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Asymptotic theory for static behavior of elastic anisotropic I-beams

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Abstract

End effects for prismatic anisotropic beams with thin-walled, open cross-sections are analyzed by the variational-asymptotic method. The decay rates for disturbances at the ends of prismatic beams are evaluated, and the most influential end disturbances are incorporated into a refined beam theory. Thus, the foundations of Vlasov's theory, as well as restrictions on its applicability, are obtained from the variational-asymptotic point of view. Vlasov's theory is proved to be asymptotically correct for isotropic I-beams. The asymptotically correct generalization of Vlasov's theory for static behavior of anisotropic beams is presented. In light of this development, various published generalizations of Vlasov's theory for thin-walled anisotropic beams are discussed. Comparisons with a numerical 3-D analysis are provided, showing that the present approach gives the closest agreement of all published theories. The procedure can be applied to any thin-walled beam with open cross-sections. © 1988 Elsevier Science Ltd. All rights reserved.

1. Introduction

In some cases classical beam theory is not sufficient for accurately predicting the internal stress–strain state. In order to explain the nature of the discrepancies let us consider a prismatic beam that occupies a domain

$$\Omega = \{0 < x_1 < l, \quad (x_2, x_3) \in S\}$$

with some prescribed cross-section S; x_1, x_2, x_3 are Cartesian coordinates; and $\partial\Omega$ is the boundary of Ω. The system of equations governing displacements of the beam consists of four sets of equations: the equilibrium equation

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$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad x_i \in \Omega \tag{1}$$

the constitutive law

$$\sigma_{ij} = A_{ijkl}\varepsilon_{kl} \tag{2}$$

the kinematic relations

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \tag{3}$$

and the free boundary conditions at the lateral surface

$$\sigma_{ij}n_j = 0 \quad \text{for } x_i \in \partial\Omega \tag{4}$$

Here components of the displacement are denoted as $u_i(x_1, x_2, x_3)$; stress and strain tensors as $\sigma_{ij}(x_1, x_2, x_3)$ and $\varepsilon_{ij}(x_1, x_2, x_3)$, respectively; the unit normal vector at the beam boundary as $n_i(x_2, x_3)$; and the tensor of material constants as $A_{ijkl}(x_2, x_3)$. Boundary conditions at the ends of the beam $x_1 = 0$ and $x_1 = l$ need not be specified at this stage. Note that material constants are independent of x_1 due to the assumed spanwise uniformity. To understand the behavior of the solutions of eqns (1)–(4) it is useful to find particular solutions of the form

$$\begin{aligned} u_i(x_1, x_2, x_3) &= u_i^0(x_2, x_3)e^{ikx_1} \\ \sigma_{ij}(x_1, x_2, x_3) &= \sigma_{ij}^0(x_2, x_3)e^{ikx_1} \\ \varepsilon_{ij}(x_1, x_2, x_3) &= \varepsilon_{ij}^0(x_2, x_3)e^{ikx_1} \end{aligned} \tag{5}$$

where k is the wave number. By substituting eqns (5) into eqns (1)–(4) one can obtain the reduced system

$$\frac{\partial \sigma_{i\alpha}^0}{\partial x_\alpha} + ik\sigma_{1i}^0 = 0, \quad (x_2, x_3) \in S \tag{6}$$

$$\sigma_{ij}^0 = A_{ijkl}\varepsilon_{kl}^0 \tag{7}$$

$$\varepsilon_{\alpha\beta}^0 = \frac{1}{2}\left(\frac{\partial u_\alpha^0}{\partial x_\beta} + \frac{\partial u_\beta^0}{\partial x_\alpha}\right) \tag{8}$$

$$\varepsilon_{\alpha 1}^0 = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_\alpha} + iku_\alpha^0\right) \tag{9}$$

$$\varepsilon_{11}^0 = iku_1^0 \tag{10}$$

$$\sigma_{\alpha\beta}^0 v_\beta = 0 \quad \text{at } (x_2, x_3) \in \partial S \tag{11}$$

where v_β is outward unit normal to the boundary ∂S of the cross-section S; indices α and β vary from 2–3.

Equations (6)–(11) determine an eigenvalue problem: a non-trivial solution of eqns (6)–(11)

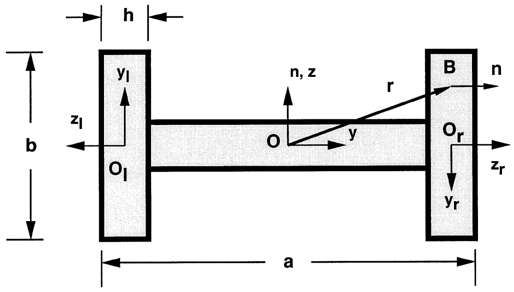

Fig. 1. Schematic of I-beam cross-section and coordinate system (axis $x_1 \equiv x$ is directed outward).

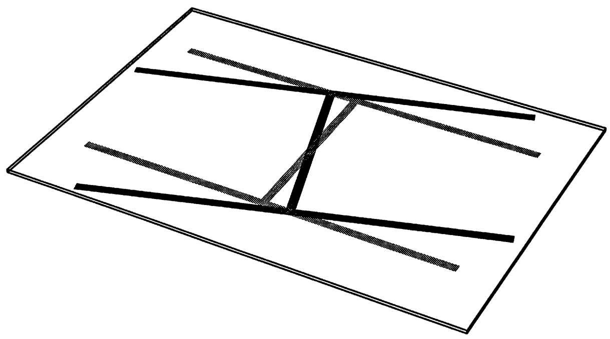

Fig. 2. First "non-classical" (Vlasov) mode for isotropic I-beam: $\Im(bk) = 0.1103$, $h/b = 0.02$.

exists only for particular values of k. The corresponding set of eigenfunctions comprises an infinite-dimensional basis for beam solutions. Analysis of eqns (6)–(11) reveals that there are only four real eigenvalues k, all zero. The four eigenfunctions corresponding to $k = 0$ represent an "interior" stress state, which is described by classical beam theory. All the other values of k are complex, and $\Im(k)$ has the sense of a decay rate from the left end if $\Im(k) > 0$ and from the right end otherwise. Eigenproblem eqns (6)–(11) is solved numerically (see Volovoi et al., 1995; Volovoi et al., 1998) for a cross-section of arbitrary geometry and material properties.

Classical beam theory with free lateral surfaces can be viewed as a truncation of the solution including only the first four of the base eigenfunctions, and it is "exponentially" exact in the sense that all corrections stem from end effects which decay exponentially as they penetrate in the interior of the beam. For some cross-sections the decay rate might be small, so that the end effects significantly influence the global elastic behavior of beams. For such beams it is important to refine classical theory by incorporation of the disturbances with the slowest decay rate.

Let us focus our attention on one such type, thin-walled beams with open cross-sections. First we consider a symmetric isotropic I-beam (a schematic of the cross-section is shown in Fig. 1). Here the web height is denoted as a, the flange width as b, and uniform thickness of both flanges and the web as h. Figures 2 and 3 depict the two "non-classical" modes with the slowest decay rates for $a/b = 0.5$, $h/b = 0.02$, and Poisson ratio $\nu = 0.42$.

Two important facts were established during parametric studies: while the numerical values of

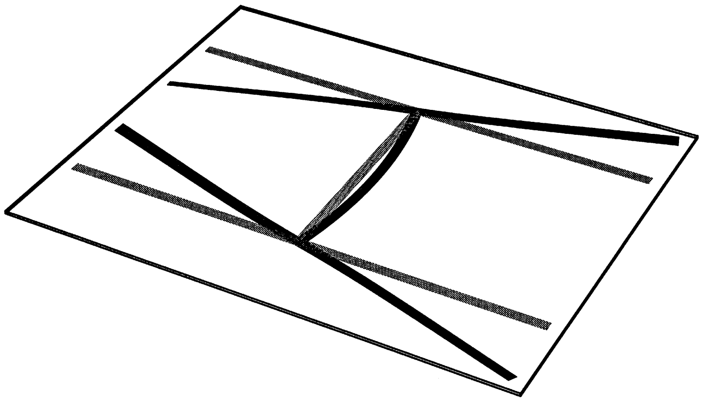

Fig. 3. Second "non-classical" (camber) mode for isotropic I-beam: $\Im(bk) = 1.8988$, $h/b = 0.02$.

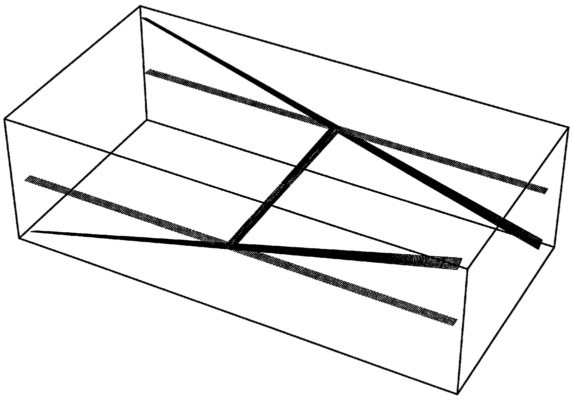

Fig. 4. Scaled up out-of-plane displacements of the first "non-classical" mode for isotropic I-beam: $\Im(bk) = 0.1103$.

the decay rate varied with the parameters of the beam, the shape of the modes remained essentially the same, and the first mode (Fig. 2) always had a much smaller decay rate for isotropic I-beams than the second mode (Fig. 3). Both modes have purely imaginary k. It is noted that in this example $\Im(bk) = 0.1103$ for the first mode implies that the amplitude of the disturbances at the end for this mode will decrease by a factor of e^{-1} at a distance from the end of about $9b$, while for the second mode ($\Im(bk) = 1.8988$) it will happen much closer, at a distance slightly exceeding $b/2$.

