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LETTER TO THE EDITOR

IMPROVED APPROXIMATIONS VIA RAYLEIGH'S QUOTIENT

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1. INTRODUCTION

Many problems can be solved approximately through the minimization of Rayleigh's quotient, which is equivalent to an ordinary differential equation (ODE) and associated boundary conditions that govern the eigenvalue (say, a free-vibration frequency or buckling load) and the "mode shape" associated therewith. In addition to this, there are many other problems for which one-term approximations are both helpful and sufficiently accurate; see, for example, the non-linear examples in reference [1]. To illustrate the methods discussed in this note, however, it is sufficient to consider only that class of problems which can be treated with Rayleigh's quotient. These two methods are Rayleigh's quotient with a free parameter and the method of Stodola and Vianello. Since they are not mutually exclusive, one can sometimes apply both methods simultaneously to obtain outstanding results.

2. RAYLEIGH'S QUOTIENT WITH A FREE PARAMETER

The idea of introducing a free parameter into the usual Rayleigh quotient approach is quite straightforward. One starts with a two-term approximation for the mode shape. However, since the amplitude of the mode shape is arbitrary, it is possible to divide the two-term approximation by either of the undetermined coefficients. This leaves a single unknown coefficient, say a (referred to below as a "free parameter") in the approximate mode shape. Rayleigh's quotient then becomes a ratio of two quadratic polynomials in a, which can be minimized with respect to a. The minimum value of this ratio with respect to a is an improved result versus the value with $a = 0$. The usual means of treating Rayleigh's quotient in most textbook treatments has no free parameter. However, some texts do treat the free parameter method in their problems. For example, reference [2], in problem 12–10 on p. 410, calls for a Rayleigh's quotient with free parameter for a beam on flexible supports. Similarly, reference [3] does the same for problem 7·3, p. 298, for a shaft in torsion. Although not addressed herein, it is noted that this technique also has applications for discrete problems. Finally, it is shown herein how this method can be used to overcome one of the apparent shortcomings of one-term approximations via Rayleigh's quotient—that of using comparison functions for problems which have frequency-dependent boundary conditions.

3. THE METHOD OF STODOLA AND VIANELLO

Modern references to the method of Stodola and Vianello are not common; see, for example, reference [4, 5]. There is a short historical summary in reference [6]. Apparently, this method was applied to technical eigenvalue problems involving continuous systems independently around the turn of the century by Vianello in 1898 and Stodola in 1903.

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If the governing ODE can be solved explicitly for the highest spatial derivative of the unknown mode shape in terms of the mode shape itself and/or some or all of its lower derivatives, then the method of Stodola and Vianello can be applied. The method is executed by substituting an admissible function in place of the mode shape on the right side of this equation. If one can integrate the equation in closed form, the result is a comparison function and a lower bound approximation of the fundamental frequency. If the same procedure is carried out by substitution of a comparison function into the right side, the result will then be an improved comparison function. Obviously, then, the method can be used to develop a sequence of comparison functions, each one of which will be closer to the exact solution than the one before. The method is not limited to eigenvalue problems; indeed, a non-linear equation is treated in reference [1]. For non-linear problems, and even for certain linear ones, the functions can become quite complicated after only a few iterations. Nevertheless, only one or two iterations are normally necessary to obtain results of engineering accuracy, and the whole process is easily automated by means of symbolic manipulation software, such as *Mathematica*. For linear equations, each step is no more complicated than solving for the static deflection. It should be noted that, while the usual application of the method provides a lower bound of the eigenvalue at every iteration, the use of the mode shape in Rayleigh's quotient gives an upper bound on the eigenvalue that is much more accurate than the lower bound.

4. RESULTS

In this section a variety of problems will be solved by both methods presented above and, in some cases, by a combination of the two methods. The intent is not to solve problems that have not been solved before but to present approximate solutions to certain non-trivial but well understood problems to give an idea of the power of the methods.

4.1. *Rayleigh's quotient with a free parameter*

4.1.1. Cantilevered beam. As a first illustration of the method, let us consider a uniform cantilevered beam. Recall that the ODE $\phi'''' = \beta^4\phi$ describes the mode shape of a vibrating beam, where $\beta^4 = \omega^2 ml^4/EI$ is an unknown constant that depends on the natural frequency ω and $(\,)'$ denotes the derivative with respect to the axial co-ordinate x of the beam made dimensionless by the length l. For the cantilevered case we have $\phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0$.

