with most of the curvature concentrated at the root of the blade as Ω increases; and
4) the geometric stiffness term is responsible for the increase of frequency; it actually dominates the elastic stiffness term.

Lead-lag equation

The lead-lag equation

$$(EI_3 v'')'' - (Tv')' - m\Omega^2 v + m\ddot{v} = 0$$

where EI_3 is the bending stiffness about the x_3 axis, is very similar to the flap equation except for the presence of an additional term that tends to partially cancel out the geometric stiffness effect. This term is often called the negative spring term. It stems from centrifugal forces acting radially thus having a component in the lag direction. The rate of increase in fundamental frequency versus rotor angular speed seen with the flapping equation is strongly reduced by this term, and the increase is now only very slight.

One would expect to see a Coriolis force in the lead-lag equation of the form $2m\Omega\dot{u}$. Since both flap and lead-lag bending lead to nonlinear terms that dominate the axial displacement due to the smallness of longitudinal strain of the elastic axis, this term turns out to be nonlinear in a small strain analysis. It is interesting to note that the tension equation (see Hodges and Dowell 1974) is

$$T' = -m(\Omega^2 x_1 + 2\Omega\dot{v})$$

This effect when substituted into the lead-lag equation leads to near cancellation of the Coriolis terms. If only one admissible function is used, there will be no nonlinear Coriolis term. If more than one is used, this term can be represented by an antisymmetric matrix that depends linearly on the generalized coordinates multiplied by the time derivatives of the generalized coordinates.

Torsion equation

The torsion equation typically found in technical analyses is rather complicated, and there are some peculiarities about which one needs to be aware. From Hodges and Dowell (1974) with corrections based on Hodges (1980) and Hodges (1987), the uncoupled torsion equation is of the form

$$(EC\phi'')'' - (GJ\phi')' - (EB\theta'^2\phi')' - \left(\frac{TEK\phi'}{EA}\right)' - \left[\frac{T(EI_1 - EJ)\theta'}{EA}\right]' + i_1\ddot{\phi} + (i_3 - i_2)\Omega^2 \sin(\theta+\phi)\cos(\theta+\phi) = 0$$

where C is the warping rigidity, J is the torsional stiffness constant, B is a cross section integral, i_α is the mass moment of inertia about the x_α axis, $i_1 = i_2 + i_3$, $EI_1 = EI_2 + EI_3$, and K is a stiffness factor that depends on the assumed constitutive law in the analysis. The constant K is of the form

$$K = EI_1 + k_1 EJ + k_2 GJ$$

The first term is important for beams with thin sections (which includes blades). This is the only term of this type in older analyses, and it is often called the trapeze effect. It was first discovered by Wagner (1929) who hypothesized that the beam consists of fibers which carry the longitudinal load during twist. Houbolt and Brooks (1958) discuss this in their classical analysis. The existence of other terms, dependent on the constitutive law (thus the uncertain constants) was first reported in Hodges (1987). For small strain analyses we may neglect the last term. The middle one may be important for materials with large E/G ratios. Warping rigidity is justifiably neglected for beams other than thin-walled, open-section beams.

The correct form of the terms involving pretwist was only ascertained in 1980 by Hodges and Rosen in separate analyses (the tension term by Hodges and the pretwist squared term by Rosen). There are also various nonlinear terms in elastic twist not shown in the above equation that have been shown to be of importance in predicting the behavior of highly twisted blades.

The inertial term with the difference in mass moments of inertia is often called the tennis racquet term. It gives rise to a linear term that stiffens the blade in torsion as Ω increases *provided that* $i_3 > i_2$. In other words it tends to bring each cross section of a blade into the plane of rotation.

Nonlinear Phenomena

Rotor blade aeroelasticity is an inherently nonlinear problem. Thus, some discussion is in order about the dominant nonlinear effects.

Flap-lag coupling

The main nonlinear coupling between flap and lag is the Coriolis coupling and is discussed by Hodges and Ormiston (1973). This coupling stems from two effects: (1) the appearance of $+2m\Omega\dot{u}$ in the v equation and (2) $-2m\Omega\dot{v}$ in the u equation. If one substitutes $v = V(t)\psi(x)$ and $w = W(t)\psi(x)$, the resulting nonlinear coupling is $-W\dot{W}$ in the V equation and $W\dot{V}$ in the W equation. If there is more than one assumed mode for v, there will also be a gyroscopic coupling among these modes of the nature $V_1\dot{V}_2$ and $V_2\dot{V}_2$ in the V_1 equation and $V_1\dot{V}_1$ and $V_2\dot{V}_1$ in the V_2 equation.

