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# Asymptotic construction of Reissner-like composite plate theory with accurate strain recovery ☆

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**Abstract**

The focus of this paper is to develop an asymptotically correct theory for composite laminated plates when each lamina exhibits monoclinic material symmetry. The development starts with formulation of the three-dimensional (3-D), anisotropic elasticity problem in which the deformation of the reference surface is expressed in terms of intrinsic two-dimensional (2-D) variables. The variational asymptotic method is then used to rigorously split this 3-D problem into a linear one-dimensional normal-line analysis and a nonlinear 2-D plate analysis accounting for classical as well as transverse shear deformation. The normal-line analysis provides a constitutive law between the generalized, 2-D strains and stress resultants as well as recovering relations to approximately but accurately express the 3-D displacement, strain and stress fields in terms of plate variables calculated in the plate analysis. It is known that more than one theory may exist that is asymptotically correct to a given order. This nonuniqueness is used to cast a strain energy functional that is asymptotically correct through the second order into a simple "Reissner-like" plate theory. Although it is not possible in general to construct an asymptotically correct Reissner-like composite plate theory, an optimization procedure is used to drive the present theory as close to being asymptotically correct as possible while maintaining the beauty of the Reissner-like formulation. Numerical results are presented to compare with the exact solution as well as a previous similar yet very different theory. The present theory has excellent agreement with the previous theory and exact results.


*Keywords:* Composite plates; Strain recovery; Stress recovery; Reissner theory; Asymptotic methods

## 1. Introduction

Although composite materials have found increasing applications in aerospace engineering due to their superior engineering properties and enhanced manufacturing technology, their application is not so extensive as one could expect. One reason is that treatment of composite materials greatly complicates structural analysis. The old tools used for design of structures made of isotropic materials are no longer

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suitable for analyzing composite structures. Although many new models have appeared in the literature, design engineers have been reluctant to accept them with confidence. This is partly because many new models are constructed for specific problems without generalization in mind and partly because some models are too complicated and computationally inefficient to be used for design purposes. Simple yet efficient and generalized methods of analysis are still needed to shorten the design period and reduce the cost of composite structures.

Many engineering structures made with composite materials have one dimension much smaller than the other two and can be modeled as plates. Plate models are generally derived from three-dimensional (3-D) elasticity theory, making use of the fact that the plate is thin in some sense. The simplest composite plate theory is the classical lamination theory, which is based on the Kirchhoff hypothesis. It is well known, however, that composite plates do not have to be very thick in order for this theory to yield extremely poor results compared to the actual 3-D solution.

Although it is plausible to take into account the smallness of the thickness of such structures, construction of an accurate two-dimensional (2-D) model for a 3-D body still introduces a lot of challenges. There have been many attempts to rationally improve upon the classical model, almost all of which have serious shortcomings. One can appreciate this by reading several recent review papers, such as Noor and Burton (1989); Noor and Burton (1990) and Noor and Malik (2000). Most of the models that have appeared in the literature, for example Reddy (1984); Touratier (1991); DiSciuva (1985) and Cho and Averill (2000), are based on ad hoc kinematic assumptions that cannot be reasonably justified for composite structures, such as an a priori distribution of displacement through the thickness.

From a mathematical point of view, the approximation in the process of constructing a plate theory stems from elimination of the thickness coordinate from the independent variables of the governing partial differential equations of equilibrium, a dimensional reduction process. This sort of approximation is inevitable if one wants to take advantage of the smallness of the thickness to simplify the analysis. However, other approximations that are not absolutely necessary should be avoided. For example, for small-strain analysis of plates and shells, it is reasonable to assume that the thickness, $h$, is small compared to the wavelength of deformation of the reference surface, $l$. However, it is not at all reasonable to assume a priori some ad hoc displacement field, although that is the way most plate theories are constructed.

In this paper we will proceed in a very different manner. We first cast the original 3-D elasticity problem in a form that introduces intrinsic variables for the plate. This can be done in such a way as to be applicable for arbitrarily large displacement and global rotation, subject only to the strain being small (see Danielson (1991) and Hodges et al. (1993)). Then, a systematic approach can be employed to reduce the dimensionality in terms of the smallness of $h/l$. The present work uses the variational asymptotic method (VAM) introduced by Berdichevsky (1979), to split the original nonlinear 3-D elasticity problem into a linear, one-dimensional (1-D), normal-line analysis and a nonlinear, 2-D, plate analysis. The normal-line analysis produces the 2-D constitutive law to be used in the 2-D plate analysis, along with recovering relations that yield the 3-D displacement, strain and stress fields using results obtained from the solution of the 2-D problem. The work is an extension of Sutyrin (1997) which was restricted to linear theory and which required the use of computerized symbolic manipulation software such as Mathematica™ to obtain results. The present version, on the other hand, is intended for eventual use in finite element analyses.

## 2. Three-dimensional formulation

A point in the plate can be described by its Cartesian coordinates $x_i$ (see Fig. 1), where $x_\alpha$ are two orthogonal lines in the reference plane and $x_3$ is the normal coordinate. (Here and throughout the paper, Greek indices assume values 1 and 2 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated.) Letting $\mathbf{b}_i$ denote the unit vector along $x_i$ for the un-

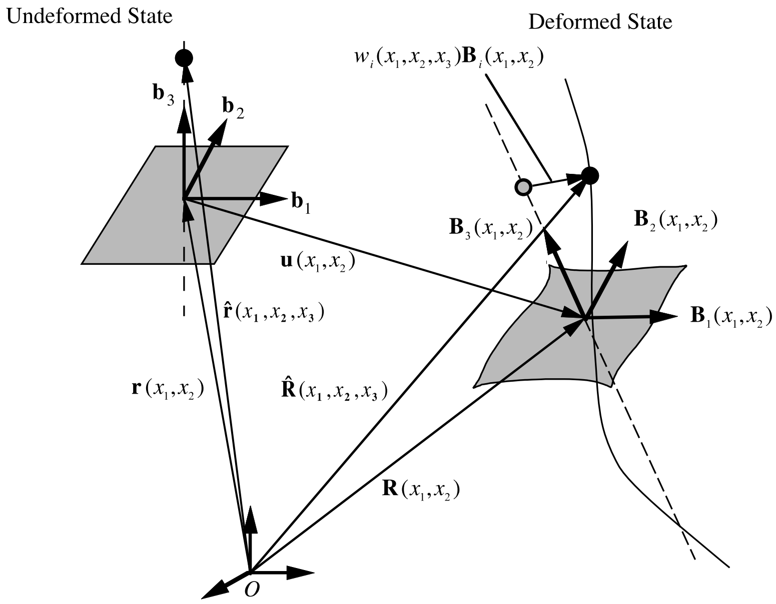


Fig. 1. Schematic of plate deformation.

deformed plate, one can then describe the position of any material point in the undeformed configuration by its position vector $\hat{\mathbf{r}}$ from a fixed point O, such that

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1, x_2) + x_3\mathbf{b}_3 \tag{1}$$

where $\mathbf{r}$ is the position vector from O to the point located by $x_\alpha$ on the reference surface. When the reference surface of the undeformed plate coincides with its middle surface, it naturally follows that

$$\langle\hat{\mathbf{r}}(x_1, x_2, x_3)\rangle = \mathbf{r}(x_1, x_2) \tag{2}$$

where the angle-brackets denote the definite integral through the thickness of the plate and will be used throughout the rest of the development.

When the plate deforms, the particle that had position vector $\hat{\mathbf{r}}$ in the undeformed state now has position vector $\widehat{\mathbf{R}}$ in the deformed plate. The latter can be uniquely determined by the deformation of the 3-D body. Similarly, another triad $\mathbf{B}_i$ is introduced for the deformed configuration. Note that the $\mathbf{B}_i$ unit vectors are not necessarily tangent to the coordinates of the deformed plate. The relation between $\mathbf{B}_i$ and $\mathbf{b}_i$ can be specified by an arbitrarily large rotation specified in terms of the matrix of direction cosines $C(x_1, x_2)$ so that

$$\mathbf{B}_i = C_{ij}\mathbf{b}_j \qquad C_{ij} = \mathbf{B}_i \cdot \mathbf{b}_j \tag{3}$$

subject to the requirement that $\mathbf{B}_i$ is coincident with $\mathbf{b}_i$ when the structure is undeformed. Now the position vector $\widehat{\mathbf{R}}$ can be represented as

$$\widehat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{R}(x_1, x_2) + x_3\mathbf{B}_3(x_1, x_2) + w_i(x_1, x_2, x_3)\mathbf{B}_i(x_1, x_2) \tag{4}$$

where $w_i$ is the warping of the normal-line element. In the present work, the form of the warping $w_i$ is not assumed, as in most plate/shell theories. Rather, these quantities are treated as unknown 3-D functions and will be solved for later. Eq. (4) is six times redundant because of the way warping introduced. Six constraints are needed to make the formulation unique. The redundancy can be removed by choosing appropriate definitions of $\mathbf{R}$ and $\mathbf{B}_i$. One can define $\mathbf{R}$ similarly as Eq. (2) to be the average position through the thickness, from which it follows that the warping functions must satisfy the three constraints

$$\langle w_i(x_1, x_2, x_3)\rangle = 0 \tag{5}$$

Another two constraints can be specified by taking $\mathbf{B}_3$ as the normal to the reference surface of the deformed plate. It should be noted that this choice has nothing to do with the famous Kirchhoff hypothesis. Indeed, it is only for convenience in the derivation. In the Kirchhoff assumption, no local deformation of the transverse normal is allowed. On the other hand, according to the present scheme we allow all possible deformation, classifying all deformation other than that of classical plate theory as warping, which is assumed to be small. This assumption is valid if the strain is small and if the order of the local rotation (i.e. the rotation of the normal line due to warping) is of the order of the strain or smaller; see Danielson (1991).

