



# Non-linear inplane deformation and buckling of rings and high arches

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## Abstract

A non-linear theory is presented for stretching and inplane-bending of isotropic beams which have constant initial curvature and lie in their plane of symmetry. For the kinematics, the geometrically exact one-dimensional (1-D) measures of deformation are specialized for small strain. The 1-D constitutive law is developed in terms of these measures via an asymptotically correct dimensional reduction of the geometrically non-linear 3-D elasticity under the assumptions of comparable magnitudes of initial radius of curvature and wavelength of deformation, small strain, and small ratio of cross-sectional diameter to initial radius of curvature ( $h/R$ ). The 1-D constitutive law contains an asymptotically correct refinement of  $O(h/R)$  beyond the usual stretching and bending strain energies which, for doubly symmetric cross sections, reduces to a stretch–bending elastic coupling term that depends on the initial radius of curvature and Poisson’s ratio. As illustrations, the theory is applied to inplane deformation and buckling of rings and high arches. In spite of a very simple final expression for the second variation of the total potential, it is shown that the only restriction on the validity of the buckling analysis is that the prebuckling strain remains small. Although the term added in the refined theory does not affect the buckling loads, it is shown that non-trivial prebuckling displacements, curvature, and bending moment of high arches are impossible to calculate accurately without this term. © 1999 Elsevier Science Ltd. All rights reserved.

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## 1. Introduction

The geometrically exact theory for analysis of large deflections of beams is well known; see Ref. [1] and the references cited therein. Elegant treatments for the planar case are presented in Refs [2, 3]. Relatively few applications of this type of

theory for stability analysis of initially curved beams appear in the literature; an analogous treatment for shell stability can be found in Ref. [4]. As noted by Ref. [3], this type of treatment also forms an excellent basis for postbuckling analysis. The intent here is to provide a geometrically exact theory for this purpose with a minimum of adhoc approximations. The theory developed herein is a special case of that which appears in Ref. [1], but it includes an asymptotic development of the 1-D constitutive law needed to have a complete theory. The asymptotic analysis works on the basis

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of small parameters related to the strain and the slenderness of the beam. Note: the case of shallow arches must be treated with a specialized theory, as pointed out by Simitse [5], because of the presence of an additional small parameter related to the ratio of beam’s length to its radius of initial curvature.

We will start with a geometric description of the undeformed and deformed states of the beam. This includes the position vectors to an arbitrary material point and definitions of the reference line and reference cross section in both states. Geometrically exact force and moment strain measures will be introduced, followed by an asymptotic reduction of the 3-D strain energy to 1-D.

Deformation and buckling of both rings and high arches will be considered for examples. For illustrating the approach, we consider a pressure loading which is a constant force per unit deformed length and acting perpendicular to the reference line of the deformed beam. This is the closest representation of hydrostatic pressure. Although it is a follower force, we will prove that for many practical cases it is conservative, having a potential, in accordance with Ref. [6].

## 2. 1-D strain energy

To form the strain energy of a planar, constant-curvature beam, we develop the geometries of both undeformed and deformed states. The beam is symmetric about the plane in which it is initially curved, and its displacement field is symmetric about that plane. We then make use of the variational-asymptotic method to reduce the 3-D strain energy to a 1-D functional for initially curved beams. This functional depends only on the geometrically exact stretching and bending measures, which we specialize for the case of small strain.

### 2.1. Undeformed state

Consider an initially curved beam with radius of curvature  $R$  in its undeformed state. The undeformed beam reference line (the line of area

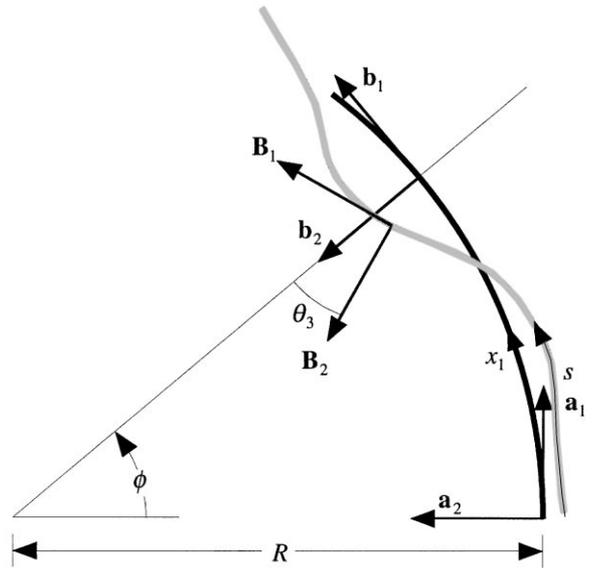


Fig. 1. Schematic of undeformed and deformed beam.

centroids will suffice in this case) is shown as the dark, heavy line in Fig. 1. The position vector from some fixed point to an arbitrary point  $p$  on the beam reference line is denoted by  $\mathbf{r}(x_1)$ , where  $x_1 = R\phi$  is the arc-length coordinate along the undeformed beam reference line. Thus, we can write the position vector to a point in the undeformed beam as

$$\bar{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1) + x_2 \mathbf{b}_2(x_1) + x_3 \mathbf{b}_3, \quad (1)$$

where the undeformed beam base vectors  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are functions of  $x_1$  and where  $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 = \mathbf{a}_3$  is not. Spatially fixed base vectors are denoted by  $\mathbf{a}_i$ , for  $i = 1, 2$ , and  $3$ , as shown in Fig. 1; note also that  $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2$ . (Here and throughout the paper, Latin indices vary from 1 to 3, while Greek indices vary from 2 to 3. Repeated indices are summed over their ranges.)

The relationship between these vectors is seen from the geometry to be

$$\begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{Bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix}. \quad (2)$$

The unit vector tangent to the curve described by  $\mathbf{r}(x_1)$  is

$$\frac{d\mathbf{r}}{dx_1} = \mathbf{r}' = \mathbf{b}_1, \tag{3}$$

where  $(\prime) = d(\ )/dx_1$ . The curvature vector for the undeformed state is defined as  $\mathbf{k} = \mathbf{b}_3/R$ , so that

$$\mathbf{b}'_i = \mathbf{k} \times \mathbf{b}_i. \tag{4}$$

The initial curvature then is exhibited, as expected, in

$$\mathbf{b}'_1 = \frac{\mathbf{b}_2}{R}, \quad \mathbf{b}'_2 = -\frac{\mathbf{b}_1}{R}. \tag{5}$$

Let us define an operator  $\langle(\bullet)\rangle$  to be the integral over the cross-sectional plane of the undeformed beam of any quantity  $(\bullet)$ , or

$$\langle(\bullet)\rangle = \int_A \int (\bullet) dA \tag{6}$$

with  $A$  as the cross-sectional area. Integrating the position vector of the undeformed beam over the cross section, we obtain

$$\langle\bar{\mathbf{r}}\rangle = A\mathbf{r} + \langle x_\alpha \rangle \mathbf{b}_\alpha. \tag{7}$$

Noting that the second term on the right-hand side vanishes if the area centroid is chosen to be the reference line location in any cross section, one finds that the vector  $\mathbf{r}$  is the average position of all points of the cross-sectional plane, viz.,

$$\frac{1}{A} \langle\bar{\mathbf{r}}\rangle \equiv \mathbf{r}. \tag{8}$$

### 2.2. Deformed state

The deformed state is a straightforward extension of the above. The position vector for the same material point in the deformed beam to which  $\bar{\mathbf{r}}$  points in the undeformed beam is

$$\begin{aligned} \bar{\mathbf{R}}(x_1, x_2, x_3) = & \mathbf{R}(x_1) + x_2\mathbf{B}_2(x_1) + x_3\mathbf{B}_3 \\ & + w_i(x_1, x_2, x_3)\mathbf{B}_i(x_1), \end{aligned} \tag{9}$$

where  $w_i(x_1, x_2, x_3)$  is the displacement of points in the reference cross-sectional plane relative to the rigid-body displacement and rotation reflected by

$\mathbf{R}(x_1)$  and  $\mathbf{B}_i(x_1)$ ; for planar deformation  $\mathbf{B}_3 = \mathbf{b}_3$ , and the curvature vector for the deformed state is  $\mathbf{K} = ((1/R) + \kappa)\mathbf{b}_3$ . In general,  $w_i$  describes both in- and out-of-plane warping of the material points which make up the reference cross-sectional plane of the undeformed beam. These functions are not known a priori; they must be calculated subject to constraints which remove redundant degrees of freedom.

