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A UNIFIED DEVELOPMENT OF BASIS REDUCTION METHODS FOR ROTOR BLADES

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ABSTRACT

The axial foreshortening effect plays a key role in rotor blade dynamics, but approximating it accurately in reduced basis models has long posed a difficult problem for analysts. Recently, though, several methods have been shown to be effective in obtaining accurate, reduced basis models for rotor blades. These methods are the axial elongation method, the mixed finite element method, and the nonlinear normal mode method. The main objective of this paper is to demonstrate the close relationships among these methods, which are seemingly disparate at first glance. First, the difficulties inherent in obtaining reduced basis models of rotor blades are illustrated by examining the modal reduction accuracy of several blade analysis formulations. It is shown that classical, displacement-based finite elements are ill-suited for rotor blade analysis because they cannot accurately represent the axial strain in modal space, and that this problem may be solved by employing the axial force as a variable in the analysis. It is shown that the mixed finite element method is a convenient means for accomplishing this, and the derivation of a mixed finite element for rotor blade analysis is outlined. A shortcoming of the mixed finite element method is that it increases the number of variables in the analysis. It is demonstrated that this problem may be rectified by solving for the axial displacements in terms of the axial forces and the bending displacements. Effectively, this procedure constitutes a generalization of the widely used axial elongation method to blades of arbitrary topology. The procedure is developed first for a single element, and then extended to an arbitrary assemblage of elements of arbitrary type. Finally, it is shown that the generalized axial elongation method is essentially an approximate solution for an invariant manifold that can be used as the basis for a nonlinear normal mode.

1 Introduction

The application of the finite element method to rotorcraft analysis over the past two decades has removed the topological restrictions on the models that can be analyzed using older, so-called "first generation" rotorcraft codes. Although final analyses are best done in finite element space for reasons of accuracy and reliability, it is often convenient to reduce to the size of the finite element model to a small number of generalized coordinates to improve execution time or to assist in interpreting and understanding the results. The basis reduction process is typically accomplished via the *modal reduction* method, which employs eigenmodes computed about some convenient state. Unfortunately, it is often quite difficult to approximate blade response accurately using just a few eigenmodes when the axial motion of the finite element model is parameterized using the Lagrangian axial displacement variable. The reason is that the near inextensibility of the blade effectively couples the blade's axial and bending motions through the so-called "axial foreshortening" effect. However, classical eigenmodes are unable to approximate this key phenomenon accurately when the blade bends more than one or two degrees away from the state about which the modes are computed. As a result, the blade's axial displacement, and more important, the blade's axial force, may be not get computed accurately. But since the blade's bending stiffness is mostly geometric stiffness generated by the axial force, incorrectly evaluating that quantity can lead to significant errors in blade response.

The computational problems caused by the axial foreshortening effect have long been recognized in the rotorcraft literature and elsewhere. The oldest and most widely used solution to the problem involves parameterizing the axial displacement using the non-Lagrangian *axial elongation* variable instead of the *axial displacement* variable, and was proposed independently in (Smith, 1992) and in (Kane, 1987). The effectiveness of the approach has been well-documented, but the non-Lagrangian nature of the variable complicates the finite element assembly process, a feature which has spurred the search for alternative approaches. Mixed finite elements, hereafter abbreviated as "mixed elements," have been proposed for rotor blade analysis in (Hodges, 1990), and in (Bauchau, 1993). In (Bauchau, 1993), the modal reduction accuracy of mixed elements was studied, but disappointing results were obtained, especially for torsion response, even when the mixed elements were supplemented by perturbation modes (Noor, 1980). But more recently, (Ruzicka, 1999) and (Ruzicka, 2001) demonstrated generally good modal reduction accuracy for all blade motions, including torsion, when mixed elements were used to model an articulated rotor blade. In (Ruzicka, 2000), the mixed element method is further refined by using the axial force equations to eliminate the Lagrangian axial displacements from the equations, which results in what may be regarded as a generalization of the axial elongation method to blades of arbitrary topology. Still another approach to basis reduction is the "nonlinear normal mode" method proposed in (Shaw, 1993) which replaces the classical eigenvector, with its fixed relationships among the degrees of freedom, with a nonlinear set of functions termed an "invariant manifold," which is extracted from the governing equations. In (Pescheck, 2001a), a preliminary step toward unifying the axial elongation variable and nonlinear normal mode methods was the recognition that the classical inextensibility approximation—which may be viewed as a special case of the axial elongation method—leads immediately to an approximation for the invariant manifold of a simple rotor blade. But the topological limitations inherent in the inextensibility condition impeded the authors from extending the unified treatment to extensible blades of arbitrary topology.