Based on the concept of decomposition of rotation tensor, Danielson and Hodges (1987) and Danielson (1991), the Jauman–Biot–Cauchy strain components for small local rotation are given by

$$\Gamma_{ij} = \tfrac{1}{2}(F_{ij} + F_{ji}) - \delta_{ij} \tag{6}$$

where $F_{ij}$ is the mixed-basis component of the deformation gradient tensor such that

$$F_{ij} = \mathbf{B}_i \cdot \mathbf{G}_k \mathbf{g}^k \cdot \mathbf{b}_j \tag{7}$$

Here $\mathbf{G}_i = \partial\widehat{\mathbf{R}}/\partial x_i$ is the covariant basis vector of the deformed configuration and $\mathbf{g}^k$ the contravariant base vector of the undeformed configuration and $\mathbf{g}^k = \mathbf{g}_k = \mathbf{b}_k$. One can obtain $\mathbf{G}_k$ with the help of the definition of so-called generalized 2-D strains, given by

$$\mathbf{R}_{,\alpha} = \mathbf{B}_\alpha + \varepsilon_{\alpha\beta}\mathbf{B}_\beta \tag{8}$$

and

$$\mathbf{B}_{i,\alpha} = (-K_{\alpha\beta}\mathbf{B}_\beta \times \mathbf{B}_3 + K_{\alpha 3}\mathbf{B}_3) \times \mathbf{B}_i \tag{9}$$

where $\varepsilon_{\alpha\beta}$ and $K_{\alpha\beta}$ are the 2-D generalized strains and $()_{,\alpha} = \partial()/\partial x_\alpha$. Here one is free to set $\varepsilon_{12} = \varepsilon_{21}$, i.e.

$$\mathbf{B}_1 \cdot \mathbf{R}_{,2} = \mathbf{B}_2 \cdot \mathbf{R}_{,1} \tag{10}$$

which can serve as another constraint to specify the deformed configuration.

With the assumption that the strain is small compared to unity, which has the effect of removing all the terms that are products of the warping and the generalized strains, one can express the 3-D strain field as

$$\begin{aligned} \Gamma_e &= \epsilon + x_3\kappa + I_1 w_{\|,1} + I_2 w_{\|,2} \\ 2\Gamma_s &= w'_\| + e_1 w_{3,1} + e_2 w_{3,2} \\ \Gamma_t &= w'_3 \end{aligned} \tag{11}$$

where $()' = \partial()/\partial x_3$, $()_\| = \lfloor ()_1 \quad ()_2 \rfloor^{\mathrm{T}}$ and where

$$\Gamma_e = \lfloor \Gamma_{11} \quad 2\Gamma_{12} \quad \Gamma_{22} \rfloor^{\mathrm{T}} \quad 2\Gamma_s = \lfloor 2\Gamma_{13} \quad 2\Gamma_{23} \rfloor^{\mathrm{T}} \quad \Gamma_t = \Gamma_{33} \tag{12}$$

$$\epsilon = \lfloor \varepsilon_{11} \quad 2\varepsilon_{12} \quad \varepsilon_{22} \rfloor^{\mathrm{T}} \quad \kappa = \lfloor K_{11} \quad K_{12} + K_{21} \quad K_{22} \rfloor^{\mathrm{T}} \tag{13}$$

and

$$I_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad I_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad e_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad e_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{14}$$

Now, the strain energy of the plate per unit area (which is the same as the strain energy for the deformation of the normal-line element) can be written as

$$U = \frac{1}{2}\left\langle \begin{Bmatrix} \Gamma_e \\ 2\Gamma_s \\ \Gamma_t \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} D_e & D_{es} & D_{et} \\ D_{es}^T & D_s & D_{st} \\ D_{et}^T & D_{st}^T & D_t \end{bmatrix} \begin{Bmatrix} \Gamma_e \\ 2\Gamma_s \\ \Gamma_t \end{Bmatrix} \right\rangle \tag{15}$$

where $D_e$, $D_{es}$, $D_{et}$, $D_s$, $D_{st}$ and $D_t$ are the appropriate partition matrices of the original 3-D $6 \times 6$ material matrix. It is noted that the material matrix should be expressed in the global coordinates of the undeformed plate $x_i$. This matrix is in general fully populated. However, if it is desired to model laminated composite plates in which each lamina exhibits a monoclinic symmetry about its own mid-plane (for which the material matrix is determined by 13 constants instead of 21) and is rotated about the local normal to be a layer in the composite laminated plate, then $D_{es}$ and $D_{st}$ will always vanish no matter what the lay-up angle is. Considering this, we can simplify the strain energy expression to the form

$$2U = \left\langle \Gamma_e^T D_e \Gamma_e + 2\Gamma_e^T D_{et}\Gamma_t + 2\Gamma_s^T D_s 2\Gamma_s + \Gamma_t^T D_t \Gamma_t \right\rangle \tag{16}$$

To deal with applied loads, we will at first leave open the existence of a potential energy and develop instead the virtual work of the applied loads. The virtual displacement is taken as the Lagrangean variation of the displacement field, such that

$$\delta\widehat{\mathbf{R}} = \overline{\delta q}_{Bi}\mathbf{B}_i + x_3\overline{\delta\psi}_{Bi}\mathbf{B}_i \times \mathbf{B}_3 + \delta w_i \mathbf{B}_i + \overline{\delta\psi}_{Bi}\mathbf{B}_i \times w_j \mathbf{B}_j \tag{17}$$

where the virtual displacement of the reference surface is given by

$$\overline{\delta q}_{Bi} = \delta\mathbf{u} \cdot \mathbf{B}_i \tag{18}$$

and the virtual rotation of the reference surface is defined such that

$$\delta\mathbf{B}_i = (-\overline{\delta\psi}_{B\beta}\mathbf{B}_\beta \times \mathbf{B}_3 + \overline{\delta\psi}_{B3}\mathbf{B}_3) \times \mathbf{B}_i \tag{19}$$

Since the strain is small, one may safely ignore products of the warping and the loading in the virtual rotation term. Then, the work done through a virtual displacement due to the applied loads $\tau_i\mathbf{B}_i$ at the top surface and $\beta_i\mathbf{B}_i$ at the bottom surface and body force $\phi_i\mathbf{B}_i$ through the thickness is

$$\overline{\delta W} = (\tau_i + \beta_i + \langle\phi_i\rangle)\overline{\delta q}_{Bi} + \overline{\delta\psi}_{B\alpha}\left[\frac{h}{2}(\tau_\alpha - \beta_\alpha) + \langle x_3\phi_\alpha\rangle\right] + \delta\left(\tau_i w_i^+ + \beta_i w_i^- + \langle\phi_i w_i\rangle\right) \tag{20}$$

where $\tau_i$, $\beta_i$, and $\phi_i$ are taken to be independent of the deformation, $e_{ijk}$ is the permutation symbol, $()^+ = ()|_{x_3=h/2}$, and $()^- = ()|_{x_3=-(h/2)}$. By introducing column matrices $\overline{\delta q}$, $\overline{\delta\psi}$, $\tau$, $\beta$, and $\phi$, which are formed by stacking the three elements associated with indexed symbols of the same names, and using Eqs. (1), (3) and (4) one may write the virtual work in a matrix form, so that

$$\overline{\delta W} = \overline{\delta q}^T f + \overline{\delta\psi}^T m + \delta(\tau^T w^+ + \beta^T w^- + \langle\phi^T w\rangle) \tag{21}$$

where

$$f = \tau + \beta + \langle\phi\rangle$$
$$m = \begin{Bmatrix} \frac{h}{2}(\tau_1 - \beta_1) + \langle x_3\phi_1\rangle \\ \frac{h}{2}(\tau_2 - \beta_2) + \langle x_3\phi_2\rangle \\ 0 \end{Bmatrix} \tag{22}$$

The complete statement of the problem can now be presented in terms of the principle of virtual work, such that

$$\delta U - \overline{\delta W} = 0 \tag{23}$$

In spite of the possibility of accounting for nonconservative forces in the above, the problem that governs the warping is conservative. Thus, one can pose the problem that governs the warping as the minimization of a total potential functional

$$\Pi = U + V \tag{24}$$

so that

$$\delta\Pi = 0 \tag{25}$$

in which only the warping displacement is varied, subject to the constraints Eq. (5). This implies that

$$V = -\tau_{\|}^T w_{\|}^+ - \tau_3 w_3^+ - \beta_{\|}^T w_{\|}^- - \beta_3 w_3^- - \left\langle \phi_{\|}^T w_{\|} \right\rangle - \langle \phi_3 w_3 \rangle \tag{26}$$

Below, for simplicity of terminology, we will refer to $\Pi$ as the total potential energy, or the total energy.