In a manner similar to the above treatment of the undeformed state, for the deformed state we can write

$$\langle\bar{\mathbf{R}}\rangle = A\mathbf{R} + \langle x_\alpha \rangle \mathbf{B}_\alpha + \langle w_i \rangle \mathbf{B}_i. \tag{10}$$

Again, the second term on the right-hand side vanishes by the above choice of the reference line. By constraining the average value of the warping to be zero, so that

$$\langle w_i \rangle = 0, \tag{11}$$

the last term also vanishes leaving

$$\frac{1}{A} \langle\bar{\mathbf{R}}\rangle \equiv \mathbf{R}. \tag{12}$$

This means that  $\mathbf{R}$  is the average position of the points that make up the cross-sectional plane of the undeformed beam when the beam is in the deformed state. Letting  $\mathbf{R} = \mathbf{r} + \mathbf{u}$ , one then finds that  $\mathbf{u}$  is the vector from a point on the undeformed beam reference line to the corresponding point (i.e., at the same value of  $x_1$ ) of the deformed beam reference line. This implies that  $\mathbf{u}$  is not the displacement of some material point in the 3-D structure. Rather, it is the average displacement of all the points contained in the undeformed beam reference cross section, i.e.,

$$\mathbf{u} = \frac{1}{A} \langle\bar{\mathbf{R}} - \bar{\mathbf{r}}\rangle. \tag{13}$$

The above development implies that the warping is measured relative to a translated and rotated planar image of the undeformed beam cross-sectional plane. One specifies the rotation in accordance with the type of theory to be derived. To derive a theory of the “classical” type, which neglects transverse shear deformation, we require the cross-sectional plane of the deformed beam to be

normal to the tangent of the local deformed beam reference line, so that

$$\mathbf{B}_x \cdot \frac{d\mathbf{R}}{ds} = 0, \quad (14)$$

where  $s$  is the running arc-length along the deformed beam reference line. Since  $d\mathbf{R}/ds$  is a unit vector, we define the local stretching strain measure  $\varepsilon$  to be such that  $s' = 1 + \varepsilon$ . It then follows that

$$\mathbf{R}' = (1 + \varepsilon)\mathbf{B}_1. \quad (15)$$

To complete the specification of the rotation, the rotation of this plane about  $\mathbf{B}_1$  must be defined. Letting  $\bullet_{,x} = \partial(\bullet)/\partial x_x$ , we can choose this rotation to satisfy

$$\langle \bar{\mathbf{R}}_{,2} \cdot \mathbf{b}_3 - \bar{\mathbf{R}}_{,3} \cdot \mathbf{B}_2 \rangle = 0, \quad (16)$$

so that the final constraint on the warping is

$$\langle w_{2,3} - w_{3,2} \rangle = 0, \quad (17)$$

making a total of four constraints on the warping. Note that our choice of constraints is not unique, but it is necessary that the constraints render the displacement field unique.

The strain is now defined based on the simplest small-strain approximation given by Danielson and Hodges [7]. This choice is appropriate for large deflection analysis of isotropic beams with closed cross sections, since there are no restrictions on the magnitudes of reference line displacement or on cross-sectional rotation – only on the local rotation, which is related to the cross-sectional warping  $w_i$ . (Indeed, as long as the magnitude of the warping remains of the order of strain compared to the cross-sectional diameter, the analysis is suitable even for stretching and bending of isotropic beams with open cross sections, since the only component of warping that would be large in such cases is that due to torsion.) The matrix of strain components  $\Gamma$  in a local Cartesian frame can be expressed in terms of the matrix of deformation gradient components in mixed bases  $\chi$  as

$$\Gamma_{ij} = \frac{\chi_{ij} + \chi_{ji}}{2} - \delta_{ij}, \quad (18)$$

where

$$\chi_{ij} = \mathbf{B}_i \cdot \mathbf{G}_k \mathbf{p}_k \cdot \mathbf{b}_j, \quad \mathbf{g}_i = \frac{\partial \bar{\mathbf{r}}}{\partial x_i},$$

$$\mathbf{G}_i = \frac{\partial \bar{\mathbf{R}}}{\partial x_i}, \quad \mathbf{p}_i \cdot \mathbf{g}_j = \delta_{ij} \quad (19)$$

and  $\delta_{ij}$  is the Kronecker symbol. In the above, the  $\mathbf{g}_i$  are the covariant base vectors for the undeformed state,  $\mathbf{G}_i$  are the covariant base vectors for the deformed state, and  $\mathbf{p}_i$  are the contravariant base vectors for the undeformed state.

For the beam under consideration,

$$\mathbf{g}_1 = \sqrt{g} \mathbf{b}_1, \quad \mathbf{g}_x = \mathbf{b}_x,$$

$$\mathbf{p}_1 = \frac{\mathbf{b}_1}{\sqrt{g}}, \quad \mathbf{p}_x = \mathbf{b}_x, \quad (20)$$

where

$$\sqrt{g} = 1 - x_2/R$$

and

$$\mathbf{G}_1 = \left[ 1 + \varepsilon - x_2 \left( \frac{1}{R} + \kappa \right) + w'_1 \right] \mathbf{B}_1$$

$$+ \left( w'_2 + \frac{w_1}{R} \right) \mathbf{B}_2 + w'_3 \mathbf{B}_3,$$

$$\mathbf{G}_2 = w_{1,2} \mathbf{B}_1 + (1 + w_{2,2}) \mathbf{B}_2 + w_{3,2} \mathbf{B}_3,$$

$$\mathbf{G}_3 = w_{1,3} \mathbf{B}_1 + w_{2,3} \mathbf{B}_2 + (1 + w_{3,3}) \mathbf{B}_3. \quad (21)$$

The strain in a local Cartesian system parallel to  $\mathbf{b}_i$  is then

$$\Gamma_{11} = \frac{\varepsilon - x_2 \kappa + w'_1 - (w_2/R)}{\sqrt{g}},$$

$$2\Gamma_{12} = w_{1,2} + \frac{w'_2 + (w_1/R)}{\sqrt{g}},$$

$$2\Gamma_{13} = w_{1,3} + \frac{w'_3}{\sqrt{g}},$$

$$\Gamma_{22} = w_{2,2}, \quad 2\Gamma_{23} = w_{2,3} + w_{3,2},$$

$$\Gamma_{33} = w_{3,3}. \quad (22)$$

For cross sections other than those that are open and thin-walled, it can be shown [8] that the first

approximation of the warping is of the order  $h\varepsilon$ , where  $h^2$  is a constant of the order of the cross-sectional area, and  $\varepsilon = \max(|\varepsilon|, h|\kappa|)$  is the maximum strain. Now, assuming  $(\cdot)$  is the order of  $(\cdot)/R$  (which requires that the wavelength of the deformation, assumed here to be of the order of the beam length, and  $R$  are of the same order) and  $h^2/R^2 \ll 1$ , one can rewrite the strain terms through  $O(h\varepsilon/R)$  as

$$\begin{aligned} \Gamma_{11} &= \varepsilon - x_2\kappa + \underline{w'_1} + (\varepsilon - x_2\kappa) \frac{x_2}{R} - \frac{w_2}{R} \\ 2\Gamma_{12} &= w_{1,2} + \underline{w'_2} + \frac{w_1}{R}, \\ 2\Gamma_{13} &= w_{1,3} + \underline{w'_3}, \\ \Gamma_{22} &= w_{2,2}, \quad 2\Gamma_{23} = w_{2,3} + w_{3,2}, \\ \Gamma_{33} &= w_{3,3}, \end{aligned} \tag{23}$$

where the non-underlined terms are all  $O(\varepsilon)$  and the underlined ones are  $O(h\varepsilon/R)$ .