The dimensionless Rayleigh quotient for a vibrating beam is

$$\mathscr{R} = \frac{\int_0^1 \phi''^2\,\mathrm{d}x}{\int_0^1 \phi^2\,\mathrm{d}x}\,, \tag{1}$$

which provides an upper bound for β^4. The simplest admissible function for this problem is $\phi = x^2$, which yields from Rayleigh's quotient an approximate natural frequency of $\omega = \sqrt{20}\sqrt{EI/ml^4} = 4{\cdot}4721\ldots\sqrt{EI/ml^4}$, compared with the exact value of $\omega = 3{\cdot}51601526850015118\ldots\sqrt{EI/ml^4}$.

Now, let us consider adding a free parameter, so that $\phi = x^2 + ax^3$, for example. Rayleigh's quotient in this case turns out to be a rational function of a given by

$$\mathscr{R} = \frac{(4 + 12a + 12a^2)}{\left(\dfrac{1}{5} + \dfrac{a}{3} + \dfrac{a^2}{7}\right)}\,. \tag{2}$$

This function can be minimized with respect to a yielding

$$\min(\mathscr{R}) = 12(51 - 8\sqrt{39}), \tag{3}$$

providing the approximation $\omega = 3{\cdot}53273\ldots\sqrt{EI/ml^4}$, which is much closer to the exact solution.

The same sort of improvement can be found when the starting point is a comparison function. For example, consider the simplest possible polynomial comparison function with one free parameter:

$$\phi = \frac{x^2}{6}(6a + 20x - 14ax - 20x^2 + 11ax^2 + 6x^3 - 3ax^3). \tag{4}$$

which yields, for Rayleigh's quotient,

$$\mathscr{R} = 594\left(\frac{100 - 30a + 11a^2}{3260 + 326a + 23a^2}\right), \tag{5}$$

the minimum of which is

$$\min(\mathscr{R}) = \frac{30}{163}(1435 - 4\sqrt{116935}). \tag{6}$$

This leads to an approximate natural frequency of $\omega = 3{\cdot}516035\ldots\sqrt{EI/ml^4}$ in agreement to five places with the exact solution.

4.1.2. Beam simply supported at left end with a roller and discrete mass at the right end.

One of the challenges to using simple one-term approximations is what to do when, because of the boundary conditions, the unknown represented by the quotient itself (the natural frequency) appears in a comparison function. Consider a beam with simply supported boundary conditions at the left end such that $\phi(0) = \phi''(0) = 0$. At the right end, impose the condition that the slope remains zero while the part of the mechanism that imposes this conditions, which moves with the beam end, has mass μml. The boundary conditions at the right end are thus $\phi'(1) = \phi'''(1) + \mu\beta^4\phi(1) = 0$. For this problem, the Rayleigh quotient has the form

$$\mathscr{R} = \frac{\int_0^1 \phi''^2\,\mathrm{d}x}{\int_0^1 \phi^2\,\mathrm{d}x + \mu\phi^2(1)}, \tag{7}$$

which, again, provides an upper bound for β^4. Notice, however, that the value of β^4 appears in one of the boundary conditions, and will thus appear in any comparison function. Therefore, it would appear on both the left and right sides of the expression for Rayleigh's quotient (on the left side as part of the Rayleigh quotient itself), thus introducing an uncertainty into how to use it.

The simplest polynomial admissible function for this problem, satisfying only $\phi(0) = \phi'(1) = 0$, is $\phi = 2x - x^2$. This yields from Rayleigh's quotient an approximate natural frequency of $\omega = \sqrt{60/(8 + 15\mu)}\sqrt{EI/ml^4}$ which, for $\mu = 1$ yields $\omega = 1{\cdot}6151457\ldots\sqrt{EI/ml^4}$, compared with the exact value of $1{\cdot}4198994\ldots\sqrt{EI/ml^4}$. For $\mu = 10$ the frequency is $\omega = 0{\cdot}616236\ldots\sqrt{EI/ml^4}$ compared with the exact solution $\omega = 0{\cdot}534878\ldots\sqrt{EI/ml^4}$.