Study of the first "non-classical mode" (Fig. 2) reveals that it has a shape closely resembling the rotation of a cross-section due to torsion. This implies that the first correction to the classical beam theory, which would include this slowly decaying mode, does not require a new degree of freedom: torsional rotation of the cross-section is already among the modes described by classical beam theory. This conclusion is further supported by studying scaled up out-of-plane displacements for this mode, which clearly correspond to classical St Venant warping (see Fig. 4). This observation agrees very well with the hypothesis made by Vlasov for thin-walled beams, Vlasov (1961). It turns out that good qualitative and remarkably good quantitative correlation was observed between the

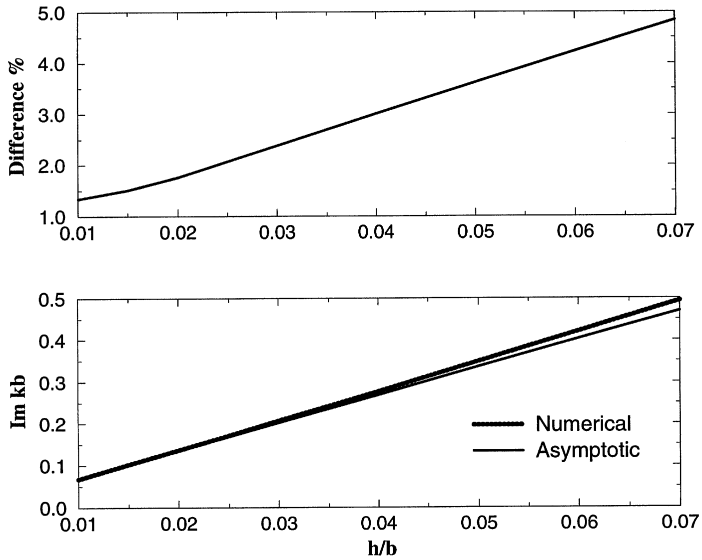

Fig. 5. Decay rate for isotropic I-beams; $a/b = 0.5$. (The top picture demonstrates the difference between the two results in the bottom picture.)

predictions of Vlasov's theory and numerical results (see Fig. 5, and Section "Comparing Theories"). All this prompted a detailed investigation of Vlasov's theory in an attempt to explain this coincidence.

The basic assumptions of Vlasov's theory, Vlasov (1961), are

(A) The cross-section remains rigid in its own plane.
(B) Shear strains are small.
(C) The Kirchhoff–Love assumption of classical shell theory remains valid.

Using assumption C, the 3-D problem is cast in terms of shell variables. Curvilinear coordinates are introduced: x_1 along the beam axis, and in the cross-sectional, y along the contour and z normal to the contour (see Fig. 1). Corresponding displacements are v_1 and v_2 and v_3, respectively. Unknowns are functions of x_1 and y, with dependence on z provided by the Kirchhoff hypothesis. From assumption A expressions for in-plane displacements are obtained as

$$v_2(x_1, y) = [U_\alpha(x_1) + \theta(x_1) x_\beta e_{\beta\alpha}] \frac{dx_\alpha}{dy}$$

$$v_3(x_1, y) = e_{\alpha\beta} U_\beta(x_1) \frac{dx_\alpha}{dy} - \frac{1}{2}\theta(x_1) \frac{d(x_\alpha x_\alpha)}{dy} \quad (12)$$

where $e_{\beta\alpha}$ is the 2-D Levi–Civita symbol; U_α and θ are the cross-section translations along the Cartesian coordinate x_α and the rotation about the x_1 axis, respectively. Assumption B implies that

$$v_{1,y} + v_{2,1} \equiv \varepsilon_{12} = 0 \quad (13)$$

Integrating eqn (13) with respect to y and substituting the expression for $v_{2,1}$, obtained by differentiation of eqns (12), one obtains an expression for the out-of-plane displacement

$$v_1(x_1, y) = U_1 - U'_\alpha x_\alpha - \theta'(x_1)\eta(y) \tag{14}$$

where U_1 is translation of a cross-section along the x_1 and $\eta(y)$ is a sectorial coordinate of point B with respect to point O (see Fig. 1):

$$\eta(y) = \int_O^B e_{\alpha\beta} x_\beta \frac{\mathrm{d}x_\alpha}{\mathrm{d}y} \equiv \int_O^B \mathbf{r} \cdot \mathbf{n} \, \mathrm{d}y \tag{15}$$

Here **r** is a radius vector from the origin, and **n** is an outward unit normal to the contour vector in the plane of a cross-section; integration is performed in the direction of the contour coordinate. When evaluated at the junction, $\eta(y)$ should have the same value in the branches meeting at this junction.

Using the displacement field represented by eqns (12) and (14) one can calculate strains, substitute them into the 3-D strain energy, and explicitly integrate over the cross-section to obtain the 1-D energy per unit length of the beam. The torsional part (which for symmetric cases is uncoupled from bending and extensional parts) has the following form

$$2\mathscr{E}_{\text{refined}} = GJ\theta'^2 + E\langle \phi^2 \rangle \theta''^2 \tag{16}$$

where G and E are the shear and Young's moduli, respectively, J is the torsional constant of the cross-section, ϕ is the St Venant warping function (which for thin-walled beams can be approximated by a sectorial coordinate), and $\langle \cdot \rangle$ refers to integration over the cross-section. The second term in eqn (16) leads to the introduction of the so-called bi-moment—which is proportional to θ'' and is represented by a system of forces statically equivalent to zero applied to a cross-section. Let us note that consistent use of the displacement field, in eqns (12) and eqn (14) in the above derivation, leads to a material coefficient $E/(1-\nu^2)$ instead of E in eqns (16). However, in his application of the constitutive relations, Vlasov first neglects stresses in the contour direction, not strains (as kinematic assumption B would require). This contradictory assumption leads to the desired coefficient in eqn (16), but it also resulted in a certain degree of confusion when attempts were made to generalize Vlasov's theory for anisotropic beams. Some researchers (Gjelsvik, 1981; Wu and Sun, 1992) tacitly recommend the neglect of stresses, correctly noting that the alternative will lead to overstiffening the structure. Others, notably Bauld and Tzeng (1984) and Chandra and Chopra (1991) consistently followed Vlasov's explicitly stated assumption A and neglected strains. While the resulting material coefficients satisfying these two contradictory assumptions differ by a factor of $1-\nu^2$ for isotropic beams (which happens to be close to unity), the difference for anisotropic beams can be dramatic for certain lay-ups. The results presented here prove the validity of neglecting stresses in the contour direction and the invalidity of assumption A.

It should be noted that Smith and Chopra (1991) derived a thin-walled beam theory for box beams which purports to be an improvement over one in which the stresses are set equal to zero. The results of these theories are very close together, and the claim that their theory is superior is based on only one experimental data point, which is only slightly closer to one curve than the other. Since exact 3-D solution contains nonzero contour stresses it is clear that such terms will appear in higher-order corrections with respect to h/a. However, such corrections have not been developed, while consideration of nonzero contour stresses as main effects is incorrect. In this paper both isotropic and anisotropic I-beams are studied. Vlasov's theory is justified from the asymptotic point of view, and consistently generalized for anisotropic beams. The derivation is based on a general procedure called the variational-asymptotic method, which was developed by Berdichevsky and his collaborators. While the basic notions of this method are given in this paper, for more detailed description of this method, as applied to shells, see Berdichevsky (1979), Berdichevsky (1983), and Sutyrin and Hodges (1996), and beams, Berdichevsky (1983), Berdichevsky (1982), and Le (1986). The method is based on small parameters, and the two small parameters that are employed in the derivation are

$$\frac{a}{l} \ll 1 \quad \frac{h}{a} \ll 1 \tag{17}$$

where l, a, and h are characteristic length, cross-sectional size, and thickness of the walls, respectively.

Discussion of existing theories and parametric comparison with 3-D numerical results are provided in section "Comparing Theories". It should be noted that classical coefficients derived in the present paper are identical to those that can be obtained following the procedure outlined in Reissner and Tsai (1972). Vlasov's coefficients are the same as in Wu and Sun (1992) if the hoop stress resultants and moments together with membrane shear resultant are set to zero, and membrane shear effects are disregarded.

2. Results

For the convenience of the reader the final results for anisotropic I-beams are provided here. The I-beam is viewed as a rigidly connected combination of three plates (where the validity of this approach is discussed below). Flanges do not have to be of identical length, and the web need not be connected to the middle of the flange (so that, for example, a channel can be treated as well).

The present derivation is based on the two small parameters introduced by eqns (17). Only the leading terms with respect to thickness parameter h/a are retained, while terms up to the second-order with respect to the beam small parameter a/l are retained.

For a general anisotropic beam and the 1-D strain energy per unit length, asymptotically correct to the second-order, has the following form:

$$\mathscr{E}_{\text{refined}} = \tfrac{1}{2}\alpha_b C_{bc} \alpha_c + \alpha_b L_{bc} \alpha'_c + \tfrac{1}{2}\alpha'_b M_{bc} \alpha'_c \tag{18}$$

The classical strain energy provides us only with the interior solution and corresponds to the first term in the expression for refined energy. Here b, $c = 1 \ldots 4$, and

$$\alpha = \begin{Bmatrix} U'_1 \\ U''_2 \\ U''_3 \\ \theta' \end{Bmatrix} \tag{19}$$

is a column matrix of 1-D strain measures, where U_1 is axial displacement due to extension, U_2 and U_3 are transverse displacements due to bending in two orthogonal directions, and θ is the section rotation due to torsion. The form of eqns (18) implies that the dominant correction to the classical theory is associated with θ, while all other corrections will be of higher-order with respect to the small parameters of the system.