### 2.3. Dimensional reduction

In order to reduce the 3-D strain energy to a 1-D functional, we employ the variational-asymptotic method of Berdichevsky [9]. For a homogeneous, isotropic beam with Young’s modulus  $E$ , shear modulus  $G$ , and Poisson’s ratio  $\nu$ , twice the strain energy per unit length  $\bar{U}$  can be written as

$$\begin{aligned} 2\bar{U} &= E\langle \Gamma_{11}^2 \sqrt{g} \rangle + G\langle [(2\Gamma_{12})^2 + (2\Gamma_{13})^2 \\ &\quad + (2\Gamma_{23})^2] \sqrt{g} \rangle + \frac{E}{(1+\nu)(1-2\nu)} \\ &\quad \times \left\langle \left\{ \begin{matrix} \nu\Gamma_{11} + \Gamma_{22} \\ \nu\Gamma_{11} + \Gamma_{33} \end{matrix} \right\}^T \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \right. \\ &\quad \left. \times \left\{ \begin{matrix} \nu\Gamma_{11} + \Gamma_{22} \\ \nu\Gamma_{11} + \Gamma_{33} \end{matrix} \right\} \sqrt{g} \right\rangle. \end{aligned} \tag{24}$$

Considering only terms that are  $O(\varepsilon)$  in the strain for now, one can write the dominant terms of  $2\bar{U}$  as

$$\begin{aligned} 2\bar{U}_0 &= EA\varepsilon^2 + EI_3\kappa^2 + G\langle w_{1,2}^2 + w_{1,3}^2 \\ &\quad + (w_{2,3} + w_{3,2})^2 \rangle + \frac{E}{(1+\nu)(1-2\nu)} \end{aligned}$$

$$\begin{aligned} &\times \left\langle \left\{ \begin{matrix} \nu(\varepsilon - x_2\kappa) + w_{2,2} \\ \nu(\varepsilon - x_2\kappa) + w_{3,3} \end{matrix} \right\}^T \right. \\ &\quad \times \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \\ &\quad \left. \times \left\{ \begin{matrix} \nu(\varepsilon - x_2\kappa) + w_{2,2} \\ \nu(\varepsilon - x_2\kappa) + w_{3,3} \end{matrix} \right\} \right\rangle. \end{aligned} \tag{25}$$

The variational-asymptotic procedure requires that this expression be minimized by the warping, as constrained above in Eqs. (11) and (17). Carrying out this minimization in the usual manner of the calculus of variations, one can show that the following warping functions uniquely satisfy all constraints and minimize the approximate energy  $2\bar{U}_0$ :

$$\begin{aligned} w_1 &\equiv 0, \\ w_2 &= -\nu\varepsilon x_2 + \frac{\nu\kappa}{2} \left( \frac{I_2 - I_3}{A} + x_2^2 - x_3^2 \right), \\ w_3 &= -\nu\varepsilon x_3 + \nu\kappa x_2 x_3. \end{aligned} \tag{26}$$

The minimized value of  $2\bar{U}_0$  is then the familiar first approximation to the 1-D energy for planar deformation

$$2\bar{U}_0^* = EA\varepsilon^2 + EI_3\kappa^2. \tag{27}$$

Notice that no adhoc assumptions such as “the rigidity of the cross section in its own plane” or “ $\nu = 0$ ” were made. Such assumptions are neither necessary nor asymptotically correct.

In order to find the next approximation to the 1-D energy, the variational-asymptotic method requires that we perturb the above “classical” approximation of the warping, so that

$$\begin{aligned} w_1 &= v_1, \\ w_2 &= -\nu\varepsilon x_2 + \frac{\nu\kappa}{2} \left( \frac{I_2 - I_3}{A} + x_2^2 - x_3^2 \right) + v_2, \\ w_3 &= -\nu\varepsilon x_3 + \nu\kappa x_2 x_3 + v_3. \end{aligned} \tag{28}$$

We must substitute the perturbed warping into the strain expressions in Eq. (23), including the underlined terms, and in turn substitute those expressions into Eq. (24), thus retaining all terms in the 3-D energy up through  $O(h/R)$  relative to the leading terms.

Carrying out these operations, one finds that  $v_i$  is  $O(h\varepsilon/R)$  and that the next approximation of the 1-D energy, including terms that are  $O(h/R)$  relative to the leading terms of the 1-D energy found in Eq. (27), can be found without having to calculate  $v_i$ . The resulting strain energy density is

$$\begin{aligned}
 2\bar{U}_1^* &= EA\varepsilon^2 + EI_3\kappa^2 + E \left\langle \frac{x_2}{R} (\varepsilon - x_2\kappa)^2 \right. \\
 &\quad \left. - 2 \frac{w_2}{R} (\varepsilon - x_2\kappa) \right\rangle + O\left(\frac{EAh^2\varepsilon^2}{R^2}\right) \\
 &= EA\varepsilon^2 + EI_3\kappa^2 - 2 \frac{EI_3}{R} \varepsilon\kappa + 2\nu E \langle x_2 w_2 \rangle \\
 &\quad + O\left(\frac{EAh^2\varepsilon^2}{R^2}\right) \\
 &= EA\varepsilon^2 + EI_3\kappa^2 \\
 &\quad - \frac{E \langle x_2^3 \rangle (1 + \nu)}{R} \kappa^2 - \frac{2(1 + \nu)EI_3}{R} \varepsilon\kappa \\
 &\quad + O\left(\frac{EAh^2\varepsilon^2}{R^2}\right). \tag{29}
 \end{aligned}$$

Based on this approximate 1-D energy, the corresponding 1-D constitutive law is then

$$\begin{aligned}
 F &= \frac{\partial \bar{U}_1^*}{\partial \varepsilon} = EA\varepsilon - \frac{(1 + \nu)EI_3}{R} \kappa \\
 &\quad + O\left(\frac{EAh^2\varepsilon}{R^2}\right), \\
 M &= \frac{\partial \bar{U}_1^*}{\partial \kappa} = - \frac{(1 + \nu)EI_3}{R} \varepsilon \\
 &\quad + \left[ EI_3 - \frac{E \langle x_2^3 \rangle (1 + \nu)}{R} \right] \kappa \\
 &\quad + O\left(\frac{EAh^3\varepsilon^2}{R^2}\right), \tag{30}
 \end{aligned}$$

where  $F$  and  $M$  are the tangential force resultant and bending moment, respectively. The underlined terms are  $O(h/R)$  relative to the leading terms and represent (a) a slight change in the bending stiffness, which vanishes for doubly symmetric cross sections, and (b) stretching–bending coupling indicative of a shift in the position of the neutral axis

away from the area centroid. This result is in agreement with that of Ref. [10]. The only approximations in the dimensional reduction are thus  $\varepsilon \ll 1$  and  $h^2/R^2 \ll 1$ . Later, it will be shown that these conditions dovetail into one condition for ring- and high-arch-buckling problems. The next approximation would produce terms in the 1-D energy which are  $O(h^2/R^2)$  relative to the leading terms. These are associated with large initial curvature and transverse shear effects, not necessary in the present treatment.

Substituting Eq. (28) into Eq. (23), one can formally write the asymptotically correct 3-D strain field to  $O(h\varepsilon/R)$ . However, without calculation of  $v_i$  the present analysis only allows recovery of  $\Gamma_{11}$  to  $O(h\varepsilon/R)$ , while recovery of  $\Gamma_{1\alpha}$  and  $\Gamma_{\alpha\beta}$  is only possible to  $O(\varepsilon)$ . This means that stresses are only recoverable to  $O(E\varepsilon)$ . It should be noted that the moment stress resultant  $M$ , when calculated from the 3-D stress based on the approximate 3-D strain, is accurate to  $O(h/R)$  relative to the leading classical term and agrees with the result in Eq. (30). This is because the terms of the approximate 3-D stress which contain  $v_i$ , when multiplied by  $x_2$  and integrated over the cross-sectional area, are zero, since the average values of the warping perturbations  $v_i$  must also vanish. In contrast, only the leading term of the tangential force resultant, when calculated from the 3-D stress based on the approximate 3-D strain, is correctly obtained without knowledge of  $v_i$ .

#### 2.4. 1-D Strain–displacement relations

From the above, the unit tangent to the reference line of the deformed beam (see Fig. 1) is

$$\frac{d\mathbf{R}}{ds} = \mathbf{B}_1, \tag{31}$$

where  $\mathbf{B}_1$  is a unit vector tangent to the reference line at  $P$  and  $s$  is the arc-length coordinate along the deformed beam. By choosing a specific set of displacement variables, one can find the relationship between  $s$  and  $x_1$ .