This paper first reprises the mixed element study presented in (Ruzicka, 2000), and then extends it by developing a relationship between the axial elongation method and the nonlinear normal mode method. The paper thereby achieves a unified development of three key methods for rotor blade basis reduction: the axial elongation method, the mixed finite element method, and the nonlinear normal mode method. The mixed finite element method is the most extensively treated of the three methods because it occupies the most central role procedurally, and because it is not well known to most investigators in rotor blade dynamics. In what follows, a rationale for the use of mixed elements is developed starting from the well-known Hodges-Dowell equations (Hodges, 1974) of a rotor blade specialized to axial and flap motions. The derivation of a mixed element is then described, and the element's effectiveness in modal reduction is illustrated by applying it to the analysis of an articulated blade model. Then, the mixed finite element is altered by employing the axial force equations to eliminate the Lagrangian axial displacements in favor of the axial forces for the case of a single element. To extend the single element analysis to an arbitrary assemblage of beam elements, the "axial displacement" notion is suitably generalized, and then a procedure is presented that solves for the generalized axial displacements in terms of the bending displacements and the axial forces. Essentially, this procedure constitutes a generalization of the axial elongation method to blades of arbitrary topology. Finally, the generalized axial elongation relationship is transformed to modal variables, and shown to have the same form of an invariant manifold used in nonlinear normal mode analysis.

2 Rationale for Mixed Elements

2.1 Preliminaries: Hodges-Dowell Flap-Axial Blade Equations

The starting point for developing a rationale for mixed elements is the Hodges-Dowell blade equations (Hodges, 1974) specialized to coupled axial-flap motions:

$$V_x' = -m\Omega^2 x - f_x \tag{2-1}$$

$$m\ddot{w} = (V_x w')' - (EI_y w'')'' + f_z \tag{2-2}$$

where V_x is the axial force, m is the mass per unit length, x is the axial coordinate, Ω is the rotor angular speed, w is the flap displacement, EI_y is the cross section flap flexural rigidity, and f_x and f_z are applied forces per unit length. Following the usual conventions in the rotorcraft literature, a prime ($'$) denotes partial differentiation with respect to the axial coordinate, and a dot ($\dot{}$) denotes partial differentiation with respect to time. If the blade is clamped at the spin axis, the boundary conditions are:

$$w|_{x=0} = w'|_{x=o} = 0 \tag{2-3}$$

These equations are particularly useful for examining blade analysis methods because they are simple enough to permit easy inspection, yet they embody the coupling between axial and flap degrees of freedom, which includes the axial foreshortening phenomenon.

In what follows, several mathematical formulations for rotor blade analysis are presented, culminating with a mixed element. The formulations are evaluated based on their treatment of axial motions, and the key features of the mixed element are introduced in a step-by-step fashion.

2.2 Method 1: The Axial Displacement Variable

This method represents all variables in terms of displacement fields, and then applies polynomial discretizations of the fields over sub-regions of the model, which are simply finite elements. The following expression for axial force is used, which is consistent with the Hodges-Dowell ordering scheme:

$$V_x = EA\left(u' + \frac{1}{2}w'^2\right) \tag{2-4}$$

where EA is the axial stiffness of the blade cross section. Equation (2-1) neglects the small axial inertia term, but this termSubstituting equation (2-4) into the Hodges-Dowell equations and restoring the unsteady axial inertia term that has been omitted from equation (2-1) gives:

$$m\ddot{u} = \left[EA\left(u' + \frac{1}{2}w'^2\right)\right]' + m\Omega^2 x + f_x \tag{2-5}$$

$$m\ddot{w} = \left[EA\left(u' + \frac{1}{2}w'^2\right)w'\right]' - \left(EI_y w''\right)'' + f_z \tag{2-6}$$

where $m\ddot{u}$ is formally negligible but is included here for consistency with other treatments in the literature. The boundary conditions for a blade clamped at the spin axis are