With a free parameter, the simplest admissible function (with the same boundary conditions satisfied as before) becomes $\phi = x(2 + 3a - x - ax^2)$ and results in a dimensionless Rayleigh quotient of

$$\mathscr{R} = 840\left(\frac{1 + 3a + 3a^2}{112 + 427a + 408a^2 + 210\mu + 840a\mu + 840a^2\mu}\right). \tag{8}$$

When minimized with respect to a, the numerical values for $\mu = 1$ and $\mu = 10$, respectively, yield approximate frequencies $\omega = 1{\cdot}42050\ldots\sqrt{EI/ml^4}$ and $\omega = 0{\cdot}5348827\ldots\sqrt{EI/ml^4}$. These agree with the exact solution to within four to five places.

Still better results are obtained with the simplest polynomial admissible function that satisfies both geometric boundary conditions and the zero bending moment at $x = 0$, augmented with a free parameter. That function is $\phi = x/30(48 - a - 24x^2 + 3ax^2 + 6x^3 - 2ax^3)$. Rayleigh's quotient is

$$\mathscr{R} = 4536\left(\frac{384 - 6a + a^2}{285\,696 + 567\,000\mu - 3489a + 19a^2}\right). \tag{9}$$

Minimization with respect to a yields numerical values for ω which, for $\mu = 1$ and $\mu = 10$, respectively, are $\omega = 1{\cdot}41992098\ldots\sqrt{EI/ml^4}$ and $\omega = 0{\cdot}534878387\ldots\sqrt{EI/ml^4}$. These agree with the exact solution to six places—amazing accuracy for such a simple one-term approximation.

Note that this last result is identical to that which would be obtained were the simplest polynomial comparison function used, in which the frequency appears explicitly. This way, it also appears explicitly in the Rayleigh quotient; this last answer is obtained when the expression for Rayleigh's quotient (i.e., the right side) is minimized with respect to the frequency as a parameter (see the treatment of the rotating beam below for a similar result).

4.2. *The Stodola and Vianello method*

4.2.1. Hanging cord. First, let us consider the free vibration of a uniform, inextensible cord of length l, hanging vertically under the influence of its own weight. The governing equation and boundary conditions are derived in [7] as

$$[(1 - x)\phi']' + \omega^2\phi = 0, \tag{10}$$

where ϕ is the mode shape for the lateral deflection of the cord, $\omega^2 = \lambda^2 l/g$, where ω is the dimensionless natural frequency, λ is the dimensional natural frequency, and g is the acceleration of gravity. The geometric boundary condition is that $\phi(0) = 0$, and the natural boundary condition is simply that the solution is bounded at $x = 1$. The solution is found in reference [7] as

$$\phi = \mathrm{J}_0(2\omega\sqrt{1 - x}), \tag{11}$$

where J_0 is the Bessel function of the first kind of order zero. The exact value of the eigenvalue is found to be $\omega = \mathrm{J}_0^{-1}(0)/2 = 1{\cdot}202412\ldots$

We look for a one-term approximation of the mode shape, ϕ, such that $\phi(1) = 1$. The dimensionless Rayleigh's quotient for this problem is

$$\mathscr{R} = \frac{\int_0^1 (1 - x)\phi'^2\,\mathrm{d}x}{\int_0^1 \phi^2\,\mathrm{d}x}, \tag{12}$$

which serves as an upper bound for ω^2. The simplest admissible function is $\phi = x$, for which $\omega = \sqrt{\mathscr{R}} = \sqrt{\frac{3}{2}} = 1{\cdot}22474\ldots$

In accordance with the method of Stodola and Vianello, one sets

$$[(1-x)\phi']' = -\omega^2\phi \tag{13}$$

and integrates twice to obtain

$$\phi = \omega^2 \int_0^x \frac{1}{1-\xi} \int_\xi^1 \phi(\eta)\, \mathrm{d}\eta\, \mathrm{d}\xi, \tag{14}$$