For anisotropic I-beams eqns (18) can be significantly simplified, and explicit expressions are obtainable. The final expression for the 1-D strain energy per unit length can be written as

$$\mathscr{E}_{\text{refined}} = \tfrac{1}{2}\alpha_b C_{bc}\alpha_c + \alpha_b C_{b5}\theta'' + \tfrac{1}{2}C_{55}\theta''^2 \tag{20}$$

The coefficients in eqns (20) are expressed in terms of the cross-sectional properties in the following manner. For each member of an I-beam the plate strain energy per unit area can be written as

$$\mathscr{E}_{\text{plate}} = \tfrac{1}{2}hE_e^{\gamma\delta\nu\mu}A_{\gamma\delta}A_{\nu\mu} + \tfrac{1}{2}h^3E_b^{\gamma\delta\nu\mu}B_{\gamma\delta}B_{\nu\mu} + h^2E_{eb}^{\gamma\delta\nu\mu}A_{\gamma\delta}B_{\nu\mu} \tag{21}$$

where Greek indices vary from 1–2; $E_e^{\gamma\delta\nu\mu}$ and $E_b^{\gamma\delta\nu\mu}$ are fourth-order tensors of 2-D material constants corresponding to membrane and bending deformation, respectively, and $E_{eb}^{\gamma\delta\nu\mu}$ corresponds to coupling between these two types of deformation (see Berdichevsky, 1983); $A_{\gamma\delta}$ and $B_{\gamma\delta}$ are 2-D (plate) membrane and bending measures of deformation, respectively, given by

$$A_{\gamma\delta} = \tfrac{1}{2}(v_{\gamma,\delta} + v_{\delta,\gamma}), \quad B_{\gamma\delta} = -v_{3,\gamma\delta} \tag{22}$$

As shown below, retaining in eqns (21) only the leading terms with respect to small parameters defined in eqns (17) will yield the following simplified expression for the plate strain energy per unit area:

$$\mathscr{E}_{\text{plate}} = \tfrac{1}{2}Q_{11}A_{11}^2 - Q_{12}A_{11}B_{12} + \tfrac{1}{2}Q_{22}B_{12}^2 \tag{23}$$

Here $Q_{\gamma\delta}$ can be found as a result of minimization of eqn (21) with respect to the unknowns A_{12}, A_{22}, and B_{22}, so that

$$Q \equiv \bar{Q} - SR^{-1}S^T \tag{24}$$

with the following formulae for $\bar{Q}$, R and S:

$$\bar{Q} = h\begin{bmatrix} E_e^{1111} & 2hE_{eb}^{1112} \\ 2hE_{eb}^{1112} & 4h^2E_b^{1212} \end{bmatrix} \tag{25}$$

$$S = h\begin{bmatrix} E_e^{1112} & E_e^{1122} & E_{eb}^{1122} \\ 2hE_{eb}^{1212} & 2hE_{eb}^{1222} & 2h^2E_b^{1222} \end{bmatrix} \tag{26}$$

$$R = h\begin{bmatrix} E_e^{1212} & E_e^{1222} & 2hE_{eb}^{1222} \\ E_e^{1222} & E_e^{2222} & hE_{eb}^{2222} \\ 2hE_{eb}^{1222} & hE_{eb}^{2222} & h^2E_b^{2222} \end{bmatrix} \tag{27}$$

The first term in eqns (20) corresponds to the classical part of the strain energy, and the coefficients are given by

$$C_{ab} = \int_{\mathrm{r+l+w}} T_{\gamma a} Q_{\gamma\delta} T_{\delta b}\,\mathrm{d}y \tag{28}$$

where w stands for the web, r and l for the right and left flanges, respectively; indices a and b vary from 1–4; 2×4 "transition" matrix $T_{\gamma a}$ expresses the relevant 2-D strain measures (A_{11} and B_{12}, associated with extension in axial direction, and torsion, respectively) as functions of the contour coordinate in terms of 1-D strain measures [eqn (19)]. Explicit formulae for the members of an I-beam are:

$$T_{\mathrm{w}} = \begin{bmatrix} 1 & -y & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{29}$$

$$T_{\mathrm{r}} = \begin{bmatrix} 1 & -\dfrac{a}{2} & y & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{30}$$

$$T_{\mathrm{l}} = \begin{bmatrix} 1 & \dfrac{a}{2} & -y & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{31}$$

Coefficients for correction terms are:

$$C_{a5} = \int_{\mathrm{r+l+w}} \eta Q_{1\gamma} T_{\gamma a}\,\mathrm{d}y$$

$$C_{55} = \int_{\mathrm{r+l+w}} \eta^2 Q_{11}\,\mathrm{d}y \tag{32}$$

Integration is performed over the contour of the cross-section; and η is a sectorial coordinate in the circumferential direction if the pole is located in the middle of the web [η is defined in eqns (15)].

It is necessary to note that the capability of the asymptotic method to describe dominant end effects, and thus obtain a refined beam theory, is somewhat counterintuitive. The end effects are related to deformations with short wavelength along the beam—deformation with wavelength of the order of a. Therefore, strictly speaking, the small parameter a/l of the asymptotic derivation ceases to be small, and procedure becomes self-contradictory. The present analysis shows, however, that in the cases when end effects are important for global elastic properties of the beam, the disturbances away from the ends have wavelengths large enough (for the I-beams under consideration, $4a < 1/k < 20a$) to be captured by the second-order terms of the variational-asymptotic procedure with very good accuracy (see Fig. 5 and Section "Comparing Theories"). Similar situations are often encountered in asymptotic derivations: asymptotic theories typically work beyond the range for which they are originally developed.

3. Outline of variational-asymptotic method

This section is intended to serve as a quick reference for the reader who is unfamiliar with the variational-asymptotic method, which is repeatedly applied to different cases in the following sections. We limit the description of the method to beams, specifying the small parameter as a/l. While consideration of the practical cases is pivotal for understanding this method and some details vary from one case to the other, it is important to view the forest instead of the trees. Therefore, it may be advisable to skip this section in the first reading, and proceed with consideration of specific examples. Then, if necessary to understand details, the reader can return to this section in order to obtain a more complete view of the method.

(A) The preliminary steps of the asymptotic procedure yield what is sometimes called the "zeroth" approximation. The sole purpose of this part is to establish the "building blocks" of the solution by eliminating terms in energy, that are "excessively large" with respect to the small parameter a/l. There are always two such terms for beams. Let us note that it is the relative order of terms, which is important in the asymptotic method, so it is customary to assume the largest term in the energy which cannot be minimized to zero to be of order unity with respect to a/l. Then the terms which can be (and should be) minimized to zero will be of order $(a/l)^{-2}$ and $(a/l)^{-1}$. Thus, the preliminary part consists of the two steps:

(i) Elimination of all terms of order $(a/l)^{-2}$. All terms in energy which contain the small parameter (i.e., containing derivative with respect to axial coordinate) are disregarded. The remaining "main" part of energy is minimized with respect to the displacement field. The general solution for the minimizing displacement field corresponds to four motions of the cross-section as a rigid body, or "classical" degrees of freedom. This step alone sometimes is also referred as "zeroth" approximation, so the caution is advised while using this term, although the desire to avoid names employing negative numbers is understandable.
(ii) Elimination of all terms of order $(a/l)^{-1}$. This step follows the general procedure, which is basically an "induction step" (see below): finding the approximation of order $N+1$ if we know the approximation of order N, with the one simplifying difference that the final calculation of the strain energy is not needed, since it is minimized to zero at this step. It is important to note, that while we eliminated terms of order $(a/l)^{-1}$, terms of order $(a/l)^0$ remain.

(B) The "induction step" consists of searching the displacement field as a perturbation of the displacement field of the previous approximation:

(i) Unknown perturbations w_i are introduced.
(ii) Strains as functions of w_i are calculated and substituted into the energy.
(iii) Only the leading terms with respect to the small parameter are retained.
(iv) Energy is minimized with respect to w_i.
(v) Minimizing the energy, we obtain w_i as functions of the "classical" degrees of freedom (which depend only on the axial coordinate) and their derivatives, with explicit dependence on the cross-sectional coordinates, so the energy density can be integrated over the cross-section to yield 1-D energy density.

(C) After the preliminary part completed, the first full cycle described in B is conducted to obtain the next approximation. If further approximations are needed, the displacement field of the previous order is perturbed again, and steps B(i)–B(v) are repeated.

Three important points have to be noted here:

(1) Orders of the perturbations are not assumed *a priori*, but rather obtained as a result of the

minimization. Generally speaking at the first full cycle of the asymptotic procedure perturbations give contribution to the energy of order $(a/l)^0$. In that case perturbations combined with the displacement obtained in the preliminary stage yield the first-order approximation, which is equivalent to classical beam theory. In order to obtain terms in the energy of second order, one has to consider perturbation of the first-order results and repeat the whole procedure of minimization. For some cases, such as torsion of isotropic strips, however, perturbations of the first-order vanish, and we obtain the terms of the second-order at the first step, whereas classical terms are fully represented by the preliminary approximation.

