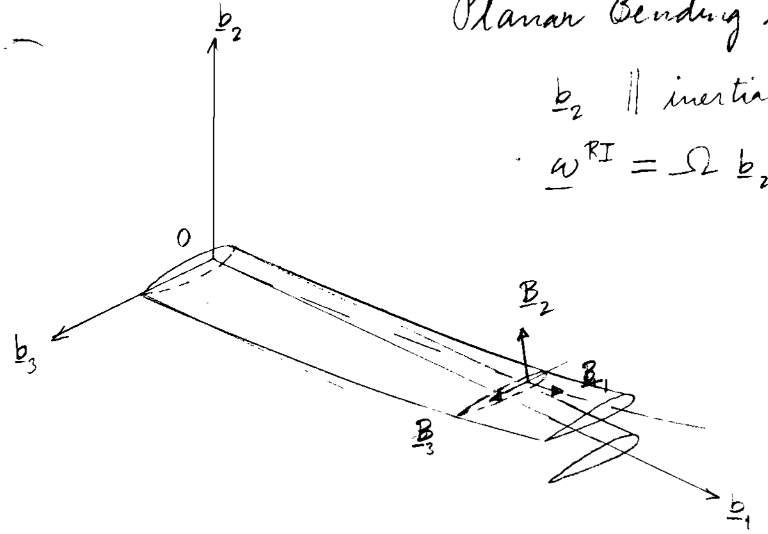
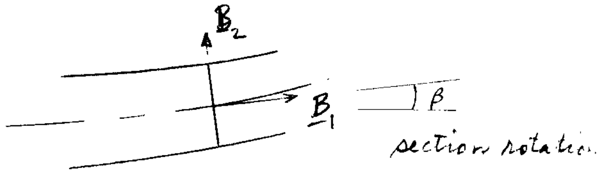
Planar Bending of a Rotating Beam

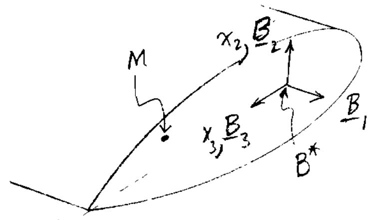
$\underline{b}_2 \parallel$ inertially fixed axis, O fixed in inertial space

$\underline{\omega}^{RI} = \Omega\, \underline{b}_2$





$\underline{B}_1$ is normal to the cross section plane ($\underline{B}_2$ lies in that plane)



arbitrary cross section

M is arbitrary point (material)
B^* is at reference axis (let reference axis be neutral axis)

Bending occurs about $\underline{b}_3 = \underline{B}_3$ (plane sections in undeformed beam remain plane in deformed beam)

$$\begin{Bmatrix} \underline{B}_1 \\ \underline{B}_2 \end{Bmatrix} = \begin{bmatrix} c_\beta & s_\beta \\ -s_\beta & c_\beta \end{bmatrix} \begin{Bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{Bmatrix}$$

$$\underline{R} = \underline{r}^{M/O} = (x_1 + u_1)\underline{b}_1 + x_2 \underline{B}_2 + u_2 \underline{b}_2 + x_3 \underline{b}_3 + \underbrace{w_1 \underline{B}_1 + w_2 \underline{B}_2 + w_3 \underline{B}_3}_{w_i \underline{B}_i}$$

$$\underline{r} = x_i \underline{b}_i \qquad (u_1 = u_2 = \beta = 0)$$

$$\underline{G}_i = \frac{\partial \underline{R}}{\partial x_i} \qquad \underline{g}_i = \frac{\partial \underline{r}}{\partial x_i} \qquad \underline{g}_i \cdot \underline{g}^j = \delta_{ij}$$

$\underline{g}_i \triangleq$ covariant base vectors for undeformed state
$\underline{G}_i \triangleq$ covariant base vectors for deformed state
$\underline{g}^i \triangleq$ contravariant base vectors for undeformed state

$$A_{ij} \triangleq (\underline{B}_i \cdot \underline{G}_k)(\underline{g}^k \cdot \underline{b}_j) \quad \text{(deformation gradient)}$$

$\underline{g}_i = \underline{g}^i = \underline{b}_i$ here (initially straight, untwisted beam)

$$\therefore\ A_{ij} = \underline{B}_i \cdot \underline{G}_j$$

$$\underline{G}_1 = (1+u_1')\,\underline{b}_1 + x_2(\underline{B}_2)' + u_2'\,\underline{b}_2 + w_i'\underline{B}_i + w_i\underline{B}_i' \qquad ()' = \frac{\partial}{\partial x_1}()$$

