In[1]:= $\delta = \text{Simplify}\left[\sqrt{(x\,\epsilon\,\text{Cos}[\psi]+1)^2+(\epsilon\,x\,\text{Sin}[\psi])^2}\right]$

Out[1]= $$\sqrt{x^2\,\epsilon^2+2\,x\,\epsilon\cos(\psi)+1}$$

In[2]:= $\text{i}\delta = \text{Integrate}\left[\frac{x\,\epsilon\,\text{Cos}[\psi]+1}{\delta^3},\left\{x,-\frac{1}{2},\frac{1}{2}\right\},\text{GenerateConditions}\to\text{False}\right]$

Out[2]= $$\frac{1}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}+\frac{1}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}$$

In[3]:= $\text{F1} = -\text{FullSimplify}\left[\frac{\text{i}\delta\,\text{G m M}}{\text{R}^2}\right]$

Out[3]= $$-\frac{G\,m\,M\left(\frac{1}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}+\frac{1}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}\right)}{R^2}$$

In[4]:= $\text{F1simp0} = \text{Simplify}[\text{F1} \,/.\, \epsilon\to 0]$

Out[4]= $$-\frac{G\,m\,M}{R^2}$$

In[5]:= $\text{F1simp1} = \text{Simplify}\left[\frac{\text{Limit}\left[\partial_{\{\epsilon,2\}}\text{F1},\epsilon\to 0\right]\epsilon^2}{2\,\text{F1simp0}}\right]$

Out[5]= $$\frac{1}{16}\epsilon^2\,(3\cos(2\,\psi)+1)$$

In[6]:= $\text{F1simp} = \text{F1simp0}\,(\text{F1simp1}+1)$

Out[6]= $$-\frac{G\,m\,M\left(\frac{1}{16}\epsilon^2\,(3\cos(2\,\psi)+1)+1\right)}{R^2}$$

In[7]:= $\text{Plot}\left[\left\{\frac{\text{F1simp}}{\text{F1simp0}}\,/.\,\epsilon\to .1,\ \frac{\text{F1}}{\text{F1simp0}}\,/.\,\epsilon\to .1\right\},\left\{\psi,0,\frac{\pi}{2}\right\}\right];$

In[8]:= $\text{i}\delta = \text{Simplify}\left[\text{Integrate}\left[\frac{x}{\delta^3},\left\{x,-\frac{1}{2},\frac{1}{2}\right\},\text{GenerateConditions}\to\text{False}\right]\right]$

Out[8]= $$\frac{\csc^2(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)}{\epsilon^2}$$

In[9]:=
```
F2 = -Simplify[iδ G m M ϵ Sin[ψ] / R^2]
```

Out[9]=
$$-\frac{G\,m\,M\csc(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)}{R^2\,\epsilon}$$

In[10]:=
```
F2simp = 1/2 Limit[∂{ϵ,2} F2, ϵ → 0] ϵ^2
```

Out[10]=
$$\frac{G\,m\,M\,\epsilon^2\sin(2\,\psi)}{8\,R^2}$$

In[11]:=
```
Plot[{F2simp / (ϵ^2 G m M / (8 R^2)), F2 / (ϵ^2 G m M / (8 R^2)) /.ϵ → .1}, {ψ, 0, π/2}];
```

In[12]:=
```
torque = Simplify[ϵ iδ Sin[ψ] G m M / R]
```

Out[12]=
$$\frac{G\,m\,M\csc(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)}{R\,\epsilon}$$

In[13]:=
```
torquesimp = 1/2 Limit[∂{ϵ,2} torque, ϵ → 0] ϵ^2
```

Out[13]=
$$-\frac{G\,m\,M\,\epsilon^2\sin(2\,\psi)}{8\,R}$$

In[14]:=
```
Plot[{-8 R torquesimp / (G m M ϵ^2), -8 R torque / (G m M ϵ^2) /.ϵ → .1}, {ψ, 0, π/2}];
```

In[15]:=
```
(*Magnitude of the force resultant:*)
```

In[16]:=
```
Fmag = PowerExpand[FullSimplify[Sqrt[F1^2 + F2^2]]]
```

Out[16]=
$$\frac{1}{R^2}G\,m\,M\sqrt{\left(\left(\frac{1}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}+\frac{1}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}\right)^2+\frac{\csc^2(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)^2}{\epsilon^2}\right)}$$

In[17]:=
```
(*Unit vector along the force resultant:*)
```

In[18]:= `u = PowerExpand[FullSimplify[(F1 a₁ + F2 a₂)/Fmag]]`

Out[18]=
$$\left(a_1\left(-\left(\frac{1}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}+\frac{1}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}\right)\right)-\frac{a_2\csc(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)}{\epsilon}\right)\Bigg/$$
$$\left(\sqrt{\left(\left(\frac{1}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}+\frac{1}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}\right)^2+\frac{\csc^2(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)^2}{\epsilon^2}\right)}\right)$$

In[19]:= `PowerExpand[Simplify[Series[u, {ϵ, 0, 2}]]]`

Out[19]=
$$-a_1+\frac{1}{8}a_2\,\epsilon^2\sin(2\,\psi)+O(\epsilon^3)$$

In[20]:= `δ2 = FullSimplify[(G M m)/Fmag]`

Out[20]=
$$R^2\Bigg/\left(\sqrt{\left(\left(\frac{1}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}+\frac{1}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}\right)^2+\frac{\csc^2(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)^2}{\epsilon^2}\right)}\right)$$

In[21]:= `cog = PowerExpand[FullSimplify[u √δ2]]`

Out[21]=
$$\left(R\left(a_1\left(-\left(\frac{1}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}+\frac{1}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}\right)\right)-\frac{a_2\csc(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)}{\epsilon}\right)\right)\Bigg/$$
$$\left(\left(\frac{1}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}+\frac{1}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}\right)^2+\frac{\csc^2(\psi)\left(\frac{2-\epsilon\cos(\psi)}{\sqrt{\epsilon^2-4\,\epsilon\cos(\psi)+4}}-\frac{\epsilon\cos(\psi)+2}{\sqrt{\epsilon^2+4\,\epsilon\cos(\psi)+4}}\right)^2}{\epsilon^2}\right)^{3/4}$$

In[22]:=
```
(*These are the components of the position vector
 from the mass center to the center of gravity.*)
```

In[23]:=
```
D[R a₁ + PowerExpand[Simplify[Series[cog, {ϵ, 0, 2}]]], a₁]
```

Out[23]=
$$\frac{1}{32} R \epsilon^2 (3 \cos(2 \psi) + 1) + O\left(\epsilon^3\right)$$

In[24]:=
```
D[R a₁ + PowerExpand[Simplify[Series[cog, {ϵ, 0, 2}]]], a₂]
```

Out[24]=
$$\frac{1}{8} R \epsilon^2 \sin(2 \psi) + O\left(\epsilon^3\right)$$