Note that use of this equation in the iterative sense described above is equivalent to setting

$$[(1-x)\phi'_{i+1}]' = -\omega^2\phi_i, \tag{15}$$

subject to $\phi_{i+1}(0) = 0$ and $\phi_{i+1}(1) = 1$, where $\phi_1(x)$ is any admissible function. This is equivalent to finding the unknown static transverse deflection of the cord, ϕ_{i+1}, given a known static load $= -\omega^2\phi_i$. Using the simplest polynomial admissible function $\phi_1 = x$ on the right side of equation (15) yields a comparison function ϕ_2 which, when normalized (by dividing by $\frac{3}{4}$) so that $\phi_2(1) = 1$, is

$$\phi_2 = \frac{x}{3}(2+x). \tag{16}$$

As can easily be shown, this comparison function yields $\omega = \sqrt{\mathscr{R}} = \sqrt{\frac{55}{38}} = 1{\cdot}20307\ldots$, which is an upper bound to the eigenvalue. Alternatively, one may obtain a lower bound by taking the inverse of the square root of the normalization factor at each step. For this step, the lower bound is $2\sqrt{3}/3 = 1{\cdot}1547\ldots$

Using ϕ_2 on the right side of equation (15) yields a new comparison function ϕ_3 which, when normalized so that $\phi_3(1) = 1$, is

$$\phi_3 = \frac{x}{19}(12 + 6x + x^2). \tag{17}$$

For ϕ_3, $\omega = \sqrt{\mathscr{R}} = \sqrt{3311/2290} = 1{\cdot}20244\ldots$ The lower bound obtained is $\sqrt{\frac{27}{19}} = 1{\cdot}192079\ldots$ It is obvious that both upper and lower bounds are converging to the exact solution, but the upper bound is more accurate. A plot of the relative error for both bounds is shown in Figure 1 indicating that both approximately follow a straight line on a semi-log plot. Thus, the relative error for the Rayleigh quotient associated with ϕ_i is quite

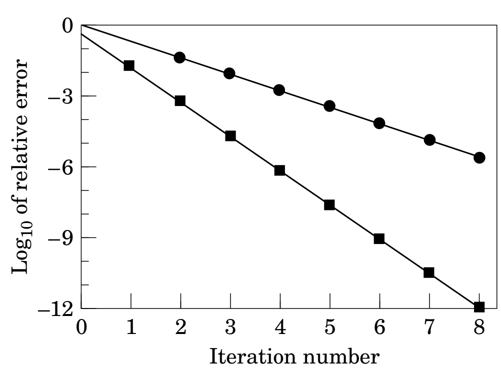

Figure 1. Log_{10} of the relative error of the upper and lower bounds versus iteration number for hanging cord problem. ■, Upper bound; ●, lower bound.

close to $10^{-(0\cdot33+1\cdot45i)}$, while the lower bound relative error is near $10^{0\cdot057-0\cdot71i}$, the straight lines shown in Figure 1. Note that for the eighth iteration the upper bound provides 12-place accuracy! One can make the relative error as small as desired by executing an appropriate number of iterations. It is emphasized here, however, that these are all one-term approximations, all polynomials, and quite easy to obtain using *Mathematica* or other symbolic manipulation software. (As upper bounds are generally more accurate, only they will be shown for subsequent results.)

4.2.2. Cantilevered beam. Similarly, one can take the formula for a freely vibrating cantilevered beam and construct a scheme to improve the comparison functions iteratively. Recall from above that the ODE $\phi'''' = \beta^4\phi$ describes the mode shape of a vibrating beam. For the cantilever case we have $\phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0$. The analog of equation (15) is then

$$\phi''''_{i+1} = \beta^4\phi_i, \tag{18}$$

where $\phi_{i+1}(1) = 1$ is imposed. Note that the imposition of $\phi_{i+1}(1) = 1$ has the effect of removing the constant β from the approximate mode shapes. As with the cord, the computational effort with each iteration of the method is equivalent to solving for the static deflection under a known load.

The dimensionless Rayleigh's quotient for a vibrating beam is

$$\mathscr{R} = \frac{\int_0^1 \phi''^2\,\mathrm{d}x}{\int_0^1 \phi^2\,\mathrm{d}x}, \tag{19}$$

which provides an upper bound for β^4. The simplest admissible function is $\phi = x^2$, which yields from Rayleigh's quotient an approximate natural frequency of $\omega = \sqrt{20}\sqrt{EI/ml^4} = 4\cdot4721\ldots\sqrt{EI/ml^4}$, compared with the exact value of $3\cdot51601526\ldots\sqrt{EI/ml^4}$.