(2) To render the solution of this minimization procedure unique, we need to impose some constraints on the perturbations. The standard option is to eliminate rigid body motions of the cross-section, already accounted for in the previous approximation. Some other choice of the constraints might be convenient from the standpoint of simplifying the calculations; and if the resulting 1-D problem is not degenerate in some sense, final results will not depend on the choice of the constraints, provided the 1-D variables are defined correctly.

(3) This procedure is valid for general beams and does not require use of the thickness small parameter h/a. There are two possibilities for using this parameter:
   (A) Consider a general beam, minimize the energy with respect to w_i and thus, reduce 3-D dimensional problem to 1-D by solving a 2-D problem over the cross-section at each step of the asymptotic procedure. In this situation the parameter h/a can be used each time the 2-D problem is solved.
   (B) Take advantage of classical plate theory from the very beginning, and apply the asymptotic procedure to an effectively 2-D problem.

As expected, both methods lead to identical results for some benchmark problems, thus confirming validity of the second approach. This approach is much more convenient from the computational point of view and, therefore, was used in this paper.

4. Torsion of isotropic strips

As a starting point isotropic strips are considered. Some results of this section will be used in the following derivation for I-beams. The system of coordinates has $x_1 \equiv x$, $x_2 \equiv y$, and $x_3 \equiv z$ as the coordinates along the beam, along the width of the strip, and through the thickness, respectively. The cross-sectional dimensions are denoted as h and a, where h is the constant thickness and a the width of the strip, respectively; $(h/a) \ll 1$. The presence of the small parameter h/a allows us to consider the strip as a plate and apply classical plate theory (with in-plane coordinates x and y, and out-of-plane coordinate z).

Then the strain energy of the strip can be expressed in terms of plate displacements v_i. The expression for plate strain energy per unit area in eqns (21) reduces for isotropic plates to

$$\mathscr{E}_{\text{plate}} = \mu h(\sigma A_{\gamma\gamma}^2 + A_{\gamma\delta}A_{\gamma\delta}) + \frac{\mu h^3}{12}(\sigma B_{\gamma\gamma}^2 + B_{\gamma\delta}B_{\gamma\delta}) \tag{33}$$

where

$$\sigma \equiv \lambda/(\lambda + 2\mu) \tag{34}$$

and where μ and λ are the Lamé constants, and strain measures $A_{\gamma\delta}$ and $B_{\gamma\delta}$ are given by eqns (22). 3-D displacements can be recovered by use of

$$u_\gamma = v_\gamma - z v_{3,\gamma}$$

$$u_3 = v_3 - \sigma z v_{\gamma,\gamma} + \frac{\sigma}{2}\left(z^2 - \frac{h^2}{12}\right) v_{3,\gamma\gamma} \tag{35}$$

4.1. *Preliminary steps*

Discarding all terms with the derivatives with respect to x_1 in eqn (33) we obtain the following general solution for the displacement field that minimizes eqn (33):

$$v_1 = U_1(x)$$

$$v_2 = U_2(x)$$

$$v_3 = U_3(x) + \theta(x) y \tag{36}$$

where U_1 is axial displacement due to extension, U_2 and U_3 are transverse displacements due to in-plane and out-of-plane bending, respectively, and θ is the section rotation due to torsion. These are the "classical" 1-D degrees of freedom.

Only the pure torsion case is considered, and thus we set $U_1 \equiv U_2 \equiv U_3 \equiv 0$. This case can be studied independently, since in the isotropic case torsion is neither coupled with bending nor extension. It could be easily checked that the second part of the preliminary procedure is not needed here, since the displacement field obtained already gives a contribution to the strain energy of needed order (i.e., in this case of order $\theta' h$). Note that for bending this is not the case (this will be demonstrated in the section for anisotropic I-beams).

One can recover 3-D displacements (leaving only the leading terms with respect to a/l) using eqns (35) to show that this displacement field indeed corresponds to the classic torsion:

$$u_1 = -yz\theta'$$

$$u_2 = -z\theta$$

$$u_3 = y\theta \tag{37}$$

4.2. *First- and second-order approximations*

The next step is to perturb the plate displacement field

$$v_1 = w_1$$

$$v_2 = w_2$$

$$v_3 = \theta y + w_3 \tag{38}$$

where w_α and $w \equiv w_3$ are the in-plane and out-of-plane perturbations, respectively, subject to constraints that eliminate rigid-body motion of the cross-section. Substituting this field into the expression for the strain energy, one can see that the problem splits into two separate ones:

(1) For the unknowns w_1 and w_2, this problem evidently leads us to the trivial solution $w_1 = w_2 = 0$, since there are no linear terms exciting these displacements.
(2) The perturbation w enters only into the bending measures, which can be rewritten as

$$hB_{11} = -h(\underset{(a/l)\varepsilon}{\theta'' y} + \underset{(a/l)^3\varepsilon}{w_{,11}})$$

$$hB_{12} = -h(\underset{\varepsilon}{\theta'} + \underset{(a/l)^2\varepsilon}{w_{,12}})$$

$$hB_{22} = -\underset{(a/l)\varepsilon}{hw_{,22}} \tag{39}$$

Bending strain measures are multiplied by h here and in the following derivation to make estimation of the orders in the energy more convenient. Here we assume that the magnitude of strains is $\varepsilon \approx \theta' h$, which allows us to calculate the order of θ. Then the orders in the strain measures are defined uniquely, as shown under each term. Orders of terms containing w are determined from retaining leading terms in strain energy. One comes from B_{22} and is proportional to $w_{,22}^2$; the other stems from B_{12} and is proportional to $\theta' w_{,12}$. To finally determine the order, we reckon that these two terms should be of the same order if w is to minimize the energy. It now becomes evident that the first-order approximation is effectively zero, and we are looking here for the second-order approximation (since w will give a contribution to the energy of order $(a/l)^2\varepsilon^2$).

Assuming that the perturbation w does not contribute to the rotation of the strip as a rigid body we obtain $\int_{-(a/2)}^{a/2} yw\,\mathrm{d}y = 0$. One can check by integration that this constraint can be rewritten as

$$\int_{-(a/2)}^{a/2} \left(y^2 - \frac{a^2}{4}\right) w_{,2}\,\mathrm{d}y = 0 \tag{40}$$

Introducing the Lagrange multiplier $\lambda_1 \theta''$ for the constraint, the integrand of the functional to be minimized in order to find w, is

$$\Phi_2 = (\sigma+1)w_{,22}^2 + 4\theta' w_{,12} + 2\sigma\theta'' y w_{,22} + \lambda_1 \theta'' \left(4\frac{y^2}{a^2} - 1\right) w_{,2} \tag{41}$$

Let us note, that the total strain energy of a strip is

$$\frac{\mu h^3}{12} \int_0^l (\Phi_1 + \Phi_2)\,\mathrm{d}x_1 \tag{42}$$

where Φ_1 contains all the terms without w. The integration by parts of eqns (41) will affect the boundary conditions of the resulting 1-D theory, but not the governing equation. Therefore, we can integrate the second term in the strain energy by parts with respect to the axial coordinate and denote $w_{,2} \equiv \psi\theta''$, so the expression for Φ_2 will become

$$\Phi_2 = \left[(\sigma+1)\psi_{,2}^2 - 4\psi + 2\sigma\psi_{,2}y + \lambda_1\left(4\frac{y^2}{a^2}-1\right)\psi\right]\theta''^2 \tag{43}$$

This leads to a differential equation

$$-2\psi_{,22}(\sigma+1) - 2\sigma - 4 + \lambda_1\left(4\frac{y^2}{a^2}-1\right) = 0 \tag{44}$$

with boundary conditions at $y = \pm(a/2)$ given as

$$[2\psi_{,2}(\sigma+1)+2\sigma y]|_{y=\pm(a/2)} = 0 \tag{45}$$

The solution of this problem will be $\lambda_1 = -6$ and

$$\psi_{,2} = \frac{1-\sigma}{1+\sigma}y - \frac{4y^3}{a^2(1+\sigma)} \tag{46}$$

Calculating the contribution to the energy of terms that contain w, one obtains

$$\min(2\Phi_2) = \int_{-(a/2)}^{a/2} (1+\sigma)w_{,22}^2\,\mathrm{d}y$$
$$= \theta''^2 a^3\left[\frac{2(9v-2)}{7\cdot 15} - \frac{\sigma^2}{(1+\sigma)\cdot 12}\right] \tag{47}$$

Adding the terms from the zeroth approximation we obtain the final expression for total 1-D energy per unit length for an isotropic strip:

$$2\mathscr{E}_{\text{refined}} = \mu(\theta'^2 J + \theta''^2\Gamma) \tag{48}$$

where

$$J = \frac{ah^3}{3}$$
$$\Gamma = \left[\frac{2(1+v)}{12^2} + \underline{\frac{(9v-2)}{6\cdot 7\cdot 15}}\right]a^3h^3 \tag{49}$$

and where J is the classical torsional rigidity. As this example shows, the terms with coefficient Γ contain the Vlasov term along with an additional correction term (underlined). This correction is of the same order as Vlasov's term with respect to h/a and may at first glance appear to be significant. However, the correction to Vlasov's term is insignificant from a practical point of view, since the entire term with coefficient Γ is of order h^3a^3/l^4, and is thus small compared to the classical term which of order h^3a/l^2.