Small for slender beams — Small for small strain

$$\underline{b}_1 = c_\beta\,\underline{B}_1 - s_\beta\,\underline{B}_2$$

$$(\underline{B}_2)' = -\beta'\,\underline{B}_1$$

$$\underline{G}_1 \cong \left[(1+u_1')c_\beta + u_2's_\beta - x_2\beta'\right]\underline{B}_1 + \left[-(1+u_1')s_\beta + u_2'c_\beta\right]\underline{B}_2$$

$$\underline{G}_\alpha = \underline{B}_\alpha + w_{i,\alpha}\,\underline{B}_i$$ We will show later how to handle warping – for now, ignore it

$$A = \begin{bmatrix} (1+u_1')c_\beta + u_2's_\beta - x_2\beta' & -(1+u_1')s_\beta + u_2'c_\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Jaumann strain components (when local rotations are neglected; strains are small relative to unity)

$$\Gamma = \frac{1}{2}(A + A^T) - I$$

all components vanish except:

$$\Gamma_{11} = (1+u_1')c_\beta + u_2's_\beta - x_2\beta' - 1$$

$$2\Gamma_{12} = -(1+u_1')s_\beta + u_2'c_\beta \quad (= 2\Gamma_{21})$$

Green strain components (more common in literature)

$$\Gamma = \frac{1}{2}(A^TA - I) \quad \text{or} \quad \Gamma_{ij} = \frac{1}{2}(\underline{G}_i\cdot\underline{G}_j - \underline{g}_i\cdot\underline{g}_j)$$

all components vanish except:

$$\Gamma_{11} = \frac{(1+u_1')^2}{2} + \frac{u_2'^2}{2} + \frac{x_2^2\beta'^2}{2} - x_2\beta'\left[(1+u_1')c_\beta + u_2's_\beta\right] - \frac{1}{2}$$

$$2\Gamma_{12} = 2\Gamma_{21} = -(1+u_1')s_\beta + u_2'c_\beta$$

For small elongations and shears, Green components will reduce to Jaumann components. This is not straightforward to show, however!

We use Jaumann for simplicity, and we make 4 geometrical approximations

Approx. I : $\Gamma_{12} = \Gamma_{21}$ is the shear strain. If we ignore shear strain (OK for slender beams that are not made of composite materials)

$$\Gamma_{12} = 0 = u_2' c_\beta - (1+u_1') s_\beta$$

$$\tan\beta = \frac{u_2'}{1+u_1'}$$

This sets $\underline{B}_1 = \frac{\partial \underline{R}^{B*/0}}{\partial s}$ ($s \triangleq$ arc length of deformed beam)

$$= \frac{1}{s'} \underline{G}_1 \Big|_{x_\alpha = 0} = \frac{(1+u_1')\underline{b}_1 + u_2' \underline{b}_2}{s'}$$

Since this is a unit vector

$$(1+u_1')^2 + u_2'^2 = s'^2 \qquad s' = \sqrt{(1+u_1')^2 + u_2'^2}$$

$$\therefore \quad \sin\beta = \frac{u_2'}{s'} \qquad \cos\beta = \frac{1+u_1'}{s'}$$

$$\beta' \cos\beta = \left(\frac{u_2'}{s'}\right)' = \frac{s' u_2'' - u_2' s''}{s'^2} \qquad s'' = \frac{(1+u_1')u_1'' + u_2' u_2''}{s'}$$

$$\beta' = \frac{(s' u_2'' - u_2' s'') s'}{s'^2 (1+u_1')} = \frac{u_2''(1+u_1') - u_2' u_1''}{s'^2}$$