Up to this point, this is simply an alternative formulation of the original 3-D elasticity problem. If we attempt to solve this problem directly, we will meet the same difficulty as solving any full 3-D elasticity problem. Fortunately, as shown below, the VAM can be used to calculate the 3-D warping functions asymptotically.

## 3. Dimensional reduction

Now, to rigorously reduce the original 3-D problem to a 2-D plate problem, one must attempt to reproduce the energy stored in the 3-D structure in a 2-D formulation. This dimensional reduction can only be done approximately, and one way to do it is by taking advantage of the smallness of $h/l$ where $h$ is the thickness of plate and $l$ is the characteristic wavelength of the deformation of the reference surface. As mentioned before, although the reduced models based on ad hoc kinematic assumptions regularly appear in the literature, there is no basis whatsoever to justify such assumptions. Rather, in this work, the VAM will be used to mathematically perform a dimensional reduction of the 3-D problem to a series of 2-D models. One can refer to Atilgan and Hodges (1992) for a brief introduction of the VAM. To proceed by this method, one has to assess and keep track of the order of all the quantities in the formulation. Following Sutyrin (1997), the quantities of interest have the following orders:

$$\epsilon_{\alpha\beta} \sim h\kappa_{\alpha\beta} \sim \varepsilon \quad f_3 \sim \mu(h/l)^2\varepsilon \quad f_\alpha \sim \mu(h/l)\varepsilon \quad m_\alpha \sim \mu h(h/l)\varepsilon \tag{27}$$

where $\varepsilon$ is the order of the maximum strain in the plate and $\mu$ is the order of the material constants (all of which are assumed to be of the same order). It is noted that $m_3 = 0$.

The VAM requires one to find the leading terms of the functional. The total potential energy consists of quadratic expressions involving the warping and the generalized strains. In addition there are terms that involve the loading along with interaction terms between the warping and both of the other types of quantities. Since only the warping is varied, one needs the leading terms that involve warping only and the leading terms that involve the warping and other quantities (i.e. the generalized strain and loading). For the zeroth-order approximation, these leading terms of Eq. (24) are

$$2\Pi_0^* = \left\langle w_{\|}'^T D_s w_{\|}' + D_t w_3'^2 + 2(\epsilon + x_3\kappa)^T D_{et} w_3' \right\rangle \tag{28}$$

The warping field that minimizes the energy expression of Eq. (28), subject to constraints Eq. (5), can be obtained by applying the usual procedure of the calculus of variations with the aid of Lagrange multipliers. The resulting warping is

$$w_{\|} = 0 \quad w_3 = D_{\perp 1}\epsilon + D_{\perp 2}\kappa = D_{\perp}\mathscr{E} \tag{29}$$

where $D_{\perp} = \lfloor D_{\perp 1} \quad D_{\perp 2} \rfloor$ and $\mathscr{E} = \lfloor \epsilon \quad \kappa \rfloor^{\mathrm{T}}$ and $D_{\perp\alpha}$ can be calculated from

$$D'_{\perp 1} = -D_{et}^T/D_t \quad D'_{\perp 2} = -x_3 D_{et}^T/D_t \quad \langle D_{\perp\alpha} \rangle = 0 \tag{30}$$

Note that inter-lamina continuity of $D_{\perp\alpha}$ must be maintained due to the continuity of warping functions to produce a continuous displacement field. This solution is the same as what is obtained in Atilgan and Hodges (1992). Substituting Eq. (29) back into the total energy Eq. (24), one can obtain the total potential energy asymptotically correct through the order of $\mu\varepsilon^2$ as

$$2\Pi_0 = \mathscr{E}^T A \mathscr{E} \tag{31}$$

where

$$A = \begin{bmatrix} \langle D_{\parallel} \rangle & \langle x_3 D_{\parallel} \rangle \\ \langle x_3 D_{\parallel} \rangle^T & \langle x_3^2 D_{\parallel} \rangle \end{bmatrix} \quad D_{\parallel} = D_e - D_{et} D_{et}^T / D_t \tag{32}$$

There are two observations to make here for the zeroth-order approximation: (1) Although the normal line of undeformed plate remains straight and normal to the deformed plate, it deforms in the normal direction in response to deformation involving $\epsilon$ and $\kappa$; (2) the total energy coincides with that of the classical laminated plate theory. Note that the transverse normal stress and shear stresses are zero, as expected. However, since the zeroth-order warping $w_3$ for each layer is a quadratic polynomial, the normal strain is not zero. To obtain this solution, no ad hoc assumptions, such as setting the transverse normal strain equal to zero, have been used.

We notice that the zeroth-order warping is of order $\varepsilon$. According to the VAM, to accept this as the zeroth-order approximation, one needs to check whether or not the order of the next approximation is higher than this one. To obtain the first-order approximation, we simply perturb the zeroth-order result, resulting in warping functions of the form

$$w_{\parallel} = v_{\parallel} \quad w_3 = v_3 + D_{\perp} \mathscr{E} \tag{33}$$

Substituting Eq. (33) back into Eq. (11) and then into Eq. (24), one can obtain the leading terms for the first-order approximation as

$$2\Pi_1^* = \left\langle v_{\parallel}'^T D_s v_{\parallel}' + D_t v_3'^2 + 2 v_{\parallel}^T C_\alpha' \mathscr{E}_{,\alpha} + 2 v_{\parallel}'^T D_s e_\alpha D_{\perp} \mathscr{E}_{,\alpha} - 2 v_{\parallel}^T \phi_{\parallel} \right\rangle - 2 v_{\parallel}^{+T} \tau_{\parallel} - 2 v_{\parallel}^{-T} \beta_{\parallel} \tag{34}$$

where $C_\alpha' = -I_\alpha^T \lfloor D_{\parallel} \; x_3 D_{\parallel} \rfloor$ and integration by parts with respect to the in-plane coordinates is used here and hereafter, whenever it is convenient for the derivation, since the goal is to obtain an interior solution for the plate without consideration of edge effects. One can observe that $v_3$ is decoupled from $v_{\parallel}$. Considering Eq. (5), $v_3$ only has a trivial solution. The Euler–Lagrange equations for the functional given in Eq. (34) are

$$(D_s v_{\parallel}' + D_s e_\alpha D_{\perp} \mathscr{E}_{,\alpha})' = C_\alpha' \mathscr{E}_{,\alpha} + g' + \Lambda \tag{35}$$

$$(D_s v_{\parallel}' + D_s e_\alpha D_{\perp} \mathscr{E}_{,\alpha})^+ = \tau_{\parallel} \tag{36}$$

$$(D_s v_{\parallel}' + D_s e_\alpha D_{\perp} \mathscr{E}_{,\alpha})^- = -\beta_{\parallel} \tag{37}$$

where $g' = -\phi_{\parallel}$ and $\Lambda$ is composed of Lagrange multipliers to enforce the constraints applied on the warping field, Eq. (5). Solving this set of equations, one obtains the following warping field

$$v_{\parallel} = \overline{C}_\alpha \mathscr{E}_{,\alpha} + \bar{g} \tag{38}$$

where

$$\overline{C}_\alpha' = D_s^{-1} C_\alpha^* \quad \langle \overline{C_\alpha} \rangle = 0 \quad C_\alpha^* = C_\alpha + \frac{x_3}{h} C_\alpha^{\mp} - \frac{1}{2} C_\alpha^{\pm} - D_s e_\alpha D_{\perp} \tag{39}$$

$$\overline{g}' = D_s^{-1} g^* \quad \langle \overline{g} \rangle = 0 \quad g^* = g + \frac{x_3}{h} g^{\mp} - \frac{1}{2} g^{\pm} + \left( \frac{x_3}{h} + \frac{1}{2} \right) \tau_{\|} + \left( \frac{x_3}{h} - \frac{1}{2} \right) \beta_{\|} \tag{40}$$

where the notation $()^{\pm} = ()^+ + ()^-$ and $()^{\mp} = ()^- - ()^+$. The warping of this approximation is of order $(h/l)\varepsilon$, which is indeed one order higher than the zeroth-order approximation. Now we are ready to obtain an expression for the total energy that is asymptotically correct through the order of $\mu(h/l)^2\varepsilon$, viz.,

$$2\Pi_1 = \mathscr{E}^T A \mathscr{E} + \mathscr{E}_{,1}^T B \mathscr{E}_{,1} + 2\mathscr{E}_{,1}^T C \mathscr{E}_{,2} + \mathscr{E}_{,2}^T D \mathscr{E}_{,2} + 2\mathscr{E}^T F + P \tag{41}$$

where

$$\begin{aligned}
B &= \left\langle D_{s_{11}} D_{\perp}^T D_{\perp} + \overline{C}_1^T C_1' + C_1^{*T} e_1 D_{\perp} \right\rangle \\
C &= \left\langle D_{s_{12}} D_{\perp}^T D_{\perp} + \frac{1}{2} \left( \overline{C}_1^T C_2' + C_1'^T \overline{C}_2 + C_1^{*T} e_2 D_{\perp} + D_{\perp}^T e_1^T C_2^* \right) \right\rangle \\
D &= \left\langle D_{s_{22}} D_{\perp}^T D_{\perp} + \overline{C}_2^T C_2' + C_2^{*T} e_2 D_{\perp} \right\rangle \\
-F &= \tau_3 D_{\perp}^{+T} + \beta_3 D_{\perp}^{-T} + \langle \phi_3 D_{\perp}^T \rangle + \frac{1}{2} \left( \left\langle D_{\perp}^T e_{\alpha}^T g_{,\alpha}^* + C_{\alpha}'^T \overline{g}_{,\alpha} - \overline{C}_{\alpha}^T \phi_{\|,\alpha} \right\rangle - \overline{C}_{\alpha}^{+T} \tau_{\|,\alpha} - \overline{C}_{\alpha}^{-T} \beta_{\|,\alpha} \right) \\
-P &= \overline{g}^{+T} \tau_{\|} + \overline{g}^{-T} \beta_{\|} + \langle \overline{g}^T \phi_{\|} \rangle
\end{aligned} \tag{42}$$