Let  $\mathbf{u} = u_1\mathbf{b}_1 + u_2\mathbf{b}_2$ . This way,  $u_1$  is the “tangential” displacement and  $u_2$  is the “radial” displacement. Using Eq. (5) to express the derivatives of the

base vectors, one finds

$$\mathbf{B}_1 = \frac{d\mathbf{R}}{ds} = \frac{\mathbf{R}'}{s'} \\ = \frac{(1 + u'_1 - (u_2/R))\mathbf{b}_1 + (u'_2 + (u_1/R))\mathbf{b}_2}{s'}. \quad (32)$$

However, it is also clear from the restriction to planar deformation that one can regard the unit vectors  $\mathbf{B}_1$ ,  $\mathbf{B}_2$ , and  $\mathbf{B}_3 = \mathbf{B}_1 \times \mathbf{B}_2 = \mathbf{b}_3 = \mathbf{a}_3$  in terms of a simple rotation by an angle, say,  $\theta_3$  such that

$$\begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{pmatrix} = \begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix}. \quad (33)$$

Now, we can determine three things by comparing the expressions for  $\mathbf{B}_1$  in Eqs. (32) and (33). The first two are that  $\cos \theta_3$  and  $\sin \theta_3$  can be found as

$$\cos \theta_3 = \frac{1 + u'_1 - (u_2/R)}{s'}, \quad \sin \theta_3 = \frac{u'_2 + (u_1/R)}{s'} \quad (34)$$

and the third is that, since  $\mathbf{B}_1$  must be a unit vector

$$s' = \sqrt{\left(1 + u'_1 - \frac{u_2}{R}\right)^2 + \left(u'_2 + \frac{u_1}{R}\right)^2}. \quad (35)$$

Since  $s' = 1 + \varepsilon$ , for completeness we note that

$$\varepsilon = \sqrt{\left(1 + u'_1 - \frac{u_2}{R}\right)^2 + \left(u'_2 + \frac{u_1}{R}\right)^2} - 1. \quad (36)$$

Finally, the state of deformation of a beam is characterized by the bending energy, which depends on the curvature. Reissner’s theory for static deformation of beams [2] makes use of a “moment strain” which is  $\kappa = \theta'_3$  rather than  $d\theta_3/ds$ . These only differ by a factor of  $s' = 1 + \varepsilon$ , and for a small strain analysis one can ignore  $\varepsilon$  compared to unity (i.e., to set  $s'$  equal to unity) in the bending measure (but *not* in the stretching energy). Thus, we can develop  $\kappa = \theta'_3$  by differentiating the second of Eqs. (34), yielding

$$\theta'_3 \cos \theta_3 = \frac{s'(u''_2 + (u'_1/R)) - s''(u'_2 + (u_1/R))^2}{s'^2}. \quad (37)$$

Now using the first of Eq. (34) and noting that

$$s'' = \frac{(1 + u'_1 - (u_2/R))(u'_1 - (u'_2/R))}{s'} \\ + \frac{(u''_2 + (u'_1/R))(u'_2 + (u_1/R))}{s'}, \quad (38)$$

we find (after a remarkable series of cancellations!)

$$\theta'_3 = \frac{(1 + u'_1 - (u_2/R))(u''_2 + (u'_1/R))}{s'^2} \\ - \frac{(u'_2 + (u_1/R))(u''_1 + (u'_2/R))}{s'^2}, \quad (39)$$

which, when specialized for  $R \rightarrow \infty$ , is in agreement with Ref. [11]. We note that to be consistent with Hooke’s law, one must restrict  $\varepsilon = \max(\varepsilon, h\kappa)$  to be small compared to unity. Thus, for small strain we may regard  $s' = 1$  in the denominator of Eq. (39), yielding a polynomial in the displacement functions and their derivatives for the moment strain [3]

$$\kappa = \left(1 + u'_1 - \frac{u_2}{R}\right) \left(u''_2 + \frac{u'_1}{R}\right) \\ - \left(u'_2 + \frac{u_1}{R}\right) \left(u''_1 - \frac{u'_2}{R}\right). \quad (40)$$

Other than small stretching strain  $\varepsilon \ll 1$ , we have made no approximations in the 1-D variables. The differences between this expression for “curvature” and that found in calculus texts is discussed fully in Ref. [11]. An alternative approximation that is discussed in Ref. [11] for straight beams is to make use of the fact that the stretching strain is essentially zero in order to altogether eliminate  $u_1$  from moment strain (see also Ref. [12]). However, this approach cannot be used to eliminate  $u_1$  for an initially curved beam; it can be used only to write  $u'_1 - (u_2/R)$  in terms of  $u'_2 + (u_1/R)$  as done in Ref. [4]. This does not serve to eliminate a variable; instead it introduces an unnecessary mathematical singularity into the formulation along with an artificial limit on the “rotation” such that  $|u'_2 + (u_1/R)| < 1$ . The difference between this approach and the singularity-free one obtained from Eq. (40) used herein is of the order of the strain compared to unity.

### 2.5. Final 1-D strain energy

To simplify the notation somewhat, we write  $I = I_3$  and consider a doubly symmetric cross section (so that  $\langle x_2^3 \rangle = 0$ ). The 1-D strain energy is then the integral over the length of  $\bar{U}_1^{\mathcal{F}}$

$$U = \frac{1}{2} \int_{\mathcal{L}} \left[ EA\varepsilon^2 - \frac{2EI(1+\nu)}{R} \varepsilon\kappa + EI\kappa^2 \right] dx_1, \quad (41)$$

where  $\mathcal{L}$  is the total length of the beam. The underlined coupling term between  $\varepsilon$  and  $\kappa$  is due to the shift of the neutral surface away from the area centroid in a curved beam. As one can see, the strain energy density becomes quite complicated when Eqs. (36) and (40) are substituted into Eq. (41). There are many problems for which the result does become tractable, however, and for this reason this approach is to be preferred over adhoc approaches in which one cannot easily assess the error associated with particular approximations.

### 3. Potential energy of applied pressure loading

In anticipation of applying the above theory to inplane deformation and buckling, here we develop the potential energy by first finding the virtual work of the applied loading. Then we establish the criteria by which the virtual work can be represented as the variation of a functional, namely the negative of the potential energy.

#### 3.1. Virtual work of pressure

We consider the case of a distributed follower force that is a constant per unit deformed beam length. This means that the local force on an element of the deformed beam is, say,  $f_2 \mathbf{B}_2 ds$  where  $f_2$  is a constant. This force does the following work through a virtual displacement

$$\bar{\delta W} = \int_{\mathcal{L}} f_2 s' \mathbf{B}_2 \cdot (\delta u_1 \mathbf{b}_1 + \delta u_2 \mathbf{b}_2) dx_1, \quad (42)$$

where the  $\bar{\delta W}$  is the virtual work and the bar over the symbol indicates that it is not necessarily equal to the variation of a functional  $W$ . We already

know that  $\mathbf{B}_2 = -\sin \theta_3 \mathbf{b}_1 + \cos \theta_3 \mathbf{b}_2$  so that, from Eqs. (34), we have

$$\bar{\delta W} = f_2 \int_{\mathcal{L}} \left[ \left( 1 + u_1' - \frac{u_2}{R} \right) \delta u_2 - \left( u_2' + \frac{u_1}{R} \right) \delta u_1 \right] dx_1. \quad (43)$$

#### 3.2. Potential energy functional

For a beam of length  $\ell$ , this can now be put into the form

$$\bar{\delta W} = f_2 \delta \int_{\mathcal{L}} \left( u_2 - \frac{u_1^2}{2R} - \frac{u_2^2}{2R} - u_1 u_2' \right) dx_1 + u_1 \delta u_2 \Big|_0^{\ell}. \quad (44)$$

It is clear then that there are situations in which the trailing term vanishes which, in turn, allows the follower force to be derived from a potential function. Namely, this is the case if the ends of the beam are not allowed to displace, or if the beam is a closed ring, for which the ends are joined so that  $u_1(\ell) \delta u_2(\ell) = u_1(0) \delta u_2(0)$ ; for a discussion of this type of “holonomicity” see pp. 159–162 of Ref. [6]. In these cases, the potential energy functional is

$$V = -f_2 \int_{\mathcal{L}} \left( u_2 - \frac{u_1^2}{2R} - \frac{u_2^2}{2R} - u_1 u_2' \right) dx_1. \quad (45)$$

### 4. Applications

In-plane deformation and buckling of circular rings and high arches are considered as applications. A simple buckling analysis will be developed from the total potential energy, and the prebuckling deflections will be determined for cases in which they are not trivial.