$s' = 1 + \gamma_{11}$ $\qquad$ γ_{11} is the strain of the elastic axis

$\gamma_{11} << 1$

$$\therefore \beta' \cong u_2''(1+u_1') - u_2' u_1''$$

$$u_2' \cong \sin\beta$$

β is now gone. The only nonzero strain is now

$$\Gamma_{11} = \gamma_{11} - x_2 [u_2''(1+u_1'') - u_2' u_1'']$$

$$\gamma_{11} = \sqrt{(1+u_1')^2 + u_2'^2} - 1 = s' - 1$$

Approx. II : γ_{11} is the extensional strain of the elastic axis.
To consider only pure bending, set $\gamma_{11}=0$

$$\gamma_{11} = \sqrt{(1+u_1')^2 + u_2'^2} - 1 = 0$$

$$(1+u_1')^2 + u_2'^2 = 1$$

$$(1+u_1')^2 = 1 - u_2'^2$$

$$1+u_1' = \sqrt{1-u_2'^2}$$

$$u_1' = \sqrt{1-u_2'^2} - 1$$

$$u_1'' = -\frac{u_2' u_2''}{\sqrt{1-u_2'^2}}$$

$$\therefore \Gamma_{11} = -\frac{x_2 u_2''}{\sqrt{1-u_2'^2}}$$

It can be shown that $\beta' = \frac{u_2''}{\sqrt{1-u_2'^2}}\left[1+O(\gamma_{11}, \ell\gamma_{11}')\right]$

u_1 is now gone.

Approx. III $\sqrt{1-u_2'^2} = 1 - \frac{u_2'^2}{2} + \cdots$ (moderate rotation)

$$u_1' \cong -\frac{u_2'^2}{2} \qquad u_1 \cong -\frac{1}{2}\int_0^{x_1} u_2'^2(\bar{x})\, d\bar{x} \qquad (u_1(0)=0)$$

$$\left.\begin{aligned} \dot{u}_1' &= -u_2'\dot{u}_2' \\ \delta u_1' &= -u_2'\delta u_2' \\ \ddot{u}_1' &= -\dot{u}_2'^2 - u_2'\ddot{u}_2' \end{aligned}\right\}$$ will need this when deriving inertial terms

Strain Energy :

$$U = \frac{1}{2}\int_0^\ell \iint_A E\,\Gamma_{11}^2\, dA\, dx_1 = \frac{1}{2}\int_0^\ell EI\left(\frac{u_2''}{\sqrt{1-u_2'^2}}\right)^2 dx_1$$

$$= \frac{1}{2}\int_0^\ell EI\left[u_2''\left(1+\underbrace{\frac{u_2'^2}{2}+\cdots}_{\text{h.o.t.'s}}\right)\right]^2 dx_1$$

Virtual Work of Inertial Forces

$$\int_0^l \iint_A \rho\, \underline{a}^{MI} \cdot \delta \underline{r}^{MI}\, dA\, dx_1$$

$$\underline{r}^{M/O} = (x_1 + u_1)\,\underline{b}_1 + x_2\,\underline{B}_2 + u_2\,\underline{b}_2 + x_3\,\underline{b}_3$$

$$\begin{aligned}
\underline{v}^{MI} = {} & \Omega\,\underline{b}_2 \times \left[(x_1+u_1)\,\underline{b}_1 + u_2\,\underline{b}_2\right] && \leftarrow \underline{\omega}^{RI} \times \underline{R}^{B^*/O} \\
& + \dot{u}_1\,\underline{b}_1 + \dot{u}_2\,\underline{b}_2 && \leftarrow {}^{R}\frac{d}{dt}\,\underline{R}^{B^*/O} \\
& + (\Omega\,\underline{b}_2 + \dot{\beta}\,\underline{b}_3) \times x_\alpha\,\underline{B}_\alpha && \leftarrow \underline{\omega}^{BI} \times \underline{R}^{M/B^*} \\
& + {}^{B}\frac{d}{dt}\, x_\alpha\,\underline{B}_\alpha \to 0 && \leftarrow {}^{B}\frac{d}{dt}\,\underline{R}^{M/B^*}
\end{aligned}$$

$$\begin{aligned}
\delta\underline{r}^{MI} &= \delta u_1\,\underline{b}_1 + \delta u_2\,\underline{b}_2 + \delta\beta\,\underline{b}_3 \times x_\alpha\,\underline{B}_\alpha \\
&= \delta u_1\,\underline{b}_1 + \delta u_2\,\underline{b}_2 - x_2\,\delta\beta\,\underline{B}_1
\end{aligned}$$