It is noted that $P$ is a quadratic term in the applied loads, and it cannot be varied in the 2-D model. When there is no load, this term will vanish. It does come from the applied load and the warping of refined approximations introduced by the applied load. Also the applied loads should not vary rapidly in the plane of the plate; otherwise, $F$ will not contain all the relevant terms to meet the requirement of asymptotical correctness.

## 4. Transforming into Reissner-like model

Although Eq. (41) is asymptotically correct through the second order and use of this strain energy expression is possible as mentioned in Sutyrin (1997), it involves more complicated boundary conditions than necessary since it contains derivatives of the generalized strain measures. To obtain an energy functional that is of practical use, one has to transform the present approximation into a Reissner-like model.

In a Reissner-like model, there are two additional degrees of freedom, which are the transverse shear strains. These are incorporated into the rotation of transverse normal. If we introduce another triad $\mathbf{B}_i^*$ for the deformed Reissner-like plate, the definition of 2-D strains becomes

$$\mathbf{R}_{,\alpha} = \mathbf{B}_{\alpha}^* + \varepsilon_{\alpha\beta}^* \mathbf{B}_{\beta}^* + 2\gamma_{\alpha 3} \mathbf{B}_3^* \tag{43}$$

$$\mathbf{B}_{i,\alpha}^* = (-K_{\alpha\beta}^* \mathbf{B}_{\beta}^* \times \mathbf{B}_3^* + K_{\alpha 3}^* \mathbf{B}_3^*) \times \mathbf{B}_i^* \tag{44}$$

where the transverse shear strains are $\gamma = \lfloor 2\gamma_{13} \quad 2\gamma_{23} \rfloor^{\mathrm{T}}$. Since $\mathbf{B}_i^*$ is uniquely determined by $\mathbf{B}_i$ and $\gamma$, one can derive the following kinematic identity between the strains measures $\mathscr{R}$ of Reissner-like plate and $\mathscr{E}$

$$\mathscr{E} = \mathscr{R} - \mathscr{D}_{\alpha} \gamma_{,\alpha} \tag{45}$$

where

$$
\begin{aligned}
\mathscr{D}_1 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}} \\
\mathscr{D}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \\
\mathscr{R} &= \lfloor \varepsilon_{11}^* \quad 2\varepsilon_{12}^* \quad \varepsilon_{22}^* \quad K_{11}^* \quad K_{12}^* + K_{21}^* \quad K_{22}^* \rfloor^{\mathrm{T}}
\end{aligned} \tag{46}
$$

Now one can express the strain energy, correct to second order, in terms of strains of the Reissner-like model as

$$2\Pi_1 = \mathscr{R}^T A \mathscr{R} - 2\mathscr{R}^T A \mathscr{D}_1 \gamma_{,1} - 2\mathscr{R}^T A \mathscr{D}_2 \gamma_{,2} + \mathscr{R}_{,1}^T B \mathscr{R}_{,1} + 2\mathscr{R}_{,1}^T C \mathscr{R}_{,2} + \mathscr{R}_{,2}^T D \mathscr{R}_{,2} + 2\mathscr{R}^T F + P \tag{47}$$

The generalized Reissner-like model is of the form

$$2\Pi_{\mathscr{R}} = \mathscr{R}^T A \mathscr{R} + \gamma^T G \gamma + 2R^T F_{\mathscr{R}} + 2\gamma^T F_\gamma \tag{48}$$

To find an equivalent Reissner-like model Eq. (48) for Eq. (47), one has to eliminate all partial derivatives of the strain. Here equilibrium equations are used to achieve this purpose. From the two equilibrium equations balancing bending moments with applied moments $m_\alpha$ which is calculated from Eq. (22), one can obtain the following formula

$$G\gamma + F_\gamma = \mathscr{D}_\alpha^T A \mathscr{R}_{,\alpha} + \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} \tag{49}$$

where the $F_{\mathscr{R},\alpha}$ terms are dropped because they are of high order. Substituting Eq. (49) into Eq. (47), one can show that $F_{\mathscr{R}} = F$ and $F_\gamma = 0$. Finally one can rewrite Eq. (47) as

$$2\Pi_1 = \mathscr{R}^T A \mathscr{R} + \gamma^T G \gamma + 2\mathscr{R}^T F + \overline{P} + U^* \tag{50}$$

where

$$U^* = \mathscr{R}_{,1}^T \overline{B} \mathscr{R}_{,1} + 2\mathscr{R}_{,1}^T \overline{C} \mathscr{R}_{,2} + \mathscr{R}_{,2}^T \overline{D} \mathscr{R}_{,2} \tag{51}$$

and

$$
\begin{aligned}
\overline{B} &= B + A\mathscr{D}_1 G^{-1} \mathscr{D}_1^T A \\
\overline{C} &= C + A\mathscr{D}_1 G^{-1} \mathscr{D}_2^T A \\
\overline{D} &= D + A\mathscr{D}_2 G^{-1} \mathscr{D}_2^T A \\
\overline{P} &= P - \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix}^{\mathrm{T}} G^{-1} \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix}
\end{aligned} \tag{52}
$$

If we can drive $U^*$ to be zero for any $\mathscr{R}$, then we have found an asymptotically correct Reissner-like plate model. For general anisotropic plates, this term will not be zero; but we can minimize the error to obtain a Reissner-like model that is as close to asymptotical correctness as possible. The accuracy of the Reissner-like model depends on how close to zero one can drive this term of the energy.

One could proceed with the optimization at this point, but the problem will lead to a least squares solution for 3 unknowns (the shear stiffness matrix $G$) from a linear system of 78 equations ($12 \times 12$ and symmetric). This optimization problem is too rigid. The solution will be better if we can bring more unknowns into the problem. As stated in Sutyrin and Hodges (1996), there is no unique plate theory of a given order. One can relax the constraints in Eq. (5) to be $\langle w_i \rangle = \text{const.}$, and still obtain an asymptotically correct strain energy. Since the zeroth-order approximation gives us an asymptotic model corresponding to classical plate theory, we only relax the constraints for the first-order approximation. This relaxation will modify the warping field to be

$$\overline{v}_{\|} = \overline{C}_\alpha \mathscr{E}_{,\alpha} + \overline{g} + L_\alpha \mathscr{E}_{,\alpha} \tag{53}$$

where $L_\alpha$ consists of 24 constants. The remaining energy $U^*$ will also be modified to be

$$U^* = \mathscr{R}^T_{,1}\widehat{B}\mathscr{R}_{,1} + 2\mathscr{R}^T_{,1}\widehat{C}\mathscr{R}_{,2} + \mathscr{R}^T_{,2}\widehat{D}\mathscr{R}_{,2} \tag{54}$$

and

$$\widehat{B} = \overline{B} + 2L_1^T\langle C_1'\rangle \quad \widehat{C} = \overline{C} + (L_1^T\langle C_2'\rangle + \langle C_1'\rangle^T L_2) \quad \widehat{D} = \overline{D} + 2L_2^T\langle C_2'\rangle \tag{55}$$

Since now we have 27 unknowns, the optimization is much more flexible. It can give us a more optimal solution for the shear stiffness matrix $G$ to fit the second-order, asymptotically-correct energy into a Reissner-like model. In other words, here we have found the Reissner-like model that describes as closely as possible the 2-D energy that is asymptotically correct through the second order in $h/l$. However, the asymptotical correctness of the warping field can only be ascertained after obtaining another higher-order approximation, which is discussed in the next section.

After minimizing $U^*$, the total energy to be used in a 2-D Reissner-like plate solver can be expressed as:

$$2\Pi_1 = \mathscr{R}^T A \mathscr{R} + \gamma^T G \gamma + 2\mathscr{R}^T F \tag{56}$$

where from Eq. (50) the quadratic loads term $\overline{P}$ is dropped because it will not enter the 2-D governing equations. Note that the load-related terms in $F$ are a new feature in the present development. Because of them, one has to modify traditional Reissner-like plate solvers to accommodate for these terms. This modification is not difficult and has a form similar to terms that must be included when considering thermal effects or actuated materials.

