Problems

1. Consider a system S shown in Fig. 1, which consists of a rocket, idealized as a rigid body B, and a system of particles F representing the fuel, the center of mass of which (P) lies on the axis of B. As B flies in a frame A, P moves during flight so that its distance from the point B^* fixed in B is denoted by $q(t)$, an unknown function of time. We introduce unit vectors fixed in A denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, and, similarly in B unit vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, where $\mathbf{b}_3$ is parallel to the axis of the rocket. From an arbitrary point O, fixed in A, the position vector of B^* is given by

$$\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$$

where $x_i = x_i(t)$ for $i = 1, 2$, and 3. Next, denote the angular velocity of B in A by

$$\boldsymbol{\omega} = \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3$$

where $\omega_i = \omega_i(t)$ for $i = 1, 2$, and 3. Finally, denote the velocity of B^* in A as

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + v_3\mathbf{b}_3$$

where $v_i = v_i(t)$ for $i = 1, 2$, and 3. The rocket control system is able to direct the rocket in flight so that the axis of the rocket is always tangent to the flight path of P (i.e., parallel to the velocity vector ${}^A\mathbf{v}^P$).

   (a) Determine the number of generalized coordinates needed to describe the configuration of S, idealizing the fuel as the single particle P. (5 points)

   (b) How many degrees of freedom are there in S? (5 points)

   (c) Regarding the quantities v_i and ω_i (i=1,2,3) as generalized speeds, express the motion constraint(s) in terms of the generalized speeds and q. (10 points)

   (d) Show that the $\mathbf{b}_1$ measure number of the acceleration of B^* in A can be written as

$$\mathbf{b}_1 \cdot {}^A\mathbf{a}^{B^*} = \omega_2(v_3 - \dot{q}) - q(\dot{\omega}_2 + \omega_3\omega_1)$$

   (15 points)

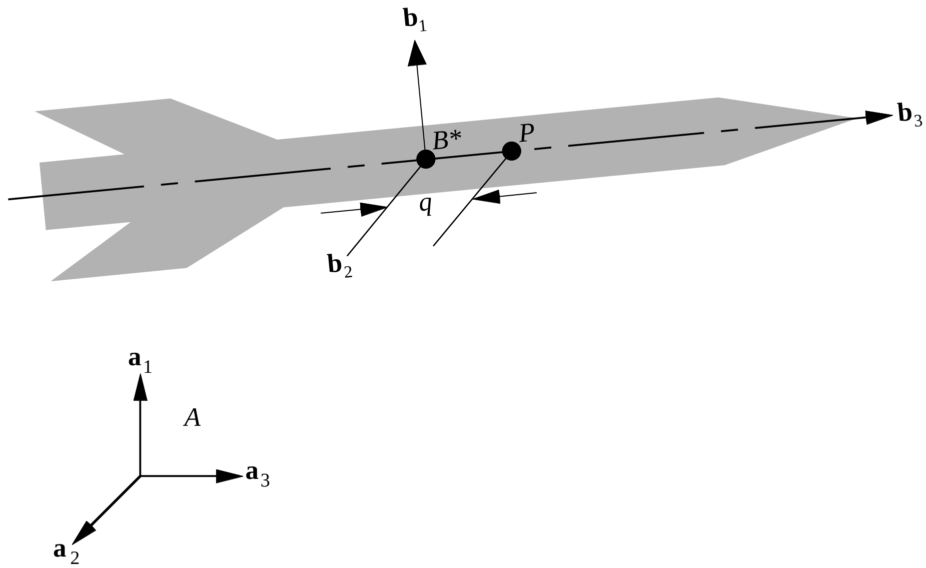

Figure 1: Schematic of rocket/fuel system

2. Referring to the system described in Problem 1, introduce the following transformation between $\mathbf{b}_i$ and $\mathbf{a}_i$: Let B be oriented so that $\mathbf{b}_i$ (i=1, 2, and 3) initially coincide with $\mathbf{a}_i$ (i=1, 2, and 3). To bring B into its actual orientation, rotate B about $\mathbf{b}_1$ by an angle θ_1; next, rotate B about $\mathbf{b}_2$ by an angle θ_2; finally, rotate B about $\mathbf{b}_3$ by an angle θ_3.

    (a) Find ω_3 in terms of $\dot{\theta}_i$ and θ_i (i=1, 2, and 3). (10 points)

    (b) Find v_3 in terms of $\dot{x}_i$ and θ_i (i=1, 2, and 3). (10 points)

    (c) Without finding all such relations, comment on the complexity of motion constraints when expressed in terms of the generalized speeds v_i and ω_i versus analogous expressions in terms of a different set of generalized speeds $u_i = \dot{x}_i$ and $u_{i+3} = \dot{\theta}_i$ (with i=1, 2, and 3). (5 points)

Solutions

1. (a) 7 generalized coordinates are needed: 6 for the rocket body and 1 for the particle representing the fuel center of mass, which is constrained to lie along the rocket axis.

    (b) There are 2 motion constraints and so there remain 5 degrees of freedom.

(c) We idealize the fuel as the single particle P so that the position vector from B^* to P is $\mathbf{P}$. Thus, we have one point P moving on a rigid body B that is, in turn, moving in A, the velocity of which is

$$^A\mathbf{v}^P = {}^B\mathbf{v}^P + {}^A\mathbf{v}^{\overline{B}}$$

where

$$^B\mathbf{v}^P = \dot{q}\mathbf{b}_3$$

and $\overline{B}$ coincides with P so that

$$\begin{aligned} ^A\mathbf{v}^P =& {}^A\mathbf{v}^{B^*} + {}^A\boldsymbol{\omega}^B \times (q\mathbf{b}_3) + {}^B\mathbf{v}^P \\ =& (v_1 + q\omega_2)\mathbf{b}_1 + (v_2 - q\omega_1)\mathbf{b}_2 + (v_3 + \dot{q})\mathbf{b}_3 \end{aligned}$$

Since this must remain parallel to $\mathbf{b}_3$ we have motion constraints as

$$\begin{aligned} ^A\mathbf{v}^P \cdot \mathbf{b}_1 =& 0 \\ ^A\mathbf{v}^P \cdot \mathbf{b}_2 =& 0 \end{aligned}$$

or

$$v_1 = -q\omega_2 \qquad v_2 = q\omega_1$$

(d) Thus, the acceleration of B^* in A can be written as

$$\begin{aligned} ^A\mathbf{a}^{B^*} =& \frac{^Ad^A\mathbf{v}^{B^*}}{dt} \\ =& \frac{^Bd^A\mathbf{v}^{B^*}}{dt} + {}^A\boldsymbol{\omega}^B \times {}^A\mathbf{v}^{B^*} \end{aligned}$$

with

$$^A\mathbf{v}^{B^*} = -q\omega_2\mathbf{b}_1 + q\omega_1\mathbf{b}_2 + v_3\mathbf{b}_3$$

so that the $\mathbf{b}_1$ measure number of the acceleration becomes

$$\mathbf{b}_1 \cdot {}^A\mathbf{a}^{B^*} = \omega_2(v_3 - \dot{q}) - q(\dot{\omega}_2 + \omega_3\omega_1)$$

2. The orientation angles are the same as those of problem 1.1, so that

(a)

$$\omega_3 = \dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2$$

(b)

$$v_3 = \dot{x}_1 \sin\theta_2 - \dot{x}_2 \sin\theta_1 \cos\theta_2 + \dot{x}_3 \cos\theta_1 \cos\theta_2$$

(c) The expressions in terms of the alternate set are more involved.