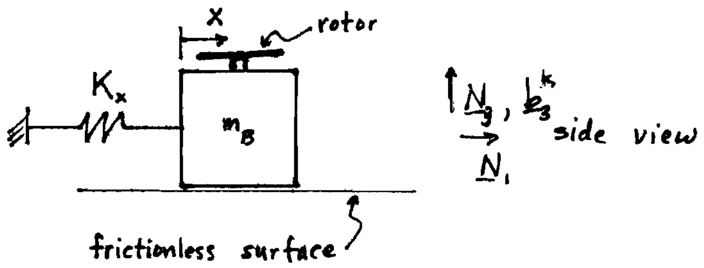
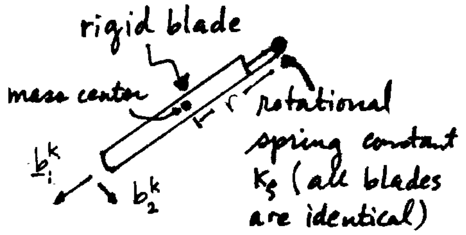
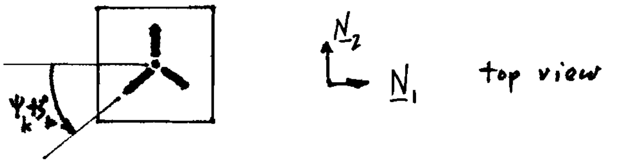
Simple Ground Resonance Model (No aerodynamics) (See Bramwell Chap. 12)




$$\psi_k = \psi + \frac{2\pi}{b}(k-1) \qquad \psi = \Omega t$$

A simple system that will exhibit the "ground resonance" instability is shown above. Let each blade have one degree of freedom $\xi_k$. The rotating unit vectors are related to the fixed ones ones

$$\begin{Bmatrix} \underline{b}_1^k \\ \underline{b}_2^k \\ \underline{b}_3^k \end{Bmatrix} = \begin{bmatrix} -C\bar{\psi}_k & -S\bar{\psi}_k & 0 \\ S\bar{\psi}_k & -C\bar{\psi}_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underline{N}_1 \\ \underline{N}_2 \\ \underline{N}_3 \end{Bmatrix} \qquad \bar{\psi}_k = \psi_k + \xi_k$$

The velocity of the body mass center is

$$\underline{V}^B = \dot{x}\,\underline{N}_1$$

The velocity of the center of mass of blade k is

$$\underline{V}^k = \dot{x}\,\underline{N}_1 + (\Omega + \dot{\xi}_k)\,\underline{b}_3^k \times r\,\underline{b}_1^k \qquad k = 1, 2, \ldots, b$$

The angular velocity of blade k is

$$\underline{\omega}^k = (\Omega + \dot{\xi}_k)\,\underline{b}_3^k \qquad k = 1, 2, \ldots, b$$

$(b \geq 3)$

$$\therefore\ \underline{V}^k = \dot{X}\,\underline{N}_1 + r\,\underline{b}_2^k(\Omega+\dot{\zeta}_k)$$
$$= [\dot{X} + (\Omega+\dot{\zeta}_k)\, r s\psi_k]\,\underline{N}_1 + [-(\Omega+\dot{\zeta}_k)\, r c\psi_k]\,\underline{N}_2$$

$$K = \frac{1}{2} m_B \dot{X}^2 + m\frac{1}{2}\sum_{k=1}^{b}\{[\dot{X} + (\Omega+\dot{\zeta}_k) r s\psi_k]^2 + (\Omega+\dot{\zeta}_k)^2 r^2 c\psi_k^2\}$$
$$+ I^*\frac{1}{2}\sum_{k=1}^{b}(\Omega+\dot{\zeta}_k)^2$$
$$= \frac{m_B}{2}\dot{X}^2 + \frac{bm}{2}\dot{X}^2 + \frac{I}{2}\sum_{k=1}^{b}(\Omega+\dot{\zeta}_k)^2 + mr\dot{X}\sum_{k=1}^{b}(\Omega+\dot{\zeta}_k)\sin(\psi_k+\zeta_k)$$
$$I = I^* + mr^2$$

$$P = \frac{k_\zeta}{2}\sum_{k=1}^{b}\zeta_k^2 + \frac{k_x}{2}X^2$$

$$M \equiv m_B + bm$$

$$K = \frac{M\dot{X}^2}{2} + \frac{I}{2}\sum_{k=1}^{b}(2\Omega\dot{\zeta}_k + \dot{\zeta}_k^2) + mr\dot{X}\sum_{k=1}^{b}(\Omega+\dot{\zeta}_k)\sin(\psi_k+\zeta_k)$$

$$\frac{\partial K}{\partial \dot{X}} = M\dot{X} + mr\sum_{k=1}(\Omega+\dot{\zeta}_k)\sin(\psi_k+\zeta_k)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{X}}\right) = M\ddot{X} + mr\sum_{k=1}^{b}[\ddot{\zeta}_k\sin(\psi_k+\zeta_k) + (\Omega+\dot{\zeta}_k)^2\cos(\psi_k+\zeta_k)]$$

$$\frac{\partial K}{\partial X} = 0 \qquad \frac{\partial P}{\partial X} = k_x X$$

$$M\ddot{X} + mr\sum_{k=1}^{b}[\ddot{\zeta}_k\sin(\psi_k+\zeta_k) + (\Omega+\dot{\zeta}_k)^2\cos(\psi_k+\zeta_k)] + K_x X = 0$$