To facilitate these analyses, it is now helpful to non-dimensionalize the equations. This we do by dividing through the total potential  $U + V$  by  $EAR$  while simultaneously changing the meaning of certain symbols. We replace  $u_1$  and  $u_2$  by  $Ru_1$  and  $Ru_2$ , respectively; we replace  $\kappa$  by  $\kappa/R$ ; and we finally let  $(\cdot)$  denote  $d(\cdot)/d\phi$ . We also introduce the new symbols  $\rho^2 = I/AR^2$  and  $\lambda = f_2 R^3/EI$ . All these

operations yield, for the non-dimensional total potential  $\Phi = (U + V)/EAR$

$$\Phi = \int_{-\alpha}^{\alpha} \left[ \frac{\varepsilon^2}{2} - (1 + \nu)\rho^2 \varepsilon \kappa + \frac{\rho^2 \kappa^2}{2} - \lambda \rho^2 \left( u_2 - \frac{u_1''}{2} - \frac{u_2''}{2} - u_1 u_2' \right) \right] d\phi, \quad (46)$$

where

$$\varepsilon = \sqrt{(1 + u_1' - u_2')^2 + (u_2'' + u_1'')^2} - 1 \quad (47)$$

and

$$\kappa = (1 + u_1' - u_2')(u_2'' + u_1'') - (u_2' + u_1')(u_1'' - u_2''). \quad (48)$$

Note that  $h/R = O(\rho)$ , so that  $\rho^2 \ll 1$ ; for a ring  $\alpha = \pi$ .

It is helpful, before proceeding further, to rewrite  $\kappa^2$  in a more compact way. To do so, we note that

$$\begin{aligned} \kappa^2 &= (1 + u_1' - u_2')^2 (u_2'' + u_1'')^2 \\ &\quad - 2(u_2' + u_1')(u_1'' - u_2'')(1 + u_1' - u_2')(u_2'' + u_1'') \\ &\quad + (u_2' + u_1')^2 (u_1'' - u_2'')^2 \end{aligned} \quad (49)$$

and that

$$(1 + \varepsilon)^2 = (1 + u_1' - u_2')^2 + (u_2'' + u_1'')^2. \quad (50)$$

Thus, Eq. (49) can be rearranged, making use of Eq. (50), to obtain

$$\begin{aligned} \kappa^2 &= [(1 + \varepsilon)^2 - (u_2' + u_1')^2] (u_2'' + u_1'')^2 \\ &\quad + [(1 + \varepsilon)^2 - (1 + u_1' - u_2')^2] (u_1'' - u_2'')^2 \\ &\quad - 2(u_2' + u_1')(u_1'' - u_2'')(1 + u_1' - u_2') \\ &\quad \times (u_2'' + u_1''), \end{aligned} \quad (51)$$

which, in light of the fact that  $\varepsilon' = s''$ , given in Eq. (38), simplifies to

$$\kappa^2 = (1 + \varepsilon)^2 [(u_2'' + u_1'')^2 + (u_1'' - u_2'')^2 - \varepsilon'^2]. \quad (52)$$

When  $\rho^2 \kappa^2$  is compared to  $\varepsilon^2$ , the last term in Eq. (52) becomes negligible because  $\rho^2 \ll 1$ . For small strain  $\kappa^2$  can finally be written as

$$\kappa^2 = (u_2'' + u_1'')^2 + (u_1'' - u_2'')^2. \quad (53)$$

### 4.1. Buckling of rings and high arches

For the first application we consider the buckling of rings and high arches. For the buckling analysis of high arches, we will follow the usual approach of assuming that the boundary conditions are such that the displacements in the prebuckled state are the same as those for a ring with the same values of  $\lambda$ ,  $\nu$ , and  $\rho$ . This has the effect of simplifying the analysis of the prebuckled state, but it does not affect the resulting bifurcation load.

#### 4.1.1. Prebuckled state

In the prebuckled state, we note that the ring remains circular so that all derivatives with respect to  $\phi$  vanish. Denoting the prebuckled state variables with overbars and noting that  $\bar{u}_2$  is the only non-zero variable, we find that  $\bar{\varepsilon} = -\bar{u}_2$ ,  $\bar{\kappa} = 0$ , and the functional reduces to

$$\bar{\Phi} = \int_{-\alpha}^{\alpha} \left[ \frac{\bar{u}_2^2}{2} - \lambda \rho^2 \left( \bar{u}_2 - \frac{\bar{u}_2''}{2} \right) \right] d\phi \quad (54)$$

from which we find, upon equating the variation to zero

$$\bar{u}_2 = \frac{\lambda \rho^2}{1 + \lambda \rho^2}. \quad (55)$$

Here, make an important observation: the strain in the prebuckled state

$$\bar{\varepsilon} = -\bar{u}_2 = -\frac{\lambda \rho^2}{1 + \lambda \rho^2} \quad (56)$$

is of the order of  $\rho^2$ . So, for a consistent small-strain analysis we need to ignore  $\rho^2$  with respect to unity. To improve on this analysis we would not only need to keep  $\rho^2$  compared to unity, we would also have to take transverse shear into account thereby improving on Eq. (41) so that  $\rho^2$  is not taken as small compared to unity in the dimensional reduction. This would entail the calculation of the perturbed warping,  $v_i$ , above, and would be much more complicated. Furthermore, to consistently keep  $\rho^2$  compared to unity would require the treatment of material non-linearities, such as retention of higher-order elastic constants of the material. Obviously, since the ring is slender and the prebuckling strain is small compared to unity, these

modifications are not necessary. This observation leads to a great simplification in the buckling analysis.

4.2. *Buckling analysis*

To further simplify the total potential, we consider that the perturbations of the prebuckled state at the onset of buckling can be regarded as arbitrarily small. We need to keep all terms of power 1 and 2 in perturbations of  $\Phi$ . Using the concept of the Taylor series to make certain all such terms are retained, we note that

$$\varepsilon = \bar{\varepsilon} + \hat{\varepsilon}_1 + \hat{\varepsilon}_2, \quad \kappa = \hat{\kappa}_1 + \hat{\kappa}_2. \tag{57}$$

The subscripts indicate the power of the perturbation displacements. Because of the non-zero value of  $\bar{\varepsilon}$ , we need both first and second order terms. For small strain, we find

$$\begin{aligned} \hat{\varepsilon}_1 &= \hat{u}'_1 - \hat{u}_2, \\ \hat{\varepsilon}_2 &= \frac{1}{2(1 + \bar{\varepsilon})} (\hat{u}'_2 + \hat{u}_1)^2 = \frac{1}{2} (\hat{u}'_2 + \hat{u}_1)^2, \\ \hat{\kappa}_1 &= \hat{u}''_2 + \hat{u}'_1, \\ \hat{\kappa}_2 &= (\hat{u}'_1 - \hat{u}_2)(\hat{u}''_2 + \hat{u}'_1) + (\hat{u}'_2 + \hat{u}_1)(\hat{u}''_1 - \hat{u}'_2). \end{aligned} \tag{58}$$

Now we can write the perturbations of the energy. First, keeping only the terms including first powers of the  $(\hat{\cdot})$  quantities, we obtain

$$\hat{\Phi}_1 = \int_{-\alpha}^{\alpha} \left[ \bar{\varepsilon} \hat{\varepsilon}_1 - (1 + \nu) \rho^2 \bar{\varepsilon} \hat{\kappa}_1 - \lambda \rho^2 (1 + \bar{\varepsilon}) \hat{u}_2 \right] d\phi, \tag{59}$$

the variation of which is identically zero, as expected. Equating to zero the variation with respect to  $\hat{u}_1$ , one obtains an identity; equating to zero the variation with respect to  $\hat{u}_2$ , one finds an equation that is satisfied given Eq. (55).