Now, substituting $\phi_1 = x^2$ into equation (18) and imposing the boundary conditions and $\phi_2(1) = 1$, one finds that

$$\phi_2 = \frac{x^2}{26}(45 - 20x + x^4). \tag{20}$$

This comparison function yields a Rayleigh quotient approximation of the natural frequency of $\omega = \sqrt{47\,320/3827}\sqrt{EI/ml^4} = 3\cdot516358\ldots\sqrt{EI/ml^4}$, which is much closer to the exact solution.

Now, substituting ϕ_2 into equation (18) and imposing the boundary conditions and $\phi_3(1) = 1$, one finds that

$$\phi_3 = \frac{x^2}{10\,576}(18\,585 - 8520x + 630x^4 - 120x^5 + x^8), \tag{21}$$

which yields a Rayleigh quotient approximation of $\omega = \sqrt{806\,181\,180/65\,212\,537}\sqrt{EI/ml^4} = 3\cdot51601548\ldots\sqrt{EI/ml^4}$, which is in agreement with the exact solution to seven places! A plot of the relative error versus iteration number for the cantilever beam is shown in Figure 2. The relative error lies very close to the straight line on the semi-log plot, given by $10^{2\cdot57-3\cdot24i}$. In other words, approximately 13-place accuracy is found on the fifth iteration! As before, these are all one-term approximations, all polynomials, and all quite easy to obtain using symbolic manipulation software.

4.2.3. Cantilevered beam with discontinuous properties. Consider a cantilevered beam with bending stiffness $EI = EI_1$ and mass per unit length $m = m_1$ in the segment $0 \leqslant x \leqslant x^*$

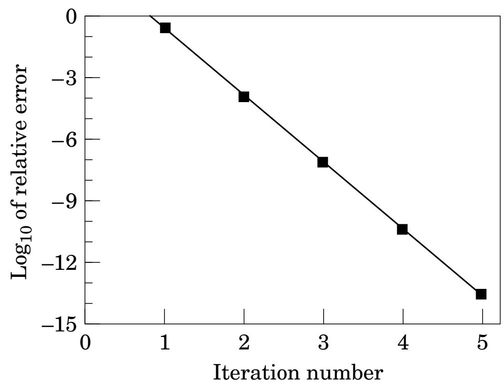

Figure 2. Log_{10} of the relative error of the upper bound versus the iteration number for the cantilevered beam problem.

and bending stiffness $EI = EI_2$ and mass per unit length $m = m_2$ in the segment $x^* < x \leqslant 1$, where x is the length co-ordinate normalized by the total length l of the beam. For simple harmonic motion, the equation of motion reduces to the ODE governing the mode shape ϕ, given by

$$(EI\phi'')'' = m\omega^2 l^4 \phi. \tag{22}$$

Rayleigh's quotient for a vibrating discontinuous beam with properties as described above is

$$\mathscr{R} = \frac{\int_0^1 EI\phi''^2 \,\mathrm{d}x}{l^4 \int_0^1 m\phi^2 \,\mathrm{d}x}. \tag{23}$$

For the case $EI_2 = 2EI_1$, $m_2 = 2m_1$, and $x^* = \frac{1}{2}$, and for the simplest admissible function $\phi = x^2$, one obtains an approximate frequency of free vibration of $\omega = \sqrt{\mathscr{R}} = 8\sqrt{5/21}\sqrt{EI_1/m_1 l^4} = 3{\cdot}9036\ldots\sqrt{EI_1/m_1 l^4}$, which is in error by over 50% relative to the exact solution of $\omega = 2{\cdot}55234722\ldots\sqrt{EI_1/m_1 l^4}$.

Equation (22) can be set up as an iterative scheme:

$$(EI\phi''_{i+1})'' = m\omega^2 l^4 \phi_i, \tag{24}$$

where the simplest admissible function $\phi_1 = x^2$ can be used to start the procedure. One must impose the same boundary conditions as given above for the uniform cantilevered beam plus continuity conditions on displacement, slope, bending moment and shear force at the discontinuity point $x = x^*$. In principle, this procedure could be carried out for any number of segments, since each iteration is no more difficult than solving a static equilibrium deflection for a given load. Again, for the case $EI_2 = 2EI_1$, $m_2 = 2m_1$, and $x^* = \frac{1}{2}$, one obtains

$$\phi_2 = \frac{4x^2}{3129}(1395 - 600x + 16x^4), \quad x \leqslant x^*; \tag{25}$$

$$\phi_2 = \frac{1}{3129}(-395 + 1860x + 2880x^2 - 1280x^3 + 64x^6), \quad x > x^*. \tag{26}$$

The Rayleigh quotient for a vibrating discontinuous beam with properties as described above yields an approximate frequency of free vibration of $\omega = \sqrt{\mathscr{R}} = 8\sqrt{90\,436\,710/888\,361\,469}\sqrt{EI_1/m_1 l^4} = 2{\cdot}552510\ldots\sqrt{EI_1/m_1 l^4}$, which agrees to four places with the exact solution.