$$u|_{x=0} = w|_{x=0} = w'|_{x=0} = 0 \tag{2-7}$$

Equations (2-5) and (2-6) allow for full discretization of the blade model in the axial direction, thereby permitting the full power of the finite element method to be brought to bear on the analysis. Unfortunately, a high price for this flexibility stems from the difficulty of approximating the axial force, which is critical for accurately representing the blade bending stiffness. The source of the difficulty is that owing to the near inextensibility of the blade, the constituent parts of the axial strain, $\varepsilon_x = \frac{V_x}{EA} = u' + \frac{1}{2}w'^2$, are opposite and nearly equal whenever the flap displacement becomes significant; i.e.,

$$u' \simeq -\frac{1}{2}w'^2 \tag{2-8}$$

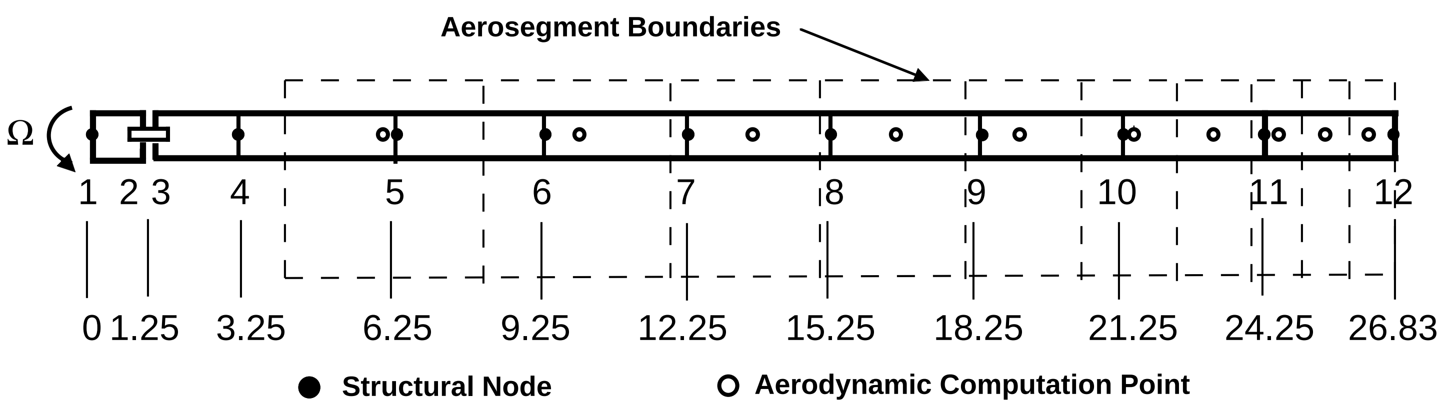
and therefore $|\varepsilon_x| \ll |u'|$, $|\varepsilon_x| \ll w'^2$. In other words, ε_x is the small difference of much larger quantities, and computing it accurately requires far more accuracy in both u' and w'^2 than can be obtained by representing either of these using a small number of eigenmodes. This situation is an example of the "small difference of large numbers" conundrum that is often encountered in computational mechanics.

A physical consequence of the blade's near inextensiblity is that the axial displacement can usually be computed accurately, for single load path blades, by simply integrating equation (2-8), viz.

$$u(x) \simeq -\int_0^x \frac{1}{2} w'^2 ds \tag{2-9}$$

In other words, the contribution of the blade's stretch to the axial displacement is generally negligible. Equation (2-9) explains why a point on a blade usually moves radially inward whenever the blade bends. The right-hand side of equation (2-9) is often the major contributor to the axial deflection, and is generally termed the "axial foreshortening," but another – and perhaps more explanatory – label is the "bending contribution to the axial displacement."

The computational difficulties stemming from use of the axial displacement variable in problems where the axial foreshortening effect is prominent will now be illustrated using several computational examples. The blade model used is shown in Figure (2-1). Consider, first, an eigenanalysis of the spinning blade. For this analysis, *in vacuo* conditions are assumed, along with zero swashplate



Structural Properties	Aerodynamic Properties	Flight Conditions
Radius - 26.83 ft. EI_y - 1.85E5 lb-ft^2 EI_z - 4.18E5 lb-ft^2 GJ - 2.08E5 lb-ft^2 EA - 6.69E8 lb 92 DOF's	SC1095 Airfoil Uniform Inflow Greenberg Aerodynamics Chord - 1.7405 ft	Ω=258 rev/minute θ_O=10.0 deg. θ_C=2.0 deg. θs=-7.0 deg. Θ=4.0 deg. $\Psi=\Phi$=0.0 deg. μ=.373, C_T/σ=.072