## 5. Recovering relations

From the above, we have obtained a Reissner-like plate model which is as close as possible to being asymptotically correct in the sense of matching the total potential energy. The stiffness matrices obtained ($A$, $G$ and the load related term $F$) can be used as input for a plate theory derived from the total energy obtained here. The geometrically nonlinear theory presented in Hodges et al. (1993) is an appropriate choice with some trivial modifications to the loading terms.

In many applications, however, while it is necessary to accurately calculate the 2-D displacement field of composite plates, this is not sufficient. Ultimately, the fidelity of a reduced-order model such as this depends on how well it can predict the 3-D results in the original 3-D structure. Hence recovering relations should be provided to complete the reduced-order model and the results then compared with those of the original 3-D model. By recovering relations, then, we mean expressions for 3-D displacement, strain, and stress fields in terms of 2-D quantities and $x_3$.

For a strain energy that is asymptotically correct through the second order, we can recover the 3-D displacement, strain and stress fields only through the first order in a strict sense of asymptotical correctness. Using Eqs. (1), (3) and (4), one can recover the 3-D displacement field through the first order as

$$\begin{aligned} U_1 &= u_1 + x_3 C_{31} + \overline{v}_1 \\ U_2 &= u_2 + x_3 C_{32} + \overline{v}_2 \\ U_3 &= u_3 + w_3 \end{aligned} \tag{57}$$

where $U_i$ are the 3-D displacement components and $u_i$ are the plate displacements. And from Eq. (11), one can recover the 3-D strain field through the first order as

$$\begin{aligned}\Gamma_e &= \epsilon + x_3\kappa \\ 2\Gamma_s &= \overline{v}_{\|}' + e_1(D_{\perp 1}\epsilon_{,1} + D_{\perp 2}\kappa_{,1}) + e_2(D_{\perp 1}\epsilon_{,2} + D_{\perp 2}\kappa_{,2}) \\ \Gamma_t &= D_{\perp 1}'\epsilon + D_{\perp 2}'\kappa\end{aligned} \tag{58}$$

Then, one can use the 3-D constitutive law to obtain 3-D stresses $\sigma_{ij}$.

Since we have obtained an optimum shear stiffness matrix $G$, some of the recovered 3-D results through the first order are better than classical theory and conventional first-order deformation theory. However, for the transverse normal component of strain and stress (i.e. $\Gamma_{33}$ and $\sigma_{33}$), the agreement is not satisfactory at all. Let us recall, that the Reissner-like theory that has been constructed only ensures a good fit with the asymptotically correct 3-D displacement, strain and stress field of the first order (while energy is approximated to the second order). Thus, in order to obtain recovering relations that are valid to the same order as the energy, the VAM iteration needs to be applied one more time.

Using the same procedure listed in previous section, the second-order warping can be obtained and expressed symbolically as

$$\begin{aligned}y_{\|} &= -L_\alpha \mathscr{E}_{,\alpha} \\ y_3 &= S_{11}\mathscr{E}_{,11} + S_{12}\mathscr{E}_{,12} + S_{22}\mathscr{E}_{,22}\end{aligned} \tag{59}$$

Eq. (59) is obtained by assuming $\overline{v}_{\|}$ to be the result of the first-order approximation. Were this assumption exact, $y_{\|}$ would be one order smaller than $\overline{v}_{\|}$. However, due to the difference between our Reissner-like theory and the asymptotically correct theory, the second-order approximation $y_{\|}$ is of the same order as $\overline{v}_{\|}$. It is remarkable to note that one does not face this problem if, instead, $v_{\|}$ is taken as the first-order approximation; $y_{\|}$ will then become trivial.

One might be tempted to use $v_{\|}$ in the recovering relations. However, the VAM has split the original 3-D problem into two sets of problems. As far as recovering relations are concerned, the normal-line analysis can at best give us an approximate shape of the distribution of 3-D results. The 2-D plate analysis will govern the global behavior of the structure. Since $\overline{v}_{\|}$ is the warping that yields a Reissner-like plate model that is as close as possible to being asymptotically correct, we should still use $\overline{v}_{\|}$ in the recovering relations instead of $v_{\|}$. By doing this, we choose to be more consistent with the constructing of a Reissner-like plate model and compromise somewhat on the asymptotical correctness of the shape of the distribution. It has been verified by numerical examples that this is a good choice.

Hence, we write the 3-D recovering relations for displacement through the second order as

$$\begin{aligned}U_1 &= u_1 + x_3 C_{31} + \overline{v}_1 \\ U_2 &= u_2 + x_3 C_{32} + \overline{v}_2 \\ U_3 &= u_3 + w_3 + S_{11}\mathscr{E}_{,11} + S_{12}\mathscr{E}_{,12} + S_{22}\mathscr{E}_{,22}\end{aligned} \tag{60}$$

and the strain field through the second order is

$$\begin{aligned}\Gamma_e &= \epsilon + x_3\kappa + I_\alpha \overline{v}_{\|,\alpha} \\ 2\Gamma_s &= \overline{v}_{\|}' + e_1(D_{\perp 1}\epsilon_{,1} + D_{\perp 2}\kappa_{,1}) + e_2(D_{\perp 1}\epsilon_{,2} + D_{\perp 2}\kappa_{,2}) \\ \Gamma_t &= D_{\perp 1}'\epsilon + D_{\perp 2}'\kappa + S_{11}'\mathscr{E}_{,11} + S_{12}'\mathscr{E}_{,12} + S_{22}'\mathscr{E}_{,22}\end{aligned} \tag{61}$$

Again the stresses through the second order can be obtained from use of the 3-D material law. It will be shown in the next section that the recovered 3-D results through the second order agree with the exact results very well.

## 6. Numerical results

In this section we give a few sample results to illustrate the power of the derived theory. The material properties of all plates analyzed are

$$E_L = 25 \times 10^6 \text{ psi} \quad E_T = 10^6 \text{ psi}$$
$$G_{LT} = 0.5 \times 10^6 \text{ psi} \quad G_{TT} = 0.2 \times 10^6 \text{ psi}$$
$$\nu_{LT} = \nu_{TT} = 0.25$$

where $L$ denotes the direction parallel to the fibers and $T$ the transverse direction. The test problem is a plate with width $L = 4$ in. along $x_1$ (the "lateral" direction) and infinite length in the $x_2$-direction (the "longitudinal" direction). The thickness of the plate is 1 in., so that the aspect ratio $L/h = 4$. The plate is simply supported and subjected to a sinusoidal surface loading of the form

$$\tau_3 = \beta_3 = \frac{p_0}{2} \sin\left(\frac{\pi x_1}{L}\right) \tag{62}$$

with $\tau_\alpha = \beta_\alpha = 0$. A geometrically linear example is considered, and the results presented here are normalized following the scheme of Pagano (1970):

$$\overline{\Gamma}_{ij} = \frac{E_L h^2 \Gamma_{ij}}{L^2 p_0} \qquad \overline{\sigma}_{ij} = \frac{\sigma_{ij}}{p_0} \tag{63}$$

In the results below all nonzero strain and stress components are shown. However, since the 3-D displacement distribution is usually of relatively little interest in analysis of composite plates, those results are not presented here for the sake of brevity. Their accuracy is comparable to that of the strains.

First, we investigate a laminated composite plate with lay-up [15°/−15°]. The results plotted in Figs. 2–7 are compared with the exact solutions from Pagano (1970) as well as with a similar yet very different theory in Sutyrin and Hodges (1996) which is referred to as the Sutyrin theory here. From the plotted results, one can observe that both the present theory and the Sutyrin theory have excellent agreement with the 3-D exact solution and produce much better results than the classical theory (the zeroth-order approximation derived herein), especially for the transverse shear and transverse normal stress and strain. The in-plane shear strain obtained herein is slightly less accurate than classical theory, although it differs from the exact solution by only 5%. Considering that the lateral transverse strain is numerically three to four times as large as the in-plane shear strain, occasional errors of that order in the numerically smaller quantities should not be unexpected.

We digress here to point out that the present theory is different from the Sutyrin theory mainly in the following two aspects. First, the Sutyrin theory is restricted to linear plate theory, while the formulation of

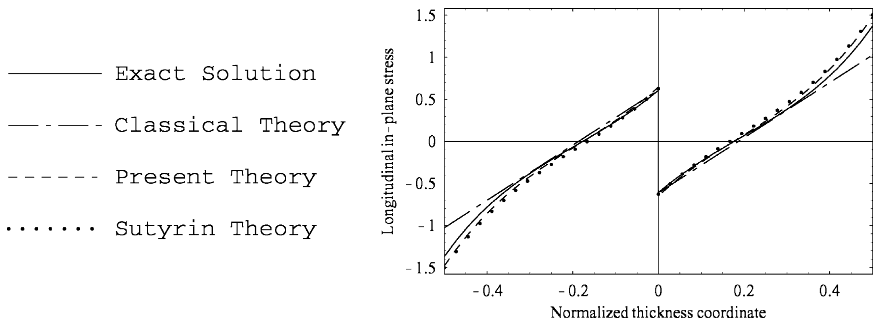


Fig. 2. Distribution of the 3-D stress $\sigma_{22}$ vs. the thickness coordinate [15°/−15°].