5. Isotropic I-beams

Let us consider symmetric isotropic I-beams. The width of the web and flanges is denoted as a and b, respectively; and the constant thickness of both flanges and the web is h. As shown in Fig. 1, for each of the plates which comprises the I-beam we introduce local coordinates y_*, z_*, which originate in the middle of the member, where * can be r for the right flange, l—for the lower flange, w—for the web. We will also use these indices to indicate any quantities pertaining to a particular member. We consider the web as the ''base'' of the cross-section, so that the global coordinates coincide with the local web coordinates (so we can omit the index w). The goal is to express all relevant quantities for each plate in terms of those of the ''base'' member. This general approach is chosen because it is also valid for open sections that are more complicated than I-beams and consist of an arbitrary number of members.

For each member the procedure described in the previous section for the strip is repeated, so that the strain energy can be written in terms of membrane and bending measures given by eqn (33). The total strain energy will now consist of three parts:

$$\int_{-(a/2)}^{a/2} F^{\mathrm{w}}\,\mathrm{d}y + \int_{-(b/2)}^{b/2} F^{\mathrm{r}}\,\mathrm{d}y_{\mathrm{r}} + \int_{-(b/2)}^{b/2} F^{\mathrm{l}}\,\mathrm{d}y_{\mathrm{l}} \tag{50}$$

Membrane and bending measures [eqns (21)] for each member can be expressed in terms of displacements of these members, and by minimizing the energy we will find these displacements at each step of the asymptotic procedure. We require for the disturbance w_i^*, as well as $w_{3,2}^*$ to vanish in the middle of each member. This corresponds to elimination of four cross-sectional rigid-body motions for each member. Only the right flange will be considered, with the implication that the procedure is identical for the left flange. The flanges and web are rigidly connected, so we use the following matching conditions for the displacements at the junction:

$$\begin{aligned} v_1^{\mathrm{r}}|_{y_{\mathrm{r}}=0} &= v_1^{\mathrm{w}}|_{y=(a/2)} \\ v_2^{\mathrm{r}}|_{y_{\mathrm{r}}=0} &= -v_3^{\mathrm{w}}|_{y=(a/2)} \\ v_3^{\mathrm{r}}|_{y_{\mathrm{r}}=0} &= v_2^{\mathrm{w}}|_{y=(a/2)} \\ v_{3,2}^{\mathrm{r}}|_{y_{\mathrm{r}}=0} &= v_{3,2}^{\mathrm{w}}|_{y=(a/2)} \end{aligned} \tag{51}$$

5.1. *Preliminary steps*

For each member of the I-beam we can write expressions for the displacement field identical to those of a strip:

$$\begin{aligned} v_1 &= U_1^*(x) + w_1^* \\ v_2 &= U_2^*(x) + w_2^* \\ v_3 &= U_3^*(x) + \theta^*(x)y + w_3^* \end{aligned} \tag{52}$$

As in the case of the strip we can consider torsion separately, thus setting U_i^{w} to zero. Terms of order $(a/l)^{-1}$ do not exist, so no minimization is needed for the terms of that order.

5.2. *First-order approximation*

Using eqns (51) we can express the classical degrees of freedom of the flange displacement field in terms of those of the web (within the precision of the first-order approximation):

$$
\begin{aligned}
U_1^{\mathrm{r}} &= 0\\
U_2^{\mathrm{r}} &= -\frac{a}{2}\theta\\
U_3^{\mathrm{r}} &= 0\\
\theta^{\mathrm{r}} &= \theta
\end{aligned}
\tag{53}
$$

The first approximation for the web will still be zero as it was in the case of the strip. For the flanges, however, we have a slightly different situation. The displacement field for the right flange is given as

$$
\begin{aligned}
v_1^{\mathrm{r}} &= w_1^{\mathrm{r}}\\
v_2^{\mathrm{r}} &= -\frac{a}{2}\theta + w_2^{\mathrm{r}}\\
v_3^{\mathrm{r}} &= \theta y^{\mathrm{r}} + w_3^{\mathrm{r}}
\end{aligned}
\tag{54}
$$

Writing expressions for the membrane measures of the plate we obtain

$$
\begin{aligned}
A_{11}^{\mathrm{r}} &= \underbrace{w_{1,1}^{\mathrm{r}}}_{(a/l)\varepsilon}\\
2A_{12}^{\mathrm{r}} &= -\underbrace{\frac{a}{2}\theta'}_{\varepsilon} + \underbrace{w_{1,2}^{\mathrm{r}}}_{\varepsilon} + \underbrace{w_{2,1}^{\mathrm{r}}}_{(a/l)^2\varepsilon}\\
A_{22}^{\mathrm{r}} &= \underbrace{w_{2,2}^{\mathrm{r}}}_{(a/l)\varepsilon}
\end{aligned}
\tag{55}
$$

Minimization of the energy with respect to w_1^{r} and w_2^{r} dictates the order of each term, written under it. Since the only linear terms disturbing these two displacement fields will come from $(a/2)\theta'$ in $A_{1,2}^{\mathrm{r}}$, $(w_{1,2}^{\mathrm{r}})^2$ is of the same order as $w_{1,2}^{\mathrm{r}}(a/2)\theta'$ and $(w_{2,2}^{\mathrm{r}})^2$ is of the same order as $w_{2,1}^{\mathrm{r}}(a/2)\theta'$. There is no change to $B_{\gamma\delta}^{\mathrm{r}}$ in the first approximation compared to the expression for the strip.

Minimizing the energy with respect to w_i^{r} we obtain $w_2^{\mathrm{r}} = w_3^{\mathrm{r}} = 0$ and $w_{1,2}^{\mathrm{r}} = (a/2)\theta'$. Satisfying the constraint $w_1^{\mathrm{r}}(0) = 0$, we obtain $w_1^{\mathrm{r}} = \frac{1}{2}y^{\mathrm{r}}a\theta' \equiv \eta\theta'$ where η is a sectorial coordinate defined in eqns (15).

5.3. *Second-order approximation*

The web problem is almost identical to the strip problem of eqns (41), just with different boundary conditions and different value of the Lagrange multiplier. Its total contribution to the energy will be also of order h^3a^3.

So we can focus our attention at the flanges, conducting the derivation only for the right flange, and taking advantage of the fact that the procedure is similar for the left flange (with the obvious changes in the matching conditions). Strictly speaking, while using matching conditions eqns (51) we need to add in eqns (53) terms of the second order. That would result in addition to U_2^{r} a term proportional to θ', which would also result in terms of order h^3a^3. Terms of this order are beyond our scope of interest, so eqns (53) can still be employed. To distinguish the new disturbances from those of the previous step we will denote a new one with a tilde, so that

$$
\begin{aligned}
v_1^{\mathrm{r}} &= y^{\mathrm{r}}\frac{a}{2}\theta + \tilde{w}_1^{\mathrm{r}} \\
v_2^{\mathrm{r}} &= -\frac{a}{2}\theta + \tilde{w}_2^{\mathrm{r}} \\
v_3^{\mathrm{r}} &= \theta y^{\mathrm{r}} + \tilde{w}_3^{\mathrm{r}}
\end{aligned} \tag{56}
$$

As in the case of the strip and the web the $\tilde{w}_3^{\mathrm{r}}$ terms will result in a contribution of order h^3b^3 in the final expression for the energy; again, the problem is almost identical to the one for a strip, eqns (41). However, for $\tilde{w}_1^{\mathrm{r}}$, and $\tilde{w}_2^{\mathrm{r}}$ we obtain for the membrane plate measures

$$
\begin{aligned}
A_{11}^{\mathrm{r}} &= \underbrace{y^{\mathrm{r}}\frac{a}{2}\theta''}_{(a/l)\varepsilon} + \underbrace{\tilde{w}_{1,1}^{\mathrm{r}}}_{(a/l)^3\varepsilon} \\
2A_{12}^{\mathrm{r}} &= \underbrace{\tilde{w}_{1,2}^{\mathrm{r}}}_{(a/l)^2\varepsilon} + \underbrace{\tilde{w}_{2,1}^{\mathrm{r}}}_{(a/l)^2\varepsilon} \\
A_{22}^{\mathrm{r}} &= \underbrace{\tilde{w}_{2,2}^{\mathrm{r}}}_{(a/l)\varepsilon}
\end{aligned} \tag{57}
$$

Minimizing $\sigma(A_{\gamma\gamma}^{\mathrm{r}})^2 + A_{\gamma\delta}^{\mathrm{r}}A_{\gamma\delta}^{\mathrm{r}}$ we immediately obtain

$$
\begin{aligned}
\tilde{w}_{1,2} &= 0 \\
\tilde{w}_{2,2} &= -\frac{\sigma}{2(\sigma+1)}\theta'' ya
\end{aligned} \tag{58}
$$

So the total contribution to the energy $A_{\alpha\beta}$ will be

$$
\frac{a^2}{4}\mu h\theta''^2\left(\sigma+1-\frac{\sigma^2}{(\sigma+1)}\right)\int_{-(b/2)}^{b/2} y^2\,\mathrm{d}y = b^3a^2h\frac{E}{48}\frac{\theta''^2}{2} \tag{59}
$$

which is precisely Vlasov's term.

Therefore, for an I-beam, where the nonclassical effects become pronounced, Vlasov's term is of order b^3a^2h and dominates the correction to Vlasov's term, which is of order $h^3(a^3+b^3)$ and thus, can be neglected. This provides a solid theoretical foundation for the validity of Vlasov's theory for isotropic I-beams: Vlasov's theory is asymptotically correct to the second-order with respect to a/l if only the leading terms with respect to h/a are kept.