There is a secondary nonlinear coupling between flap and lag bending that only explicitly appears in the equations when they are specialized for a torsionally rigid blade. This coupling stems from the fact that a blade undergoing bending may undergo a pitch rotation due to bending only as discussed by Peters and Ormiston (1975) and by Hodges, Ormiston, and Peters (1980). This is most easily seen by setting the torsional rigidity to infinity, solving for the torsion variable, and substituting the result into the bending equations. The only second order term that appears is in the aerodynamics affecting the lift on the blade. It introduces some pitch-flap and pitch-lag coupling and so has an effect on stability.

Nonlinear bending-torsion coupling

The main nonlinear bending-torsion coupling is from the so-called Mil terms in the torsion equation and their corresponding terms in the bending equations. These terms have the form $(EI_2 - EI_3)\kappa_2\kappa_3 = (EI_3 - EI_2)(v''w'' + \ldots)$ in the torsion equation.

There can be other nonlinear bending-torsion coupling that is proportional to GJ since the torsional moment strain is nonlinear, involving bending kinematical variables. Some analyses have these nonlinear GJ terms and others do not. The reason for this is that different analyses use different torsional kinematical variables. Kaza and Kvaternik (1978) have these terms because they use the third angle of a sequence of orientation angles as their torsional variable. Hodges and Dowell (1974) do not have these terms because they use a quasi-coordinate as a torsional kinematical variable (basically the integral of the moment strain). Use of the quasi-coordinate has the effect of bringing the torsional moment into the bending equations in place of the GJ terms. In this form, it was clear that these terms are negligible for the ordering scheme used by Hodges and Dowell (1974). Later studies by Kaza et al showed their nonlinear GJ terms to be negligible as well, an independent validation of their absence in Hodges and Dowell (1974).

Intrinsic quantities

To discuss this subject adequately, one needs to consider a beam under a given load. It is sufficient to restrict the discussion to statics. The deformed beam may be described by certain quantities (often called "intrinsic" quantities) whose values are independent of the way displacement is described. Our matrix C^{Bb} is one such quantity. Although one could select any number of ways to express the elements of this matrix, no matter what variables are chosen the numerical values are set once the beam is loaded and deflected. The same is true of the strains of the reference axis γ and the moment strains κ. Certainly it is true that one could choose a different coordinate system for the displacements, and it would change the appearance of the expressions for γ, but it would in no way affect the numerical values of γ! It is possible to write a set of beam equations in intrinsic form; Love (1944) considers only the static equations, while beams attached to moving frames were considered by Hodges (1990). A set of fully intrinsic equations was presented by Hodges (2003).

The above discussion gives only the tip of the iceberg concerning the controversy over torsion variables that took place in the 1970's. An in-depth discussion of this controversy may be found in Hodges, Ormiston, and Peters (1980) and in Hodges, Crespo da Silva, and Peters (1988). Essentially, those in error tried to say that their equations would predict different behavior if they were based on a different set of orientation angles. They derived sets of equations based on two different sets of angles, claiming that they were different models, and further claimed that previous analyses were wrong. They evidently did not understand the meaning of "intrinsic" quantities and thought that a choice of different variables would somehow change the outcome of the analysis.

Quasi-coordinates

One of the types of variables used by some investigators (such as Kaza and Kvaternik (1978) for the axial displacement is also a quasi-coordinate, just as the torsional kinematical variable of Hodges and Dowell (1974). It is instructive to analyze the effect of such variables in an analysis. First, these variables are analogous to generalized speeds in dynamics. Their use leads to discretizing strains rather than displacements in a Ritz analysis. Their use leads to shifting terms around in the equations in ways that can sometimes be advantageous. For example, if the axial strain is used as the primary variable instead of a Lagrangean displacement variable in the axial direction, there are two effects: (1) the axial equation is simplified considerably; and (2) the other equations automatically contain the tension force that would be obtained from solving the axial equation and substituting it. These variables are not commonly used in finite element analyses, however, because of complications in the coupling relations. See the example on pp. 32 – 35, Hodges, Ormiston, and Peters (1980).