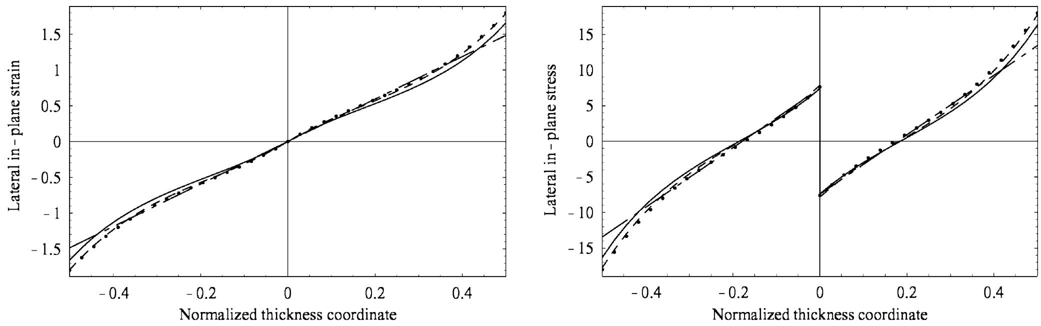


Fig. 3. Distribution of the 3-D strain $\Gamma_{11}$ and stress $\sigma_{11}$ vs. the thickness coordinate [15°/−15°].

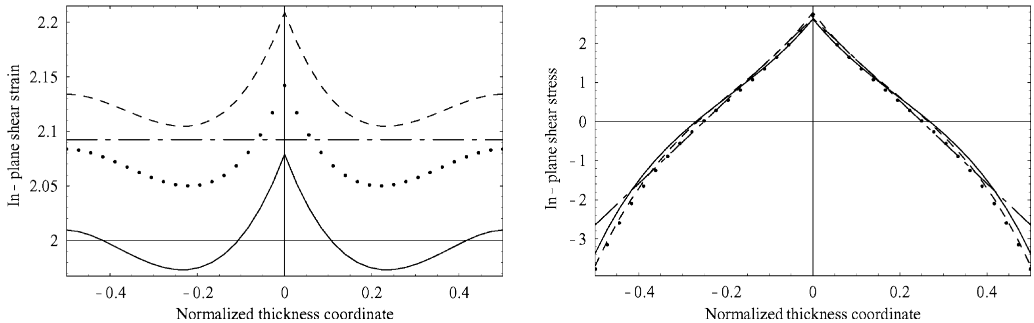


Fig. 4. Distribution of the 3-D strain $2\Gamma_{12}$ and stress $\sigma_{12}$ vs. the thickness coordinate [15°/−15°].

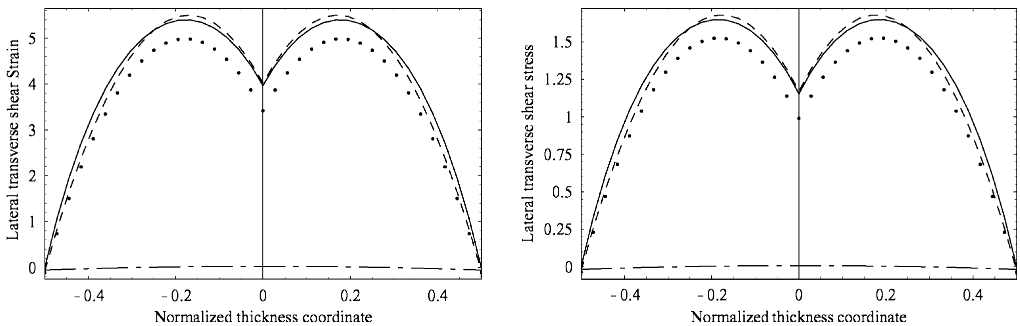


Fig. 5. Distribution of the 3-D strain $2\Gamma_{13}$ and stress $\sigma_{13}$ vs. the thickness coordinate [15°/−15°].

present theory is an intrinsic form which is good for both linear and geometrically nonlinear plate analysis. Second, in order to solve for the required warping functions, the Sutyrin theory assumes a priori a general form for the warping, taking the higher-order warping as a parameter to set interaction terms equal to zero. In the present work, however, the warping functions are solved by usual procedure of calculus of variations. In spite of the differences, it is expected that these two theories should be nearly equivalent, which is clearly verified by the presented results. In fact, for some values in this special case (in-plane shear strain, transverse shear strains) the present results are even better than those of the Sutyrin theory. Also, Sutyrin and

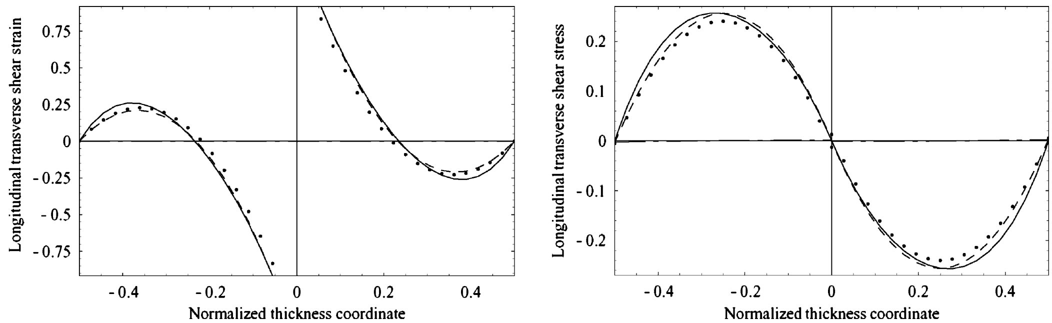


Fig. 6. Distribution of the 3-D strain $2\Gamma_{23}$ and stress $\sigma_{23}$ vs. the thickness coordinate [15°/−15°].

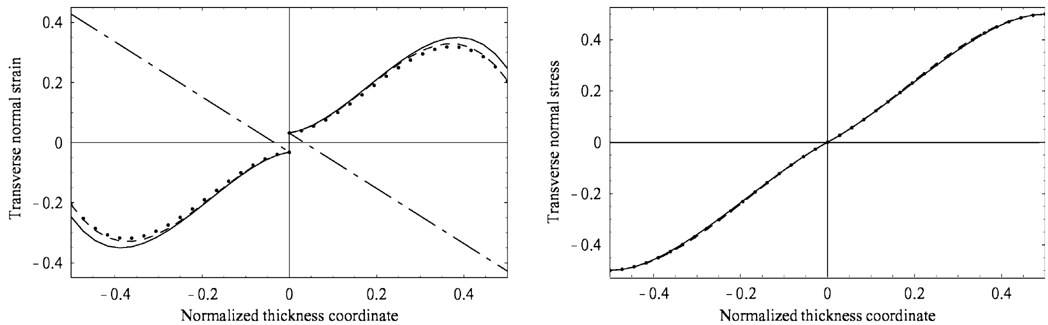


Fig. 7. Distribution of the 3-D strain $\Gamma_{33}$ and stress $\sigma_{33}$ vs. the thickness coordinate [15°/−15°].

Hodges (1996) uses a so-called smart minimization procedure without a known mathematical basis to produce the good agreement obtained. If one simply applies the least squares technique as is done for the present theory, the accuracy of the results produced by the theory of Sutyrin and Hodges (1996) will be slightly degraded.

Next, we take another laminated composite plate with lay-up [30°/−30°/−30°/30°] and the same material properties as the previous plate. The results are shown in Figs. 8–13. The power of present theory is clearly exemplified by the excellent agreement with exact 3-D solutions. Indeed, even though there are more layers in this example, the agreement is still excellent. Recall that the recovering relations use results from a

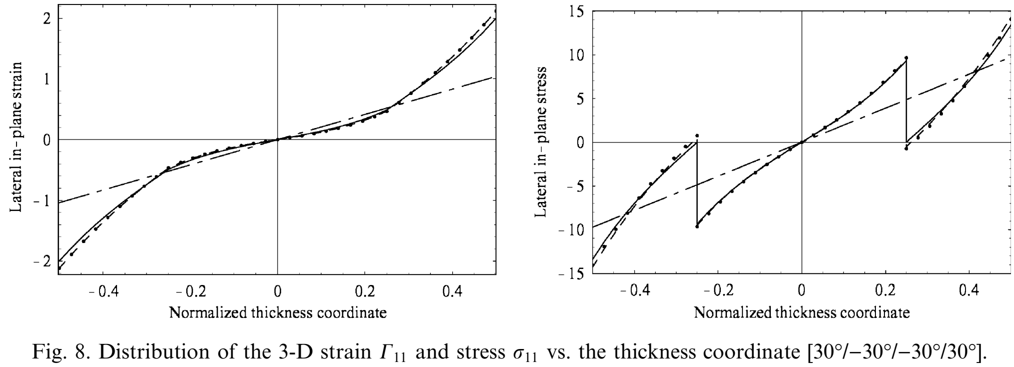


Fig. 8. Distribution of the 3-D strain $\Gamma_{11}$ and stress $\sigma_{11}$ vs. the thickness coordinate [30°/−30°/−30°/30°].