$$\frac{\partial K}{\partial \dot{\zeta}_k} = I(\Omega+\dot{\zeta}_k) + mr\dot{X}\sin(\psi_k+\zeta_k)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\zeta}_k}\right) = I\ddot{\zeta}_k + mr\ddot{X}\sin(\psi_k+\zeta_k) + mr\dot{X}(\Omega+\dot{\zeta}_k)\cos(\psi_k+\zeta_k)$$

$$\frac{\partial K}{\partial \zeta_k} = mr\dot{X}(\Omega+\dot{\zeta}_k)\cos(\psi_k+\zeta_k) \qquad \frac{\partial P}{\partial \zeta_k} = k_\zeta\zeta_k$$

$$I\ddot{\zeta}_k + mr\ddot{X}\sin(\psi_k+\zeta_k) + k_\zeta\zeta_k = 0 \qquad k = 1, 2, \ldots, b$$

*Linearize*

$$M\ddot{X} + mr\left(\sum_{k=1}^{b} \ddot{\zeta}_k \sin\psi_k - \Omega^2 \sum_{k=1}^{b} \zeta_k \sin\psi_k + + \sum_{k=1} 2\Omega\dot{\zeta}_k \cos\psi_k\right) + K_x X = 0$$

$$I\ddot{\zeta}_k + mr\ddot{X}\sin\psi_k + K_\zeta \zeta_k = 0 \qquad Q_k = 0$$

$$\zeta_k = \zeta_c \cos\psi_k + \zeta_s \sin\psi_k + \zeta_0 + \zeta_d (-1)^k + \cdots$$

$$\frac{b}{2}\zeta_s = \sum_{k=1}^{b} \zeta_k s_{\psi_k} \qquad \frac{b}{2}\zeta_c = \sum_{k=1}^{b} \zeta_k c_{\psi_k}$$

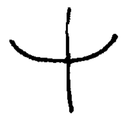
$$\sum_{k=1}^{b} \ddot{\zeta}_k c_{\psi_k} = \frac{b}{2}(\ddot{\zeta}_c + 2\Omega\dot{\zeta}_s - \Omega^2\zeta_c) \qquad \delta\zeta_k = \delta\zeta_c \cos\psi_k + \delta\zeta_s \sin\psi_k$$

$$\sum_{k=1}^{b} \ddot{\zeta}_k s_{\psi_k} = \frac{b}{2}(\ddot{\zeta}_s - 2\Omega\dot{\zeta}_c - \Omega^2\zeta_s) \qquad \sum_{k=1}^{b} Q_k \delta\zeta_k = 0$$

revert to weak form

$$M\ddot{X} + \frac{mb}{2} r \ddot{\zeta}_s + K_x X = 0$$
$$I(\ddot{\zeta}_c + 2\Omega\dot{\zeta}_s - \Omega^2\zeta_c) + K_\zeta \zeta_c = 0$$
$$I(\ddot{\zeta}_s - 2\Omega\dot{\zeta}_c - \Omega^2\zeta_s) + mr\ddot{X} + K_\zeta \zeta_s = 0$$

$\zeta_s$ lateral (long shift)

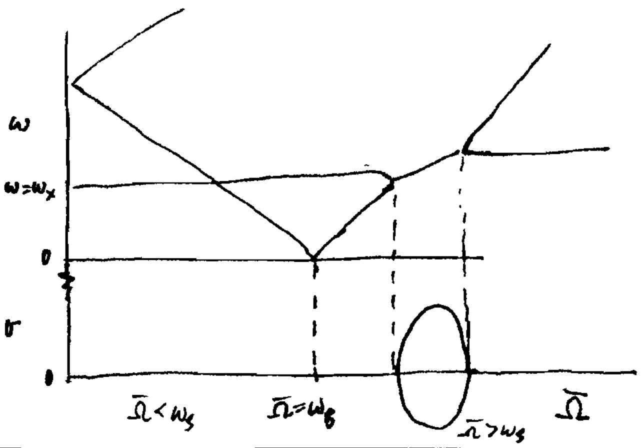


$\zeta_c$ long. (lateral shift)

Let $I = 1/3\, mR^2$
$r = R\bar{r}$

$$\frac{K_x}{M\Omega_0^2} = \omega_x^2 \qquad \frac{K_\zeta}{I\Omega_0^2} = \omega_\zeta^2 \qquad \bar{X} = \frac{X}{R} \qquad \frac{mb}{M} = \mu \text{ mass ratio}$$

$$\begin{bmatrix} 6/\mu & 0 & 3\bar{r} \\ 0 & 1 & 0 \\ 3\bar{r} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\bar{X}} \\ \ddot{\zeta}_c \\ \ddot{\zeta}_s \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\bar{\Omega} \\ 0 & -2\bar{\Omega} & 0 \end{bmatrix} \begin{Bmatrix} \dot{\bar{X}} \\ \dot{\zeta}_c \\ \dot{\zeta}_s \end{Bmatrix} + \begin{bmatrix} \frac{6}{\mu}\omega_x^2 & 0 & 0 \\ 0 & \omega_\zeta^2 - \bar{\Omega}^2 & 0 \\ 0 & 0 & \omega_\zeta^2 - \bar{\Omega}^2 \end{bmatrix} \begin{Bmatrix} \bar{X} \\ \zeta_c \\ \zeta_s \end{Bmatrix} = 0$$




STABLE IF $\bar{\Omega} < \omega_\zeta$

$s = \sigma \pm i\omega$