Now, let us consider the second-order terms (which amounts to a second variation):

$$\begin{aligned} \hat{\Phi}_2 &= \frac{1}{2} \int_{-\alpha}^{\alpha} \left[ 2\bar{\varepsilon} \hat{\varepsilon}_2 + \hat{\varepsilon}_1^2 - 2(1 + \nu) \rho^2 \bar{\varepsilon} \hat{\kappa}_2 \right. \\ &\quad - 2(1 + \nu) \rho^2 \hat{\varepsilon}_1 \hat{\kappa}_1 + \rho^2 (\hat{u}''_2 + \hat{u}'_1)^2 \\ &\quad + \rho^2 (\hat{u}''_1 - \hat{u}'_2)^2 + \lambda \rho^2 (\hat{u}_1^2 + \hat{u}_2^2 \\ &\quad \left. + 2\hat{u}_1 \hat{u}'_2) \right] d\phi. \end{aligned} \tag{60}$$

When  $\bar{\varepsilon} = -\lambda \rho^2$  is substituted into Eq. (60), the third term drops out, being  $O(\rho^4)$  relative to the leading term. It should be clear that all the remaining terms in  $\hat{\Phi}_2$  are proportional to  $\rho^2$  except the  $\hat{\varepsilon}_1^2$  term; thus, that term must be killed. Minimization of  $\hat{\Phi}_2$  with respect to  $\hat{u}_1$  shows that

$$\hat{u}'_1 = \hat{u}_2 + \rho^2 \nu (\hat{u}''_2 + \hat{u}_2) + \dots, \tag{61}$$

or alternatively

$$\hat{u}_2 = \hat{u}'_1 - \rho^2 \nu (\hat{u}'''_1 + \hat{u}'_1) + \dots, \tag{62}$$

so that

$$\begin{aligned} \hat{\varepsilon}_1 &= \rho^2 \nu (\hat{u}''_2 + \hat{u}_2) + \dots \\ &= \rho^2 \nu (\hat{u}'''_1 + \hat{u}'_1) + \dots \end{aligned} \tag{63}$$

Either  $\hat{u}_1$  or  $\hat{u}_2$  can be eliminated completely from the energy using these relations. Considering first the elimination of  $\hat{u}_2$ , substitution of Eq. (62) into Eq. (60), one obtains

$$\begin{aligned} \hat{\Phi}_2 &= \frac{\rho^2}{2} \int_{-\alpha}^{\alpha} \left[ (\hat{u}'''_1 + \hat{u}'_1)^2 - \lambda (\hat{u}''_1 + \hat{u}_1)^2 \right. \\ &\quad \left. + \lambda (\hat{u}_1^2 + \hat{u}'_1{}^2 + 2\hat{u}_1 \hat{u}'_1) \right] d\phi, \end{aligned} \tag{64}$$

which simplifies to

$$\hat{\Phi}_2 = \frac{\rho^2}{2} \int_{-\alpha}^{\alpha} \left[ (\hat{u}'''_1 + \hat{u}'_1)^2 - \lambda \hat{u}''_1{}^2 + \lambda \hat{u}'_1{}^2 \right] d\phi. \tag{65}$$

The essential boundary conditions on  $\hat{u}_2$  must be transferred over as essential boundary conditions on  $\hat{u}'_1$  in order to make proper use of this energy functional.

Alternatively, the variable  $\hat{u}_1$  cannot be eliminated entirely without a somewhat unusual treatment of the boundary conditions. Integrating both sides of Eq. (61) for a ring, when specialized for  $\rho^2 \ll 1$ , one finds

$$\hat{u}_1|_{-\alpha}^{\alpha} = 0 = \int_{-\alpha}^{\alpha} \hat{u}_2 d\phi. \tag{66}$$

This equation is satisfied for typical comparison functions used in predicting ring buckling, because whether such functions are “symmetric” or “antisymmetric” about  $\phi = 0$  is immaterial. However, for buckling of high arches in which  $\hat{u}_1(\pm\alpha) = 0$  one must be careful. Although antisymmetric functions automatically satisfy this condition for high

arches, symmetric functions do not, in general. This condition is an essential (i.e., a displacement) boundary condition, and it is therefore mandatory that any admissible/comparison function satisfy it; else, the results from Rayleigh’s method, for example, will be wrong. Using Eq. (61), one can write the energy functional in terms of  $\hat{u}_2$  only as

$$\hat{\Phi}_2 = \frac{\rho^2}{2} \int_{-\alpha}^{\alpha} [(\hat{u}_2'' + \hat{u}_2)^2 - \lambda(\hat{u}_2'^2 - \hat{u}_2^2)] d\phi \quad (67)$$

subject to

$$\int_{-\alpha}^{\alpha} \hat{u}_2 d\phi = 0. \quad (68)$$

These expressions for the second variation of the total potential provide from henceforth very simple treatments relative to published work. In spite of this simplicity, the only approximation employed was that  $\bar{\epsilon} \ll 1$ , which, because of the prebuckling state, is equivalent to  $\rho^2 \ll 1$ .

Now, using either Eq. (65) or Eqs. (67) and (68) one can derive an upper bound for the buckling load from Rayleigh’s quotient. For example, using the latter

$$\lambda_{cr} = \frac{\int_{-\alpha}^{\alpha} (\hat{u}_2'' + \hat{u}_2)^2 d\phi}{\int_{-\alpha}^{\alpha} (\hat{u}_2'^2 - \hat{u}_2^2) d\phi} \quad (69)$$

and assuming that  $\hat{u}_2 = \sin m\phi$ , which satisfies Eq. (68), one finds that

$$\lambda_{cr} = m^2 - 1. \quad (70)$$

Since  $m = 1$  is a rigid-body mode, as shown in Ref. [5], the critical load is then at  $m = 2$  so that

$$\lambda_{cr} = 3 \quad (71)$$

in agreement with published results [5].

High arches are often treated approximately by allowing the boundaries to move in the prebuckling problem, yielding a simplified prebuckling state identical to that of the ring. For those cases described in Ref. [5], one can quite easily verify that Eq. (69), subject to Eq. (68), as well as its analog in terms of  $\hat{u}_1$ , provide upper bounds for the published symmetric or antisymmetric buckling loads when either symmetric or antisymmetric admissible or comparison functions are substituted therein.

### 4.3. Analysis of in-plane deformation

The stretch–bending elastic coupling term which was found as a correction to the classical 1-D strain energy, i.e., the underlined term in Eq. (41), does not affect the above buckling analysis at all. This is clearly revealed since the functional Eq. (69) does not depend on  $v$ . Due to similarity of the present equations for analysis of curved beams and those contained in published work for shell analysis [13], it is quite likely that when  $|\epsilon| \gg \rho|\kappa| > 0$  one must retain the full expression for the strain energy in Eq. (41). This hypothesis is examined presently.

Recall that the derivation only assumes that the strain is small compared to unity ( $\epsilon \ll 1$ ) and that  $\rho^2 \ll 1$ . For in-plane deformation subject to the external pressure specified by the parameter  $\lambda$ , the Euler–Lagrange equations and boundary conditions for  $u_1$  and  $u_2$  can be obtained by taking the variation of Eq. (46) and executing appropriate integrations by parts. These equations are non-linear and quite lengthy, and they need not be written here. One can simplify the resulting analysis immensely by making the following changes of variable:

$$\begin{aligned} u_1 &= \rho^2 \bar{u}_1, \\ u_2 &= \rho^2 (\lambda + \bar{u}'_1) + \rho^4 (z - \lambda^2). \end{aligned} \quad (72)$$

and ignoring  $\rho^2$  compared to unity. Doing so, one finds that the  $u_1$  equation becomes

$$\begin{aligned} z' + v\bar{u}_1'' - \bar{u}'_1\bar{u}_1'' - \bar{u}_1''\bar{u}_1'' - \bar{u}_1(\bar{u}'_1 + \bar{u}_1''') \\ + v\bar{u}_1^{iv} = 0 \end{aligned} \quad (73)$$

and the  $u_2$  equation simplifies to allow one to solve for  $z$ , so that

$$\begin{aligned} z = \frac{(\bar{u}_1'' + \bar{u}_1)^2}{2} - (1 + v + \lambda)\bar{u}_1' \\ - (2 + v + \lambda)\bar{u}_1'' - \bar{u}_1^y. \end{aligned} \quad (74)$$

After eliminating  $z$  from Eq. (73) and after a remarkable series of cancellations, one finds that

$$\bar{u}_1^{vi} + (2 + \lambda)\bar{u}_1^{iv} + (1 + \lambda)\bar{u}_1'' = 0, \quad (75)$$

which is independent of Poisson’s ratio. For a curved beam with  $-\alpha \leq \phi \leq \alpha$ , the boundary

conditions at  $\phi = \pm \alpha$  are

$$\begin{aligned} \bar{u}_1 &= 0, \\ \bar{u}'_1 + \lambda &= 0 \quad \text{or} \quad \bar{u}_1^{iv} + (1 + \lambda)\bar{u}''_1 = 0, \\ \bar{u}''_1 &= 0 \quad \text{or} \quad \bar{u}'''_1 + \bar{u}'_1 + \lambda(1 + \nu) = 0, \end{aligned} \tag{76}$$

which are *not* independent of Poisson’s ratio.