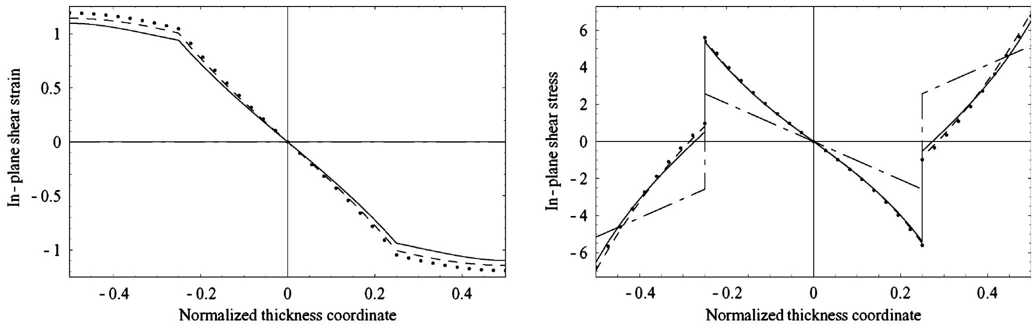


Fig. 9. Distribution of the 3-D strain $2\Gamma_{12}$ and stress $\sigma_{12}$ vs. the thickness coordinate [30°/−30°/−30°/30°].

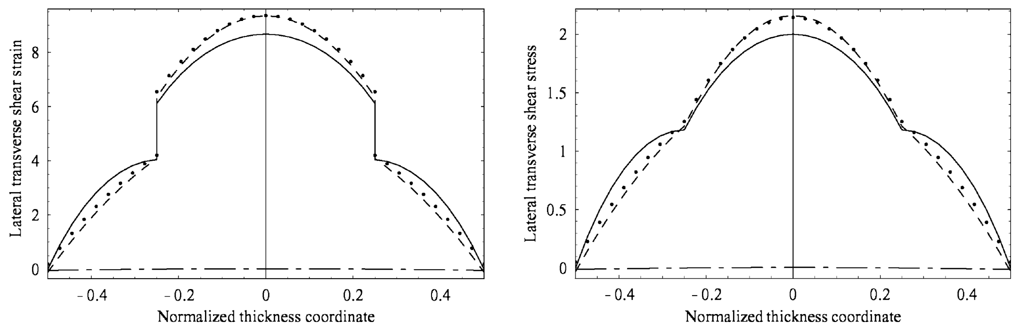


Fig. 10. Distribution of the 3-D strain $2\Gamma_{13}$ and stress $\sigma_{13}$ vs. the thickness coordinate [30°/−30°/−30°/30°].

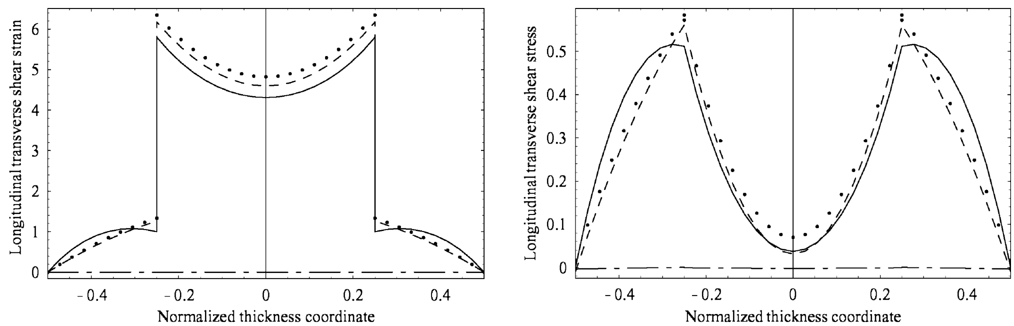


Fig. 11. Distribution of the 3-D strain $2\Gamma_{23}$ and stress $\sigma_{23}$ vs. the thickness coordinate [30°/−30°/−30°/30°].

standard Reissner-like plate model. The large number of degrees of freedom in the layer-wise models depends on the number of layers and is obviously not needed to achieve the level of accuracy shown here.

Lastly, let us try a very challenging lay-up, [0.5°/90.5°/90.5°/0.5°]. The properties and behavior of a plate with this kind of lay-up are very close to that of a sandwich plate with a very soft core. For the present theory to approximate the exact results very closely shows it to be suitable for the modeling of such structures. The stress and strain results are shown in Figs. 14–19. As one can observe from the results, even for this case, the present theory agrees with exact solution very well. This clearly proves that one can use the

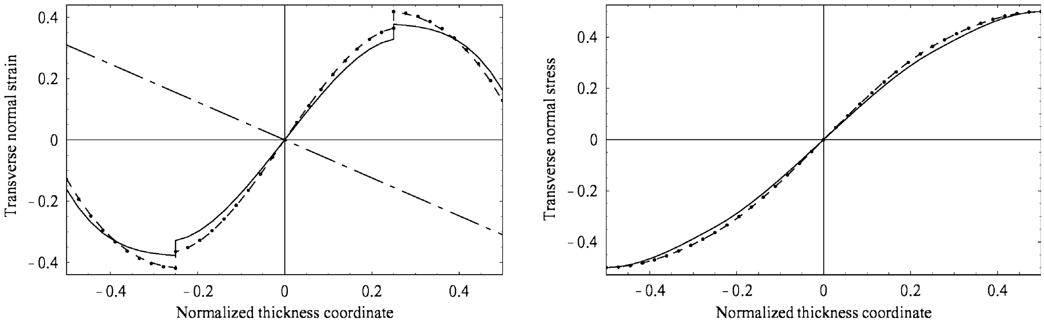


Fig. 12. Distribution of the 3-D strain $\Gamma_{33}$ and stress $\sigma_{33}$ vs. the thickness coordinate [30°/−30°/−30°/30°].

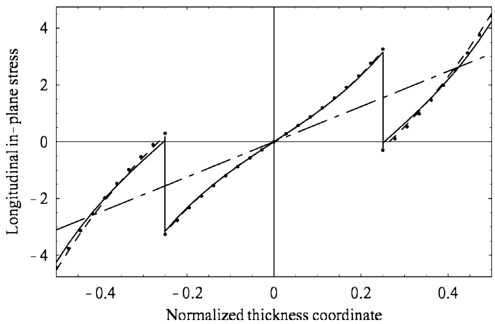


Fig. 13. Distribution of the 3-D stress $\sigma_{22}$ vs. the thickness coordinate [30°/−30°/−30°/30°].

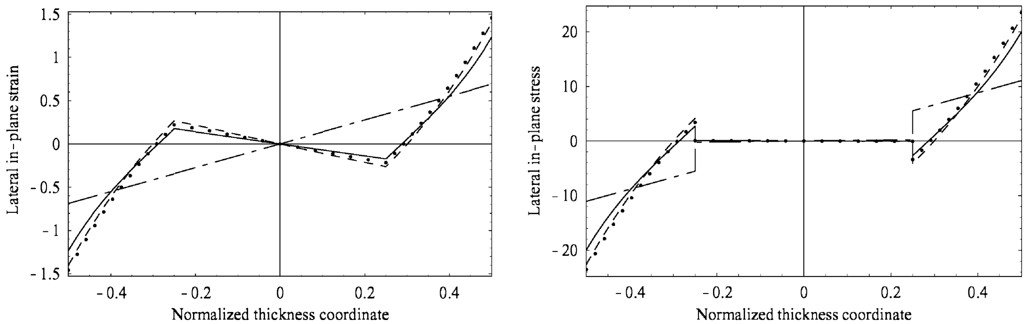


Fig. 14. Distribution of the 3-D strain $\Gamma_{11}$ and stress $\sigma_{11}$ vs. the thickness coordinate [0°/90°/90°/0°].

present theory to model laminated plates confidently to get great accuracy with much less computational effort.

Although we have not presented results from a shear deformation composite plate theory based on ad hoc kinematic assumptions (such as first-order shear deformation theory, or third-order shear deformation theory), it is to be expected that the present theory is far better than those ad hoc models. Mathematically, the accuracy of the present theory should be comparable to that of a layer-wise plate theory with assumed

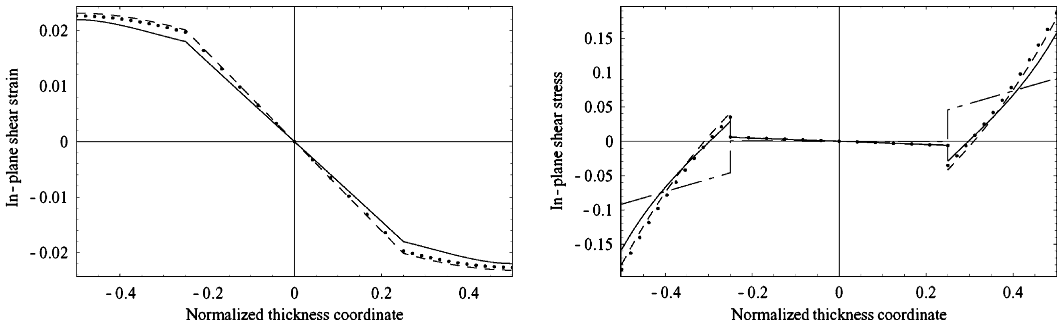


Fig. 15. Distribution of the 3-D strain $2\Gamma_{12}$ and stress $\sigma_{12}$ vs. the thickness coordinate [0°/90°/90°/0°].