6. Anisotropic I-beams

The most general case is considered, i.e., when the magnitude of strains due to extension, bending and torsion are of the same order $U_1' \approx aU_2'' \approx aU_3'' \approx h\theta' \approx \varepsilon$. Note that here the width of the flanges, b, is assumed to be of the same order as the height of the web, a.

6.1. *Preliminary steps*

Now the plate energy density will be given by eqns (19). The first step will be identical to that for isotropic I-beams, resulting in eqns (52). However, unlike the torsion of isotropic I-beams, there are some terms of order $(a/l)^{-1}$ which have to be eliminated. If we disturb the original displacement field, eqns (52), there will be only one strain measure which has terms of that order, namely

$$2A_{12}^* = \underset{(a/l)^{-1}\varepsilon}{U_2'} + \underset{?}{\hat{w}_{1,2}^*} + \underset{?}{\hat{w}_{2,1}^*} \tag{60}$$

where the hats (^) refer to disturbances of the displacements associated with the preliminary step. It is evident, that the only way to eliminate this large term is to set $\hat{w}_{1,2}^* = -U_2'^*$ (note, that the third term in the right hand side of eqn (60) cannot be of the order $(a/l)^{-1}$ because that would imply a term of order $(a/l)^{-2}$ in A_{22}^*). Using the constraint $\hat{w}_1^*(0) = 0$, we obtain $\hat{w}_1^* = -U_2'^* y^*$; $\hat{w}_2^* = \hat{w}_3^* = 0$. It needs to be noted that the preliminary steps are actually independent of material properties; we did not encounter this term for isotropic I-beams only because we were able to limit our consideration to torsion since it is uncoupled from bending and extension.

6.2. *First-order approximation*

We now perturb the above displacement field so that the plate measures for each member now are

$$A_{11}^* = \underset{\varepsilon}{\alpha_1^*} - \underset{\varepsilon}{y^*\alpha_2^*} + \underset{(a/l)\varepsilon}{w_{1,1}^*}$$

$$2A_{12}^* = \underset{\varepsilon}{w_{1,2}^*} + \underset{(a/l)\varepsilon}{w_{2,1}^*}$$

$$A_{22}^* = \underset{\varepsilon}{w_{2,2}^*}$$

$$hB_{11}^* = -h(\underset{(h/a)\varepsilon}{\alpha_3^*} + \underset{(a/l)\varepsilon}{\alpha_4'^* y^*} + \underset{(a/l)^2\varepsilon}{w_{3,11}^*})$$

$$hB^*_{12} = \underset{\varepsilon}{-h\alpha^*_4} + \underset{(a/l)\varepsilon}{w^*_{3,12}}$$

$$hB^*_{22} = \underset{\varepsilon}{-hw^*_{3,22}} \tag{61}$$

where α^*_i are given by eqns (19).

In accordance with the variational-asymptotic method, we need to find the w^*_i in terms of α^*_i by substituting the strain measures, eqns (61), into eqns (20) and minimizing the resulting expression, leaving only the leading terms with respect to parameter a/l. We will also take advantage of the small parameter h/a. Orders of each term in eqns (61) are calculated by evaluating the order of all possible combinations of terms in the strain energy. Let us repeat that orders of perturbations are defined uniquely, once the orders of α^*_i are given. Since the leading terms do not include the derivative of w^*_i along the contour, minimization can be conducted algebraically and independently for each point of the contour. It is convenient to introduce matrix notation. Known quantities are:

$$H^* \equiv \begin{Bmatrix} \bar{A}^*_{11} \\ \bar{B}^*_{12} \end{Bmatrix} = \begin{Bmatrix} \alpha^*_1 - y^*\alpha^*_2 \\ -h\alpha^*_4 \end{Bmatrix} \tag{62}$$

where the bar refers to the main part of the strain measure in an asymptotic sense, i.e., terms of order ε.

Let us note that $\bar{B}^*_{11}$ is also known but can be neglected since it is of order $(h/a)\varepsilon$. Minimization with respect to w^*_i requires expression A^*_{12}, A^*_{22}, B^*_{22} in terms of those known quantities:

$$\begin{Bmatrix} w^*_{1,2} \\ w^*_{2,2} \\ w^*_{3,22} \end{Bmatrix} \equiv \begin{Bmatrix} 2\bar{A}^*_{12} \\ \bar{A}^*_{22} \\ \bar{B}^*_{22} \end{Bmatrix} = -R^{*-1}S^*H^* \tag{63}$$

where matrices of material coefficients S^* and R^* are given by eqns (26) and (27), respectively. Substituting eqns (63) into the plate strain energy per unit area, eqn (21), and denoting

$$Q^* = (\bar{Q}^* - S^*R^{*-1}S^{T*}) \tag{64}$$

where $\bar{Q}^*$ is given in eqn (25), we can rewrite it as

$$\mathscr{E}_{\text{plate}} = \tfrac{1}{2}Q_{11}A^2_{11} + Q_{12}A_{11}B_{12} + \tfrac{1}{2}Q_{22}B^2_{12} \tag{65}$$

If H^* is expressed in terms of the chosen 1-D strain measures α for the beam, so that

$$H^* = T^*\alpha \tag{66}$$

then the final expression for the strain energy per unit length will be given by

$$2\mathscr{E}_{\text{classical}} = \int_{\text{r+l+w}} H^{*T}Q^*H^*\,\mathrm{d}y \tag{67}$$

where integration is carried out over all members of the I-beam. While T^{w} (web) can be immediately obtained from eqn (63), which yields eqns (29), the matching conditions, eqns (51), are required to express α^{r} in terms of $\alpha^{\mathrm{w}} \equiv \alpha$ in order to obtain T^{r}.

Up to terms of first order, where terms are multiplied by the corresponding characteristic dimension to make them of the same order as they contribute to the strains, we now have

$$\begin{aligned}
\alpha_1^{\mathrm{r}} &\equiv U_1^{\mathrm{r}\prime} = \underline{U_1' - \tfrac{1}{2} a U_2''} + \tfrac{1}{2} a w_{1,1} \\
&\qquad\quad \varepsilon \qquad\quad (a/l)\varepsilon \\
a\alpha_2^{\mathrm{r}} &\equiv a U_2^{\mathrm{r}\prime\prime} = -\underline{a U_3''} - \underline{\underline{\tfrac{1}{2} a^2 \theta''}} - \tfrac{1}{2} a^2 w_{3,11} \\
&\qquad\quad \varepsilon \quad (a/h)(a/l)\varepsilon \quad (a/h)(a/l)^2\varepsilon \\
h\alpha_3^{\mathrm{r}} &\equiv h U_3^{\mathrm{r}\prime\prime} = \underline{h U_2''} + h w_{2,11}(a/2) \\
&\qquad\quad (h/a)\varepsilon \quad (h/l)(a/l)\varepsilon \\
h\alpha_4^{\mathrm{r}} &\equiv h\theta^{\mathrm{r}\prime} = \underline{h\theta'} + h w_{3,12}(a/2) \\
&\qquad\quad \varepsilon \qquad (a/l)\varepsilon
\end{aligned} \tag{68}$$

Only the underlined terms will contribute to the leading terms in energy, and therefore only those terms are needed to provide eqns (66). The explicit expressions for T^{r} and T^{t} are given by eqns (30) and eqns (31). Substituting these expressions into the strain energy density we obtain the classical energy coefficients C_{bc} from eqn (18) as given by eqn (28).

This derivation yields results identical to those obtained by the procedure outlined by Reissner and Tsai (1972) where equilibrium equations are used, and where N_{ss}, N_{zs} and M_{ss} are assumed to be negligibly small. This results in partial inversion of the 6×6 matrix of 2-D material constants from classical laminated plate theory.

6.3. *Second-order approximation*

In a situation similar to the isotropic case, the contribution to the second-order terms from the web will be of order h^3a^3. For the flanges, however, the double underlined term for α_2 in eqns (68) will induce dominant terms in the correction since there is an inverse of another small parameter h/a which distinguishes this term from all the others.

The following analysis pertains to the flanges only. To distinguish from the disturbances of the first order, we denote the new disturbances by $\tilde{w}_i^{\mathrm{r}}$. The plate strain measures are now

$$\begin{aligned}
A_{11}^{\mathrm{r}} &= \bar{\alpha}_1^{\mathrm{r}} - y^{\mathrm{r}} \bar{\alpha}_2^{\mathrm{r}} + \underline{y \frac{a}{2} \theta''} + w_{1,1}^{\mathrm{r}} + \tilde{w}_{1,1}^{\mathrm{r}} \\
&\qquad \varepsilon \quad \varepsilon \quad (a/h)(a/l)\varepsilon \quad (a/l)\varepsilon \quad (a/l)^2\varepsilon \\
2A_{12}^{\mathrm{r}} &= w_{2,1}^{\mathrm{r}} + w_{2,1}^{\mathrm{r}} + \underline{\tilde{w}_{1,2}^{\mathrm{r}}} + \tilde{w}_{2,1}^{\mathrm{r}} \\
&\qquad \varepsilon \quad (a/l)\varepsilon \quad (a/h)(a/l)\varepsilon \quad (a/l)^2\varepsilon
\end{aligned}$$

$$A^{\mathrm{r}}_{22} = w^{\mathrm{r}}_{2,2} + \underline{\tilde{w}^{\mathrm{r}}_{2,2}}$$

$$\varepsilon \quad (a/h)(a/l)\varepsilon$$

$$hB^{\mathrm{r}}_{11} = -h(\bar{\alpha}^{\mathrm{r}}_3 + \bar{\alpha}^{\mathrm{r}\prime}_4 y + w^{\mathrm{r}}_{3,11})$$