The expressions needed for recovery of the radial displacement  $u_2$ , rotation  $\theta_3$ , curvature  $\kappa$ , bending moment  $M$ , and shear force  $V$  are

$$\begin{aligned} u_2 &= \rho^2(\bar{u}'_1 + \lambda), \\ \theta_3 &= \rho^2(\bar{u}''_1 + \bar{u}_1), \\ \kappa &= \rho^2(\bar{u}'''_1 + \bar{u}'_1), \\ \frac{M}{EAR} &= \rho^4[\bar{u}''_1 + \bar{u}'_1 + \lambda(1 + \nu)], \\ \frac{V}{EA} &= -\rho^4[\bar{u}_1^{iv} + (1 + \lambda)\bar{u}''_1]. \end{aligned} \tag{77}$$

It should be noted that the above boundary-value problem can also be posed as the minimization of a functional

$$\begin{aligned} \Phi_0 &= \frac{1}{2} \int_{-\alpha}^{\alpha} [2\lambda(1 + \nu)\bar{u}''_1 + (\bar{u}'''_1 + \bar{u}'_1)^2 \\ &\quad - \lambda(\bar{u}''_1 - \bar{u}'_1)] d\phi \end{aligned} \tag{78}$$

subject to  $\bar{u}_1 = 0$  at both ends. This functional, as well as the corresponding boundary-value problem, is valid only when  $\rho^2\bar{u}_1 \ll 1$ . Relaxation of this restriction would allow the treatment of postbuckling deflections within the same framework.

Obviously, Poisson’s ratio will affect the prebuckling deflections to a non-negligible extent whenever the bending moment is prescribed at either end. Although the deflections are not affected by Poisson’s ratio when the rotation is fixed, the resulting bending moment (and thus calculation of prebuckling stress) is affected. It is interesting to note that the quadratic terms in  $\Phi_0$  are of the same form as Eq. (65). Thus, the bifurcation loads are the same for rings and high arches as would be predicted by the buckling analysis in the previous section. It is the presence of the linear term that affects prebuckling deflections (through non-

homogeneous boundary conditions), however, and one can observe a variety of situations. As examples, high arches with five sets of boundary conditions are considered.

*4.3.1. Roller–roller (rr) case*

Ignoring  $\rho^2$  compared to unity, for the roller–roller case (Case rr) one obtains the following boundary conditions, applied at both ends,  $\phi = \pm \alpha$ :

$$\bar{u}_1 = \bar{u}'_1 = \bar{u}_1^{iv} = 0. \tag{79}$$

Here both ends are clamped to rollers (i.e., so that there is zero rotation, shear force, and tangential displacement). Case rr has only homogeneous boundary conditions, so there is zero prebuckling deflection  $u_1$  and curvature  $\kappa$ . However, again ignoring  $\rho^2$  compared to unity, the radial deflection and bending moment are constants given by

$$\begin{aligned} u_2 &= \lambda\rho^2, \\ \frac{M}{EAR} &= \lambda\rho^4(1 + \nu). \end{aligned} \tag{80}$$

Although the bending moment is small, it is not zero unless  $\nu = -1$  (which is the same as if the Poisson effect is neglected from the strain energy in Eq. (41)). Thus, predicted prebuckling bending stresses are completely different (for almost all materials) from what they would be without the stretching–bending coupling term in the strain energy. The bifurcation load is easily found to be  $\lambda_{cr} = \pi^2/(4\alpha^2) - 1$  for  $\alpha > \pi/2$ .

*4.3.2. Clamped–clamped (cc) case*

In the clamped–clamped case (Case cc) we consider these boundary conditions

$$\bar{u}_1 = \bar{u}'_1 + \lambda = \bar{u}''_1 = 0, \tag{81}$$

at both ends,  $\phi = \pm \alpha$ . Here both ends of the arch are fixed to the ground and rotation is constrained to be zero. The prebuckling deflections  $u_2$  are symmetric about  $\phi = 0$ ; thus, without any non-symmetric imperfections, the lowest bifurcation load, which has an antisymmetric mode, is not picked up. The formulae for prebuckling deflections and curvature are somewhat lengthy and are not given

here, but they can be easily developed using Mathematica [14], as described above. The prebuckling deflections and curvature  $u_1$ ,  $u_2$ , and  $\kappa$ , respectively, are not functions of  $v$ , but the prebuckling bending moment  $M$  does depend on  $v$  and is given by

$$\frac{M}{EAR} = \lambda \rho^4 \times \frac{[k^2 \alpha (1 + v) \cos(\alpha) - \lambda v \sin(\alpha)] \sin(k\alpha)}{[k^2 \alpha \cos(\alpha) - \lambda \sin(\alpha)] \sin(k\alpha) - k\alpha \cos(k\alpha) \sin(\alpha)} - \frac{k\alpha [(1 + v) \cos(k\alpha) + \lambda \cos(k\phi)] \sin(\alpha)}{[k^2 \alpha \cos(\alpha) - \lambda \sin(\alpha)] \sin(k\alpha) - k\alpha \cos(k\alpha) \sin(\alpha)}, \tag{82}$$

where  $k = \sqrt{1 + \lambda}$ . The extent to which  $v$  changes the prebuckling bending moment is dependent on  $\alpha$  and  $\lambda$ . It is noted that the characteristic equation can be found by equating the denominator to zero. Results for the bifurcation load are in agreement with those of Ref. [5].

4.3.3. Pinned–pinned (pp) case

In the pinned–pinned case (Case pp) we consider boundary conditions of the form

$$\bar{u}_1 = \bar{u}'_1 + \lambda = \bar{u}'''_1 + \lambda v = 0, \tag{83}$$

at both ends,  $\phi = \pm \alpha$ . Here the ends of the arch are fixed to the ground and rotation is unconstrained. The antisymmetric bifurcation load is the critical one, and the value is  $\lambda_{cr} = \pi^2/\alpha^2 - 1$  for  $\alpha > \pi$ , in agreement with Ref. [5]. As in Case cc, the prebuckling deflections  $u_2$  are symmetric about  $\phi = 0$ ; thus, without any non-symmetric imperfections, the lowest bifurcation load, which has an antisymmetric mode, is not picked up. Like Case cc, the lengthy prebuckling deflection, curvature, and bending moment formulae can be easily developed using Mathematica [14]. However, unlike Case cc, the prebuckling deflections  $u_1$  and  $u_2$  as well as the curvature and bending moment are significantly affected by  $v$ ; see, for example, Figs. 2 and 3. Note that the magnitude of the bending moment is off by a factor of two without the stretch-bending coupling term in Eq. (41).

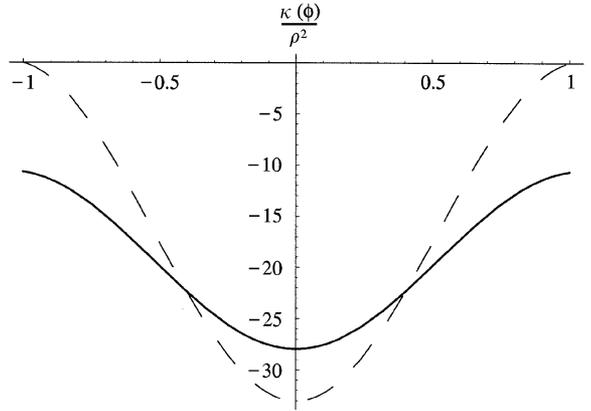


Fig. 2. Normalized prebuckling curvature  $\kappa$  for Case pp,  $\lambda = 5$ ,  $\alpha = 1$ ; the solid line is for the case with the stretch–bending coupling term ( $v = \frac{1}{3}$ ); the dashed line is for the case without the stretch–bending coupling term ( $v = -1$ ).

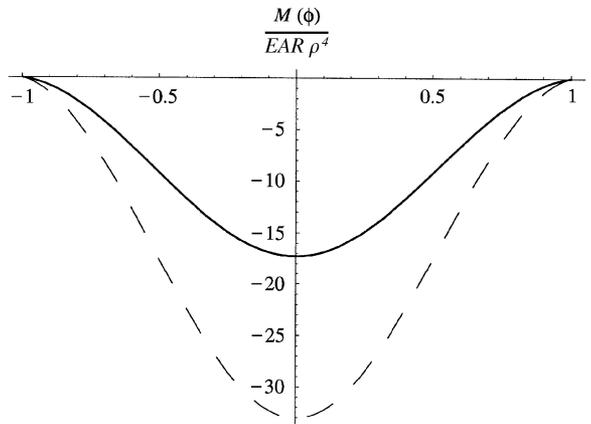


Fig. 3. Normalized prebuckling bending moment  $M$  for Case pp,  $\lambda = 5$ ,  $\alpha = 1$ ; the solid line is for the case with the stretch–bending coupling term ( $v = \frac{1}{3}$ ); the dashed line is for the case without the stretch–bending coupling term ( $v = -1$ ).