$$\begin{aligned}
\underline{a}^{MI} = {} & \Omega\,\underline{b}_2 \times \left[\Omega\,\underline{b}_2 \times (x_1+u_1)\,\underline{b}_1\right] \\
& + 2\Omega\,\underline{b}_2 \times \dot{u}_1\,\underline{b}_1 \\
& + \ddot{u}_1\,\underline{b}_1 + \ddot{u}_2\,\underline{b}_2 \\
& + (\Omega\,\underline{b}_2 + \dot{\beta}\,\underline{b}_3) \times \left[(\Omega\,\underline{b}_2 + \dot{\beta}\,\underline{b}_3) \times x_\alpha\,\underline{B}_\alpha\right] \\
& + \ddot{\beta}\,\underline{b}_3 \times x_\alpha\,\underline{B}_\alpha \\
= {} & -\Omega^2 (x_1+u_1)\,\underline{b}_1 - 2\Omega\dot{u}_1\,\underline{b}_3 + \ddot{u}_1\,\underline{b}_1 + \ddot{u}_2\,\underline{b}_2 - \ddot{\beta}\,x_2\,\underline{B}_1 \\
& + (-\Omega s_\beta\,\underline{B}_1 + \Omega c_\beta\,\underline{B}_2 + \dot{\beta}\,\underline{b}_3) \times \left[(-\Omega s_\beta\,\underline{B}_1 + \Omega c_\beta\,\underline{B}_2 + \dot{\beta}\,\underline{b}_3) \times x_\alpha\,\underline{B}_\alpha\right] \\
= {} & -\Omega^2 (x_1+u_1)\,\underline{b}_1 - 2\Omega\dot{u}_1\,\underline{b}_3 + \ddot{u}_1\,\underline{b}_1 + \ddot{u}_2\,\underline{b}_2 - \ddot{\beta}\,x_2\,\underline{B}_1 \\
& + (\Omega c_\beta x_2 + \dot{\beta} x_3)(\Omega s_\beta\,\underline{B}_1 + \Omega c_\beta\,\underline{B}_2 + \dot{\beta}\,\underline{b}_3) \\
& - (\Omega^2 + \dot{\beta}^2)\, x_\alpha\,\underline{B}_\alpha
\end{aligned}$$

Approx IV: Terms involving $\iint_A \rho x_\alpha^2\, dA$ are usually neglected based on slenderness

assume $\iint \rho x_\alpha\, dA = 0$

$$\delta W_{inertial} = \int_0^l m\,[\ddot u_1 \underline{b}_1 + \ddot u_2 \underline{b}_2 - 2\Omega \dot u_1 \underline{b}_3 - \Omega^2 (x_1 + u_1)\underline{b}_1] \cdot (\delta u_1 \underline{b}_1 + \delta u_2 \underline{b}_2)\, dx_1$$

$$= \int_0^l m\{[\ddot u_1 - \Omega^2 (x_1 + u_1)]\delta u_1 + \ddot u_2 \delta u_2\}\, dx_1$$

$$\cong \int_0^l m\Big\{\Big[\int_0^{x_1} (\dot u_2'^2 + u_2' \ddot u_2')\, dx + \Omega^2 \int_0^{x_1} \Big(1 - \frac{u_2'^2}{2}\Big) dx\Big] \int_0^{x_1} u_2' \delta u_2' \, dx$$
$$+ \ddot u_2 \delta u_2 \Big\}\, dx_1$$

$$\cong \int_0^l \Big(m\Omega^2 \int_0^x u_2' \delta u_2' \, dx + m \ddot u_2 \delta u_2\Big)\, dx_1 + \text{h.o.t.'s}$$

$$= \int_0^l (m \ddot u_2 \delta u_2 + T u_2' \delta u_2')\, dx_1 + \text{h.o.t.'s}$$

$$T = \Omega^2 \int_{x_1}^l m\, dx \quad (\triangleq \text{tension})$$

$$\int_{t_1}^{t_2} \int_0^l (m \ddot u_2 \delta u_2 + T u_2' \delta u_2' + EI u_2'' \delta u_2'')\, dx_1\, dt = 0$$

Introduce $u_2 = v e^{i\omega t}$

$$\int_0^l (EI v'' \delta v'' + T v' \delta v' - \omega^2 m v \delta v)\, dx_1 = 0$$

T term arises from so-called geometric stiffness. Linearization too early would have annihilated the term!

$$(EI v'')'' - (T v')' - \omega^2 m v = 0$$

Discuss bc

essential $\quad v(0) = 0 \qquad\qquad v(0) = v'(0) = 0$

derive natural $\quad v''(0) = v''(l) = 0 \qquad\qquad v''(l) = 0$

$T v' - (EI v'')'|_l = 0 \qquad\qquad T v' - (EI v'')'|_l = 0$