$$(h/a)\varepsilon \quad (a/l)\varepsilon \quad (a/l)^2\varepsilon$$

$$hB^{\mathrm{r}}_{12} = -h(\bar{\alpha}^{\mathrm{r}}_4 + w^{\mathrm{r}}_{3,12} + \tilde{w}^{\mathrm{r}}_{3,12})$$

$$\varepsilon \quad (a/l)\varepsilon \quad (a/l)^2\varepsilon$$

$$hB^{\mathrm{r}}_{22} = -h(w^{\mathrm{r}}_{3,22} + \underline{\tilde{w}^{\mathrm{r}}_{3,22}})$$

$$\varepsilon \quad (a/h)(a/l)\varepsilon \tag{69}$$

Here the over-bars are used to denote the "main" part (i.e., of order ε) of the α_i from eqns (68), and the double underlined term is written out explicitly. None of the other terms from eqns (68) contribute to the leading terms of the energy and therefore are neglected. Leaving only the leading terms (underlined) with respect to h/a in eqns (69) we can calculate terms of order $(a/l)^2\varepsilon^2$. Minimization in order to find $\tilde{w}^{\mathrm{r}}_i$ is identical to the determination of w^{r}_i with the one simplifying difference being that the only "driving" term for the disturbances will be A^{r}_{11}, so that

$$C_{55} = \frac{1}{4}\int_{\mathrm{r}+1} y^2 a^2 Q_{11}\,\mathrm{d}y \tag{70}$$

This corresponds to the only non-zero term M_{44} in eqns (17). The physical meaning of the obtained material coefficient Q_{11} corresponds to the effective (averaged through the thickness of the wall) Young's modulus in the axial direction. The contributions to this term from $\tilde{w}_{2,2}$, $\tilde{w}_{2,2}$, and $\tilde{w}_{3,22}$ will be of the order a^2b^3h, the same as the main term, and thus cannot be disregarded. This implies that the commonly invoked assumption of the cross-sections of thin-walled beams being rigid in their own planes is incorrect.

In addition we will have some terms of order $(a/l)\varepsilon^2$, which correspond to cross terms between the dominant term in A_{11} and classical terms giving the only non-zero terms L_{b4} from in eqns (2):

$$C_{b5} = \frac{1}{2}\int_{\mathrm{r}+1} yaQ_{1\gamma}T_{\gamma b}\,\mathrm{d}y \tag{71}$$

Note that the $\tilde{w}_i$ do not contribute to this cross-term due to the nature of the variational-asymptotic procedure (since the first approximation itself was obtained as a result of minimization procedure at the previous step). Equations (70) and (71) can be written in the form of eqns (32) by using the sectorial coordinate.

7. Comparing theories

First let us compare decay rates given by the 3-D computer code of Volovoi et al. (1995), Volovoi et al. (1998) and Vlasov theory as described in Vlasov (1961) for the isotropic case. We consider $v = 0.42$ and $a/b = 2$. Figure 5 shows prediction of the decay rate as a function of thickness, given by asymptotic analysis and 3-D code. As expected, the difference between the two diminishes as we decrease the thickness (and thus, decrease the decay rate, so that assumption of long waves becomes more valid). Let us note that as $\Im(bk)$ tends to zero, precision of the numerical method deteriorates due to ill-conditioning, but the trend within the range of reliable results is clear.

Next, it is interesting to compare the decay rate as predicted by asymptotic theory and numerical results for different values of a/b and h/b. Figure 6 presents a contour plot of the decay rate for I-beams with both short and long flanges. Figure 7 gives the percentage error for the decay rate. In both cases the thickness is normalized with respect to the larger of the two cross-sectional dimensions. While it is true that, as noted previously, for a given a/b ratio the difference between two predictions decreases together with thickness, another trend is apparent from these results: the correlation is much better for I-beams with short flanges; indeed, one has to be careful in applying Vlasov's theory for I-beams with long flanges, unless they are very thin-walled, whereas for beams with short flanges Vlasov's theory provides excellent results even for relatively thick walls.

As an example of anisotropic I-beams we consider the case exhibiting bending–torsion coupling which was studied quite extensively (see Chandra and Chopra, 1991; Badir et al. 1993), since some experimental data were available as well as numerical results. Its symmetric cross-section is made from graphite-epoxy material and had a $[0°/90°]_4$ lay-up in the web and a $[(0°/90°)_3/(\theta°)_2]$. The angle θ of the two top plies for both top and bottom flanges is the varying parameter (see Fig. 8). To validate the asymptotic procedure h/b was also varied, while $a/b = 0.5$ is kept constant, and predictions for the decay rate were compared to the numerical 3-D results. Note that in accordance with the asymptotic theory the rate of the decay varies linearly with the thickness, so it is convenient to normalize the decay rate with respect to the decay rate of some reference thickness (we have chosen $h = 0.04b$ as such reference point, because that was the thickness of the beam studied by

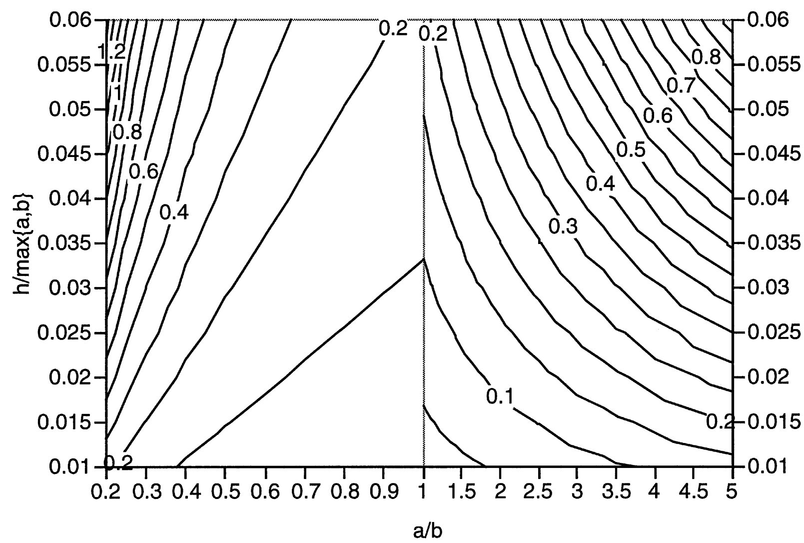

Fig. 6. Lines of constant decay rate for isotropic I-beams from 3-D analysis.

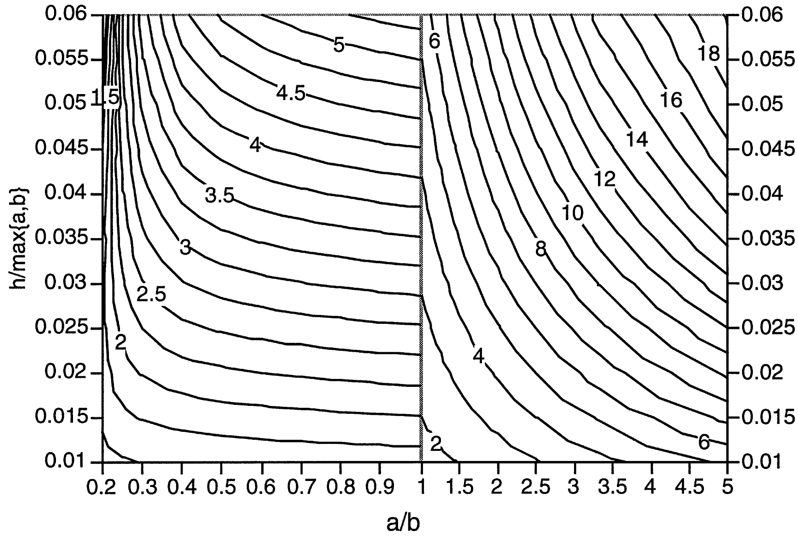

Fig. 7. Lines of constant percentage difference between the decay rate for isotropic I-beams from asymptotic theory and 3-D analysis.

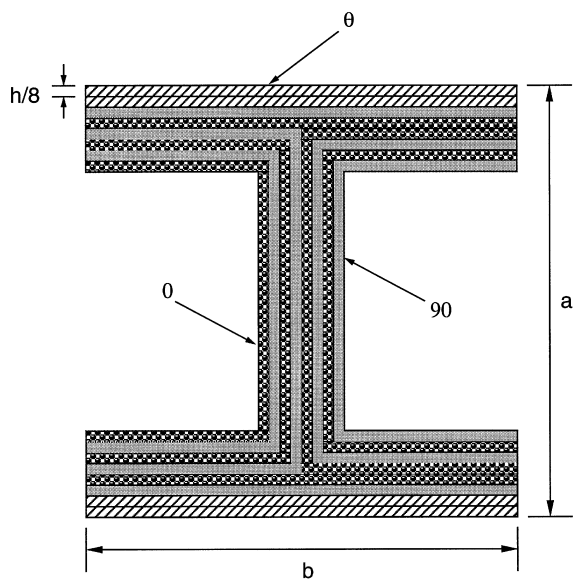

Fig. 8. Lay-up of anisotropic I-beams.

the cited references). After this normalization all asymptotic curves will collapse into one (Fig. 9). As expected correlation is the best for low h/b ratios, and the difference between asymptotic and 3-D results is indeed of order h/b. It is interesting to notice the decreasing sensitivity of the decay rate with respect to the varying ply angle as the thickness increases.