The comparison function ϕ_2 can now be used to continue the procedure. For the same case, $EI_2 = 2EI_1$, $m_2 = 2m_1$, and $x^* = \frac{1}{2}$, one obtains

$$\phi_3 = \frac{4x^2}{38\,677\,129}(17\,447\,535 - 7\,705\,200x + 312\,480x^4 - 57\,600x^5 + 256x^8), \quad x \leqslant x^*, \tag{27}$$

$$\phi_3 = \frac{1}{38\,677\,129}(-4\,839\,315 + 22\,919\,100x + 36\,066\,240x^2 - 15\,914\,880x^3$$
$$-1\,327\,200x^4 + 1\,249\,920x^5 + 645\,120x^6 - 122\,880x^7 + 1024x^{10}), \quad x > x^*. \tag{28}$$

Rayleigh's quotient for ϕ_3 yields an approximate frequency of free vibration of $\omega = \sqrt{\mathscr{R}} = 24\sqrt{41\,127\,420\,573\,060\,870/3\,636\,423\,294\,69\,684\,203}\sqrt{EI_1/m_1l^4} = 2{\cdot}55234726\ldots\sqrt{EI_1/m_1l^4}$, which agrees with the exact solution to about eight places. The same trend as before, approximately linear on a semi-log plot, is thus exhibited.

4.2.4. Continuous beam with simple supports at the left end and mid-span. The ODE that governs the mode shape and the expression for Rayleigh's quotient are the same as those for the cantilevered beam. The boundary conditions are $\phi(0) = \phi''(0) = \phi''(1) = \phi'''(1) = 0$. In addition to this, the displacement vanishes at $x = \frac{1}{2}$, resulting in $\phi(\frac{1}{2}) = 0$. Finally, slope and bending moment are continuous at $x = \frac{1}{2}$. The first approximation, using the simplest admissible function $\phi = x(\frac{1}{2} - x)$ is off by 20%, but with the next iteration one obtains a relative error nearly five orders of magnitude lower! Solution by the Stodola–Vianello method yields excellent convergence to the exact solution of $\omega = 9{\cdot}07112547\ldots\sqrt{EI/ml^4}$. The third iteration gives eight-place accuracy, while the fifth iteration yields 14-place accuracy! The convergence trend is shown in Figure 3 and lies fairly close to the straight-line approximation $10^{2\cdot17 - 3\cdot24i}$, also shown.

4.2.5. Rotating cantilevered beam. Accurate estimates of the natural frequencies of rotating beams can be challenging to obtain, especially for rapidly spinning beams [8]. The main obstacle involves rapidly varying displacement near the root of the beam. It is expected that polynomial approximations will have difficulty. It is interesting to see how well the present one-term approximation can do. The ODE for the mode shape is

$$(EI\phi'')'' - (T\phi')' - m\omega^2\phi = 0, \tag{29}$$

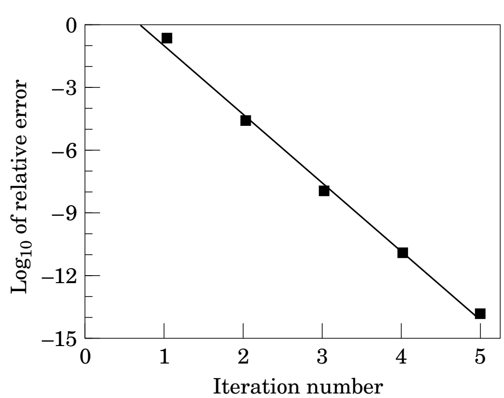

Figure 3. Log_{10} of the relative error of the upper bound versus the iteration number for the continuous beam problem.