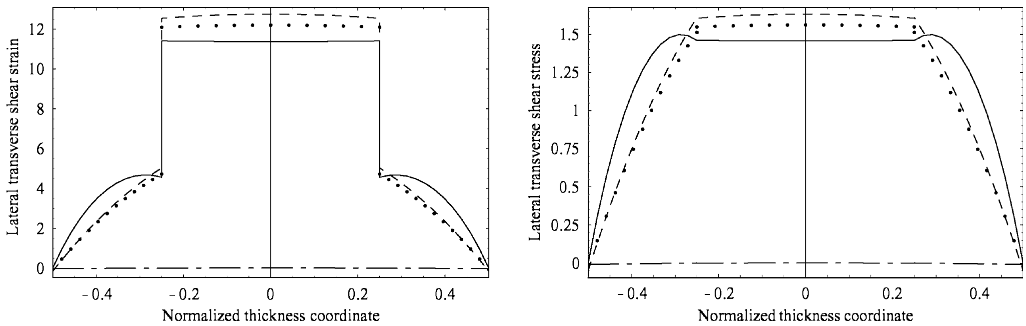


Fig. 16. Distribution of the 3-D strain $2\Gamma_{13}$ and stress $\sigma_{13}$ vs. the thickness coordinate [0°/90°/90°/0°].

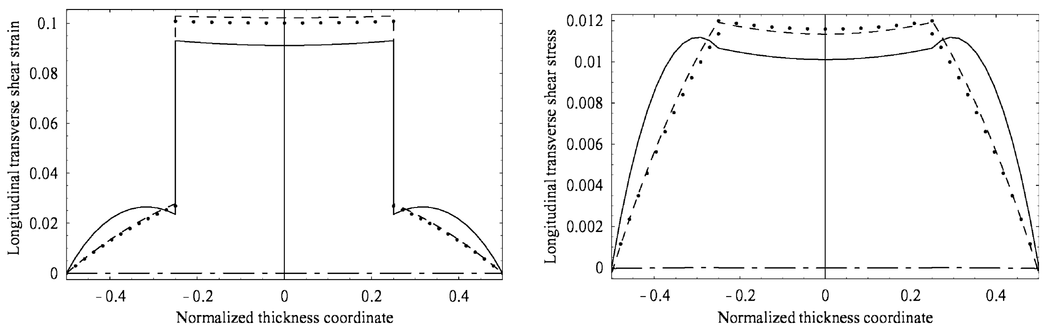


Fig. 17. Distribution of the 3-D strain $2\Gamma_{23}$ and stress $\sigma_{23}$ vs. the thickness coordinate [0°/90°/90°/0°].

in-plane displacements as layer-wise cubic polynomials of the thickness direction and transverse displacement as a layer-wise fourth-order polynomial. However, the present theory is still an equivalent single-layer theory, and the computational requirement is much less than for layer-wise theories. Moreover, it is not necessary to use integration of the 3-D equilibrium equations through the thickness to get the transverse normal and transverse shear strain and stress results presented herein.

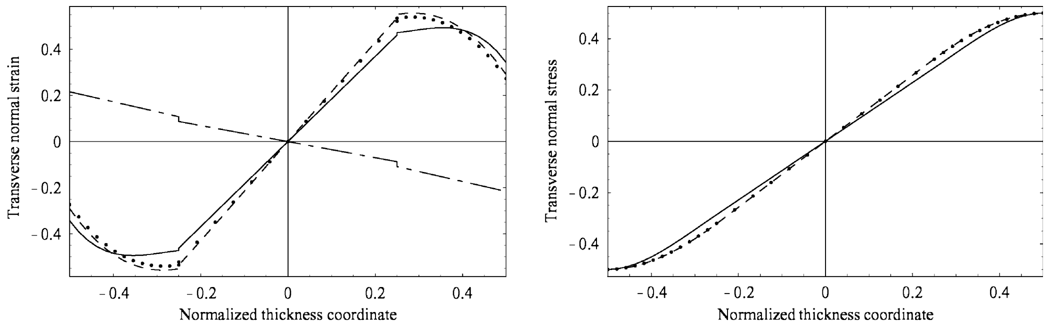


Fig. 18. Distribution of the 3-D strain $\Gamma_{33}$ and stress $\sigma_{33}$ vs. the thickness coordinate [0°/90°/90°/0°].

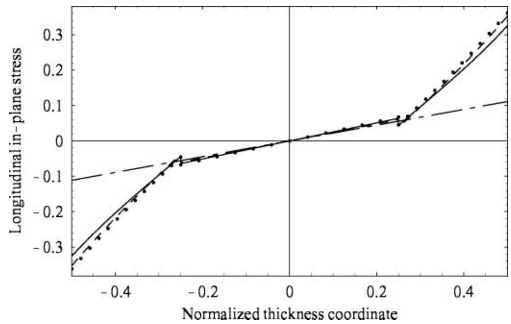


Fig. 19. Distribution of the 3-D stress $\sigma_{22}$ vs. the thickness coordinate [0°/90°/90°/0°].

## 7. Conclusion

A complete Reissner-like plate theory that is as close as possible to asymptotical correctness is developed for composite laminated plates. The theory is applicable to plates for which each layer is made with a monoclinic material. Although the resulting plate theory is as simple as a single-layer, first-order shear deformation theory, the recovered 3-D displacement, strain and stress results have excellent accuracy, comparable to that of higher-order, layer-wise plate theories that have many more degrees of freedom. The present paper has built on the work of several previous works mainly represented in Sutyrin (1997); Atilgan and Hodges (1992) and Sutyrin and Hodges (1996). The main contributions are

1. the present theory formulates the original 3-D elasticity problem in an intrinsic form which is suitable for geometrically nonlinear plate theory as well as linear theory;
2. the present theory solves the unknown 3-D warping functions asymptotically by using the VAM and the principle of minimum total potential energy, a procedure which is systematic and easy to apply iteratively. In Sutyrin and Hodges (1996), the warping functions are calculated by an equivalent but involved analytical formulation;
3. the present theory includes all potential terms in the formulation to find a total potential energy for minimizing instead of only the strain energy as presented in Atilgan and Hodges (1992). Having a total potential energy asymptotically correct to a certain order, one can derive a geometrically nonlinear plate theory in a straightforward manner using an energy method as in Hodges et al. (1993).

Since all the formulas are given in an explicit analytical form, the solution procedure becomes very tedious for plates with a large number of layers. Although a symbolic manipulator such as Mathematica™ or Maple™ can help in dealing with the complexity, the computational cost can soar in such cases. This can be remedied by hard-coding a finite element solution of the governing equations of the normal-line analysis. Since this is merely a 1-D problem, such a code will execute very rapidly, enabling these very accurate recovering relations to be cheaply included in standard plate finite element codes. This approach will be presented in a later paper.

## Acknowledgements

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## References

Atilgan, A.R., Hodges, D.H., 1992. On the strain energy of laminated composite plates. International Journal of Solids and Structures 29, 2527–2543.

Berdichevsky, V.L., 1979. Variational-asymptotic method of constructing a theory of shells. PMM 43, 664–687.

Cho, Y.B., Averill, R.C., 2000. First-order zig-zag sublaminate plate theory and finite element model for laminated composite and sandwich panels. Composite Structures 50, 1–15.

Danielson, D.A., 1991. Finite rotation with small strain in beams and plates. In: Proceedings of the Second Pan American Congress of Applied Mechanics. Valparaiso, Chile.

Danielson, D.A., Hodges, D.H., 1987. Nonlinear beam kinematics by decomposition of the rotation tensor. Journal of Applied Mechanics 54, 258–262.

DiSciuva, M., 1985. Development of anisotropic multilayered shear deformable rectangular plate element. Computers and Structures 21, 789–796.

Hodges, D.H., Atilgan, A.R., Danielson, D.A., 1993. A geometrically nonlinear theory of elastic plates. Journal of Applied Mechanics 60, 109–116.

Noor, A.K., Burton, S.W., 1990. Assessment of computational models for multilayered composite shells. Applied Mechanics Review, 43.

Noor, A.K., Burton, W.S., 1989. Assessment of shear deformation theories for multilayered composite plates. Applied Mechanics Review 41, 1–13.

Noor, A.K., Malik, M., 2000. An assessment of five modeling approaches for thermo-mechanical stress analysis of laminated composite panels. Computational Mechanics 25, 43–58.

Pagano, N.J., 1970. Influence of shear coupling in cylindrical bending of anisotropic laminates. Journal of Composite Materials 4, 330–343.

Reddy, J.N., 1984. A simple higher-order theory for laminated composite plates. Journal of Applied Mechanics 51, 745–752.

Sutyrin, V.G., 1997. Derivation of plate theory accounting asymptotically correct shear deformation. Journal of Applied Mechanics 64, 905–915.

Sutyrin, V.G., Hodges, D.H., 1996. On asymptotically correct linear laminated plate theory. International Journal of Solids and Structures 33, 3649–3671.

Touratier, M., 1991. An efficient standard plate theory. International Journal of Engineering Science 29, 901–916.