4.3.4. Free–free (ff) case

In Case ff we consider as boundary conditions

$$\bar{u}_1 = \bar{u}'''_1 + \bar{u}'_1 + \lambda(1 + v) = \bar{u}^{iv}_1 + (\lambda + 1)\bar{u}''_1 = 0, \tag{84}$$

in which case both ends,  $\phi = \pm \alpha$ , are pinned to rollers (i.e., so that there is zero bending moment and shear force with zero tangential displacement). The solutions for the prebuckling deflections,

curvature, and bending moment are simple. The deflections are given by

$$u_1 = \lambda \rho^2 (1 + \nu) \frac{\alpha \sin(k\phi) - \phi \sin(k\alpha)}{\lambda k \alpha \cos(k\alpha) + \sin(k\alpha)},$$

$$u_2 = \lambda \rho^2 \frac{k\alpha [(1 + \nu) \cos(k\phi) + \lambda \cos(k\alpha)] - \nu \sin(k\alpha)}{\lambda k \alpha \cos(k\alpha) + \sin(k\alpha)}. \tag{85}$$

The curvature is

$$\kappa = -\lambda \rho^2 (1 + \nu) \frac{\lambda k \alpha \cos(k\phi) + \sin(k\alpha)}{\lambda k \alpha \cos(k\alpha) + \sin(k\alpha)} \tag{86}$$

and the bending moment is

$$\frac{M}{EAR} = -(1 + \nu) k \alpha \lambda^2 \rho^4 \frac{\cos(k\phi) - \cos(k\alpha)}{\lambda k \alpha \cos(k\alpha) + \sin(k\alpha)}. \tag{87}$$

The shear force is identically zero through  $O(\rho^4)$ . Note that the tangential displacement  $u_1$ , the curvature  $\kappa$ , and the bending moment  $M$  are all identically zero for  $\nu = -1$ , which corresponds to the result when the stretch–bending coupling term is not present in the strain energy. When  $\nu$  assumes any other value, one sees significant changes, both qualitative and quantitative, in all prebuckling quantities; for example, in Fig. 4 note the large effect for the radial displacement.

For this case, the characteristic equation reduces to

$$\lambda k \alpha \cos(k\alpha) + \sin(k\alpha) = 0, \tag{88}$$

which, not surprisingly, tends to yield critical loads lower than the for the other sets of boundary conditions considered. What is somewhat unusual is that the critical load corresponds to a mode which is symmetric in  $u_2$ . That is, the bifurcation load for the symmetric mode is lower than that of the anti-symmetric mode.

4.3.5. Pinned–free (pf) case

Finally, we consider a non-symmetric case, the pinned–free case (Case pf). The boundary conditions are written as

$$\bar{u}_1(-\alpha) = \bar{u}'_1(-\alpha) + \lambda = \bar{u}'''_1(-\alpha) + \lambda \nu = 0,$$

$$\bar{u}_1(\alpha) = \bar{u}'''_1(\alpha) + \bar{u}'_1(\alpha) + \lambda(1 + \nu) = \bar{u}^{iv}_1(\alpha)$$

$$+ (\lambda + 1)\bar{u}''_1(\alpha) = 0. \tag{89}$$

For this case the characteristic equation reduces to

$$\sin(k\alpha) \{ 2k\alpha\lambda \cos[(k - 2)\alpha] + 2k\alpha\lambda \cos[(k + 2)\alpha] + (k^3 + 2) \sin[(k - 2)\alpha] - (k^3 - 2) \sin[(k + 2)\alpha] \} = 0, \tag{90}$$

where the bracketed quantity vanishes at a lower value of  $\lambda$  than does the  $\sin(k\alpha)$  term outside the brackets for determination of the critical bifurcation load. What is of most interest for the present, however, is the profound impact of the stretch–bending coupling term of the refined theory. In Fig. 5, for example, one sees the qualitatively different results for the prebuckling radial displacement with and without the coupling term.

Realistic arches, of course, have imperfections; the boundary conditions are not purely symmetric, nor is it possible for the displacements or rotations to be held exactly to zero values. This means that in general one must have the stretch–bending coupling which depends on the initial radius of curvature and Poisson’s ratio in order to accurately predict prebuckling deflections, bending moment and shear force.

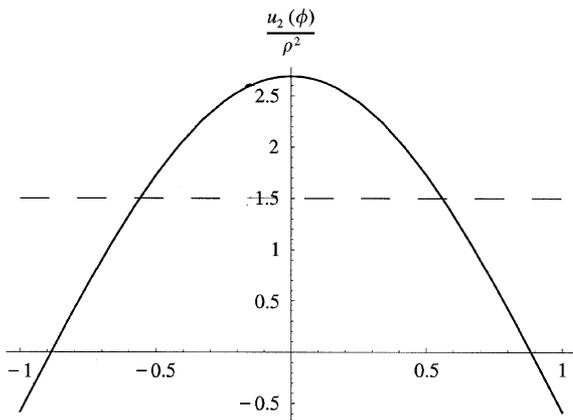


Fig. 4. Normalized prebuckling displacement  $u_2$  for Case ff,  $\lambda = 2$ ,  $\alpha = 1$ ; the solid line is for the case with the stretch–bending coupling term ( $\nu = \frac{1}{3}$ ); the dashed line is for the case without the stretch–bending coupling term ( $\nu = -1$ ).

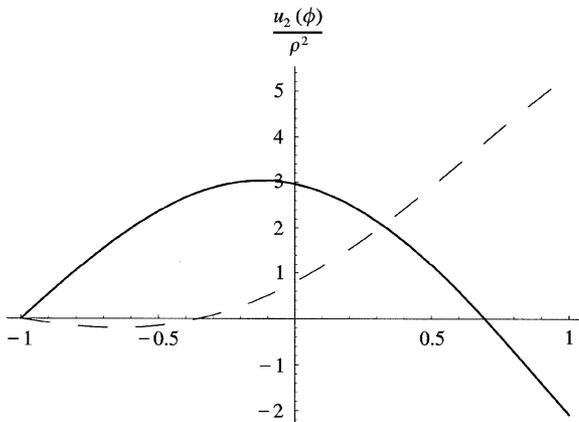


Fig. 5. Normalized prebuckling displacement  $u_2$  for Case pf,  $\lambda = 1.5$ ,  $\alpha = 1$ ; the solid line is for the case with the stretch–bending coupling term ( $\nu = \frac{1}{3}$ ); the dashed line is for the case without the stretch–bending coupling term ( $\nu = -1$ ).

## 5. Concluding remarks

The present paper considers homogeneous isotropic beams with cross sections that are symmetric about the plane in which the undeformed beam exhibits constant initial curvature. A theory is derived from geometrically non-linear 3-D elasticity for deformation of such beams in the plane of their symmetry. The dimensional reduction is performed via the variational-asymptotic method of Berdichevsky [6]. The resulting theory is subject only to the restrictions that the strain and the ratio  $\rho^2 = I/(AR^2)$  are small compared to unity. The theory contains a term in the 1-D strain energy which couples stretching and bending and which depends on the initial radius of curvature and Poisson's ratio.

When applied to the buckling of rings (and of high arches when one uses a somewhat artificial treatment of the boundary conditions so that the prebuckling state is the same as that for rings), the theory shows that the prebuckling strain is of the order of  $\rho^2$ . This means that there is really only one restriction on the theory for this application, that of prebuckling strain being small compared to unity. The buckling analysis which follows is quite simple,

boiling down to the minimization of a simple functional. This buckling analysis follows from a theory which has fewer restrictions, and exhibits a considerably simpler final form, than those typically found in textbooks. Finally, it has been shown that when non-trivial prebuckling deflections, curvature, and bending moment of high arches exist, they are impossible to calculate accurately without the stretching–bending coupling term in the strain energy which depends on the initial radius of curvature and Poisson's ratio.

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