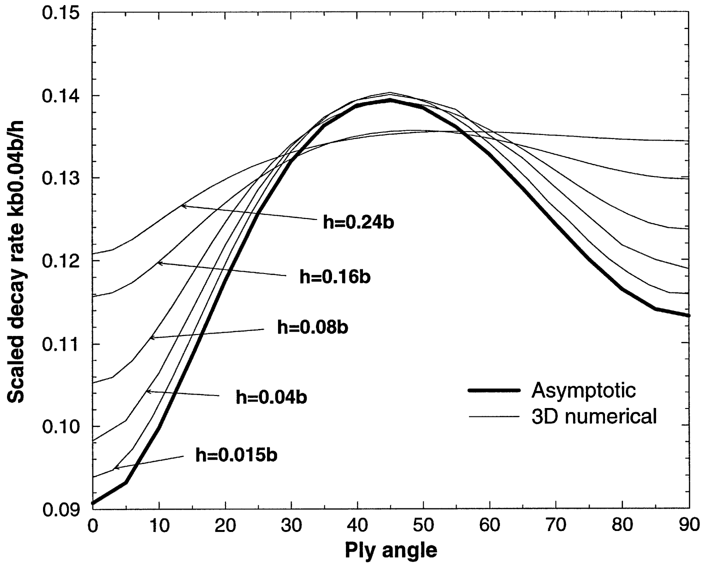

Fig. 9. Decay rate for anisotropic I-beams.

Let us now compare the asymptotic solution with the analytical solutions provided in Chandra and Chopra (1991) and Badir et al. (1993). The first method assumed a rigid cross-section and started from 2-D equilibrium equations. The second, however, provided more general treatment of thin-walled beams where, similar to the present approach, the variational-asymptotic method was employed. Unfortunately, a miscalculation of the orders of some terms was made there. The numerical consequence of this error is insignificant for most material properties; however, for certain lay-ups the difference can be noticeable, as shown below. It should also be mentioned that the variational-asymptotic procedure was carried out in Badir et al. (1993) only to recover the terms of the first-order, and Vlasov's term was included in an *ad hoc* manner. No rigorous evaluation of other terms of the order $(a/l)^2\varepsilon^2$ was conducted.

While all three methods differ in their respective approaches, it is possible (and in fact, quite convenient) to pin-point the source of the quantitative differences in terms of notations used in the present paper. Effectively, in Chandra and Chopra (1991) the second term in eqn (24) is neglected, so that $Q_{\gamma\delta} = \bar{Q}_{\gamma\delta}$. The difference between the present asymptotically correct theory and that given in Badir et al. (1993) is more subtle: while calculating $Q_{\gamma\delta}$ by using eqn (24), the importance of the terms related to B_{22} (the bending strain measure in the contour direction) was neglected, which resulted in crossing out the last column in the matrix S [eqn (26)], and both last column and last row in the matrix R [eqn (27)].

Figure 10 demonstrates the decay rate for $h = 0.015b$ as predicted by the three theories. It is interesting to see how the difference in the predicted decay rate will influence the results of 1-D theory for specific boundary conditions. For all considered examples one end is clamped, and the warping is restrained at the free end. Results for the variation of induced tip twist under tip unit vertical shear load are shown in Fig. 11. The results labelled "Gandhi and Lee" are taken from Gandhi and Lee (1992), generated therein by use of the 3-D code described in Stemple and Lee

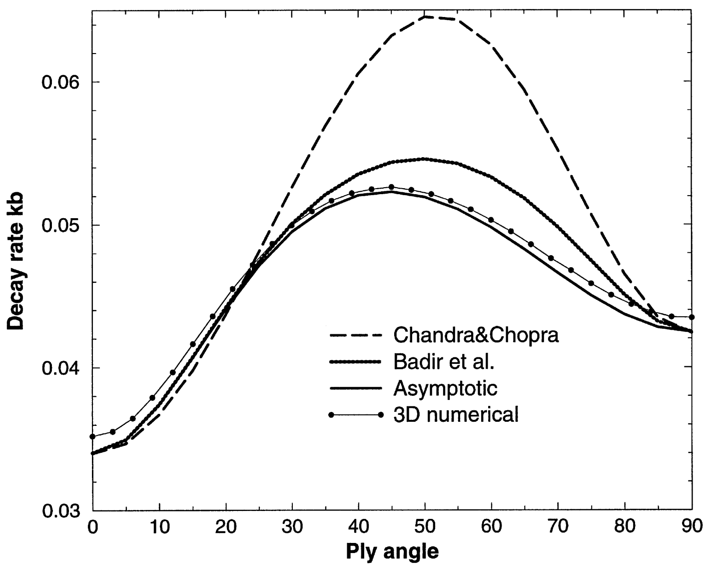

Fig. 10. Decay rate for anisotropic I-beams from different analytical theories ($h = 0.015b$).

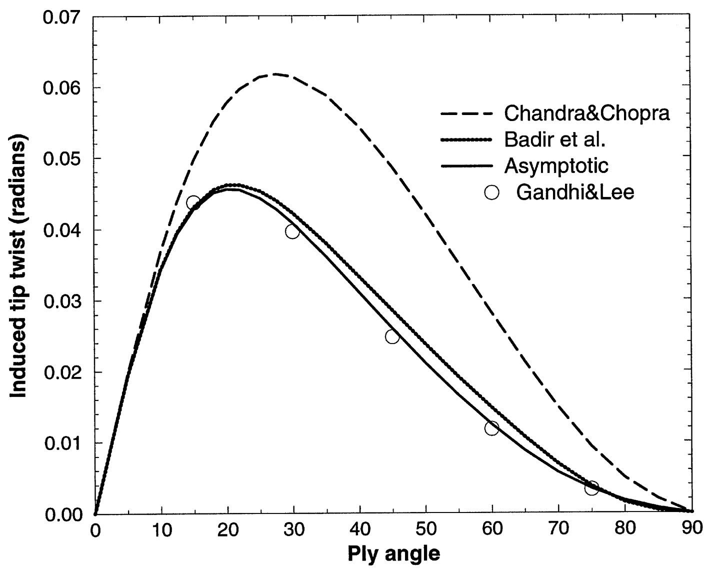

Fig. 11. Induced tip twist for a unit tip shear load ($l = 36''$).

(1988). Results for tip twist under a unit tip torsional load are given in Fig. 12. It can be observed by studying Figs 10–12 that the decay rate is quite sensitive to configuration parameters, and differences in the predicted decay rate strongly correlate with differences in the predictions of the resulting 1-D theories. Experimental results reported in Chandra and Chopra (1991) mostly pertain

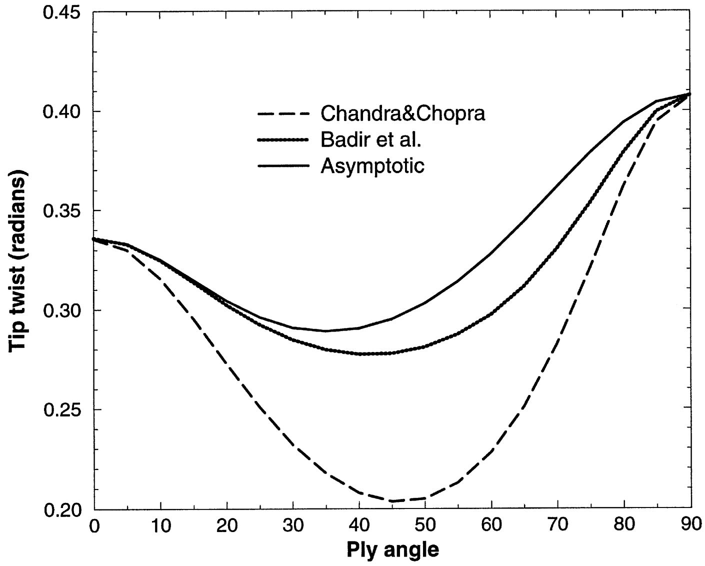

Fig. 12. Tip twist for a torsional tip unit load ($l = 36''$).

to the cases of $\theta = 0^\circ$ or $\theta = 90^\circ$ where all three theory give identical predictions. However, the difference becomes quite significant for other ply angles, especially in the range $30^\circ < \theta < 70^\circ$.

It has to be mentioned that any prediction which has a relative error less than the order of h/b is but a mere coincidence, since the latter is the magnitude of an error intrinsic to the asymptotic procedure.

8. Conclusions

This paper has presented an asymptotic treatment of the statics of thin-walled, anisotropic beams. The following conclusions have been reached:

(1) Classical laminated plate theory has been shown to be a suitable starting point for development of a theory for describing end-effects in thin-walled I-beams. This results in a simpler development than would be obtained were the theory started from 3-D elasticity or refined shell theory, with no effect on the resulting theory as long as $(h/a)^2 \ll 1$.

(2) An asymptotic verification of Vlasov's theory is presented for isotropic I-beams. It is shown to be asymptotically incorrect to assume that the cross-section is rigid in its own plane. One cannot consistently neglect both stresses and strains associated with local plate membrane and bending effects. The asymptotically correct theory can be derived by neglecting stresses. This is especially important for asymptotically correct recovery of 3-D field variables.

(3) Vlasov's theory is consistently extended to anisotropic beams. As in the isotropic case, the asymptotically correct theory can only be derived by neglecting stresses associated with local plate membrane and bending effects.

(4) A theory in which asymptotic terms of second-order in a/l are retained is capable of describing

end-effects in beams for slowly decaying disturbances. If the decay rate is large, the theory ceases to be valid; but this is a moot point, since end-effects are not significant in that case anyway

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