where $T = \Omega^2 l^2 \int_x^1 m\xi \, \mathrm{d}\eta$, Ω is the angular speed of the frame to which the beam is attached at its root, and ω is the natural frequency of free vibration. For a uniform beam this reduces to $T = m\Omega^2 l^2(1 - x^2)/2$. Rayleigh's quotient for such a beam is given by

$$\mathscr{R} = \frac{\int_0^1 (EI\phi''^2 + T\phi'^2)\,\mathrm{d}x}{l^4 \int_0^1 m\phi^2\,\mathrm{d}x}\,. \tag{30}$$

Using the simplest possible admissible function, $\phi = x^2$, one obtains $\omega = \sqrt{\mathscr{R}} = \sqrt{20}\sqrt{EI/ml^4 + 4\Omega^2/3}$ as the natural frequency from Rayleigh's quotient. The first term is the natural frequency of the same beam, but non-rotating; and the second term reflects the effect of rotation. The exact solution, found in reference [9], is a complicated function of $m\Omega^2 l^4/EI$. Naturally, it is the same as for the non-rotating cantilevered beam for $\Omega = 0$, given by $\omega = 3{\cdot}516015268\ldots\sqrt{EI/ml^4}$. For $m\Omega^2 l^4/EI = 100$ the result is $\omega = 11{\cdot}2023\ldots\sqrt{EI/ml^4}$.

It is now possible to use

$$(EI\phi''_{i+1})'' = (T\phi'_i)' + m\omega^2\phi_i \tag{31}$$

as an iterative procedure, with $\phi_1 = x^2$, to obtain a comparison function and a more accurate approximation for the natural frequency:

$$\phi_2 = \frac{x^2}{26\omega^2 - 33\Omega^2}(45\omega^2 - 45\Omega^2 - 20\omega^2 x + 15\Omega^2 x^2 + \omega^2 x^4 - 3\Omega^2 x^4). \tag{32}$$

Unfortunately, one does not know ω; that is to be found from the Rayleigh quotient. There are at least three ways in which one could calculate the natural frequency at this point: (1) find the value of ω^2 that minimizes the Rayleigh quotient; (2) set the Rayleigh quotient equal to ω^2 and solve the resulting equation for ω^2; (3) substitute the value from (1) back into the Rayleigh quotient to obtain the minimum value of the Rayleigh quotient.

The first method does not actually use the minimum value of the Rayleigh quotient; as expected, its results are very poor, and it is clear that one should never view the value of ω^2 that minimizes the quotient as a good approximation of the frequency. The second method yields much better results than the first, but results from both the first and second methods are inferior to those of the third. Note that the third method is equivalent to substituting $\omega^2 = a\Omega^2$ and minimizing the Rayleigh quotient with respect to a. Note that here, however, unlike the situation above with the pinned roller – discrete mass boundary condition, the free parameter is a natural consequence of applying the Stodola–Vianello method. Although easily obtained, the result is a complicated function of Ω. The approximate natural frequency agrees with the exact solution to about six places for $\Omega = 0$ and deteriorates as Ω increases; the agreement is about three places at $m\Omega^2 l^4/EI = 100$. Additional iterations can be carried out, but are not reported here. The deterioration in accuracy with increasing Ω is expected [8], but engineering accuracy is still obtained.

5. CONCLUDING REMARKS

Two methods for obtaining excellent results based on one-term approximations via Rayleigh's quotient are described, along with their applications. A combination of these methods can be used to obtain accurate results for a wide class of problems as well. After a few iterations, depending on the starting function, the Stodola–Vianello method sometimes leads to overly complicated formulae, so that symbolic manipulation software may become bogged down. At this point, the addition of a free parameter may be helpful to provide a more accurate answer. Similarly, an existing admissible or comparison function with a free parameter can often be easily improved by using one iteration of the

Stodola–Vianello method. Recall that the usual application of the Stodola–Vianello method gives a lower bound of the eigenvalue at every iteration. However, as shown herein, use of the resulting improved functions in Rayleigh's quotient gives an upper bound on the frequency with an accuracy better than that of the lower bound obtained directly by the method itself. Finally, sometimes the unknown eigenvalue shows up in the approximate mode shape; results obtained herein indicate that it needs to be considered as a free parameter in order to obtain the most accurate results. Since both of these methods, and their combination, are straightforward to apply, authors of textbook descriptions of Rayleigh's quotient are urged to include a treatment of them.

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