Figure 2-1. Articulated Blade Model

angles, and the blade's modes are computed about steady-state spin. Then, the static response of the blade, in modal space, is computed for tip flap loads of 2539.6 pounds and 5079.2 pounds, which correspond to coning angles of 4° and 8°, respectively. These are typical of the mean coning angles in rotorcraft under flight conditions (Johnson, 1980). The modal basis contains the first two flap modes, while the number of axial modes were varied to study the convergence of the first two flap frequencies. The flap frequencies are plotted against the number of axial modes in Figure (2-2). Note that thirty axial modes completely fills the axial displacement subspace of the model. It

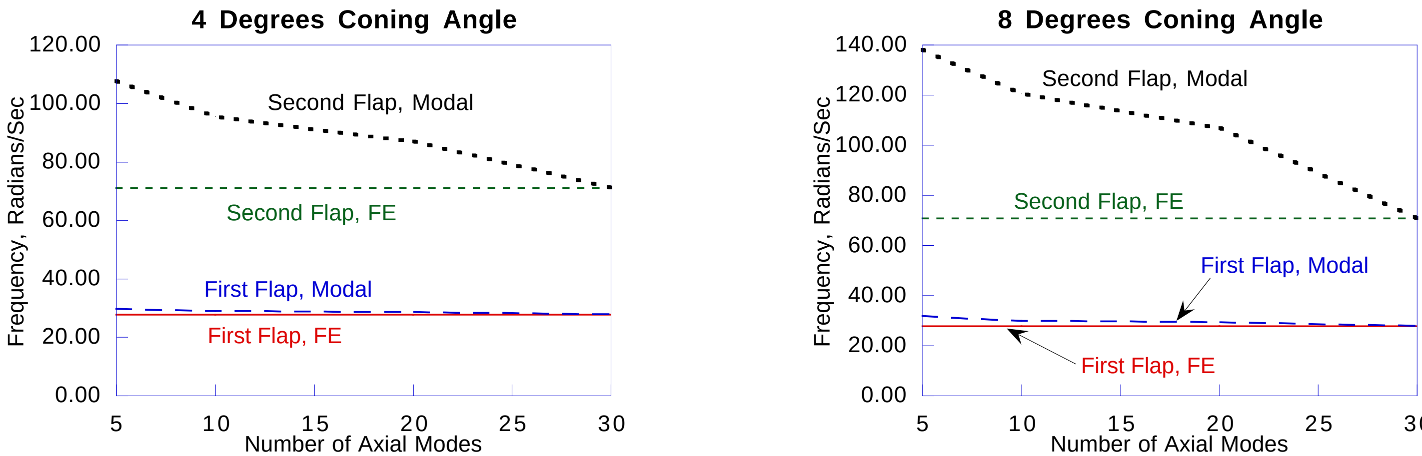


Figure 2-2. Articulated Blade Flap Frequencies: Displacement Elements

may be seen that while reasonable results are obtained for the first flap frequency with just a few axial modes, the accuracy of the second flap frequency is much poorer, and its accuracy does not become acceptable until almost the entire axial subspace is filled.

To further illustrate the problems of the axial displacement variable, consider the periodic solution of the model shown in Figure (2-1). The *in vacuo* eigenmodes of this model are given in Table (2-1), and the modal bases used in the calculations are given in Table (2-2). Note that in contrast to the theory presented earlier, the model is constrained so that all motions – not just flap and

Mode ID	Frequency (/rev)
First Lag	0.27
First Flap	1.03
Second Flap	2.63
Second Lag	4.09
First Torsion	4.86
Second Torsion	14.59
First Axial	22.12
Second Axial	66.41

Table 2-1. Articulated Blade Modes

axial – are permitted. Periodic solutions for the degrees-of-freedom at the tip node are shown in Figure (2-3). It may be seen that the agreement between the modal and finite element solutions is quite poor for the flap, lag and axial displacements, as was expected. But surprisingly, the agreement between the finite element and modal solutions is quite good for the pitch rotation. The reasons for the good modal approximations are the high torsion stiffness of the blade, combined with the low degree of bending-torsion coupling in stiff, articulated blades of the type analyzed here. Since the pitch response of the blade is the main contributor to the aerodynamic angle of attack, the good modal approximation for pitch implies that the poor results for the bending and axial motions must come from errors

Modal Basis	Number of Modes			
	Lag	Flap	Torsion	Axial
1l,1f,1t,1a	1	1	1	1
2l,2f,2t,2a	2	2	2	2

Table 2-2. Description of Modal Bases

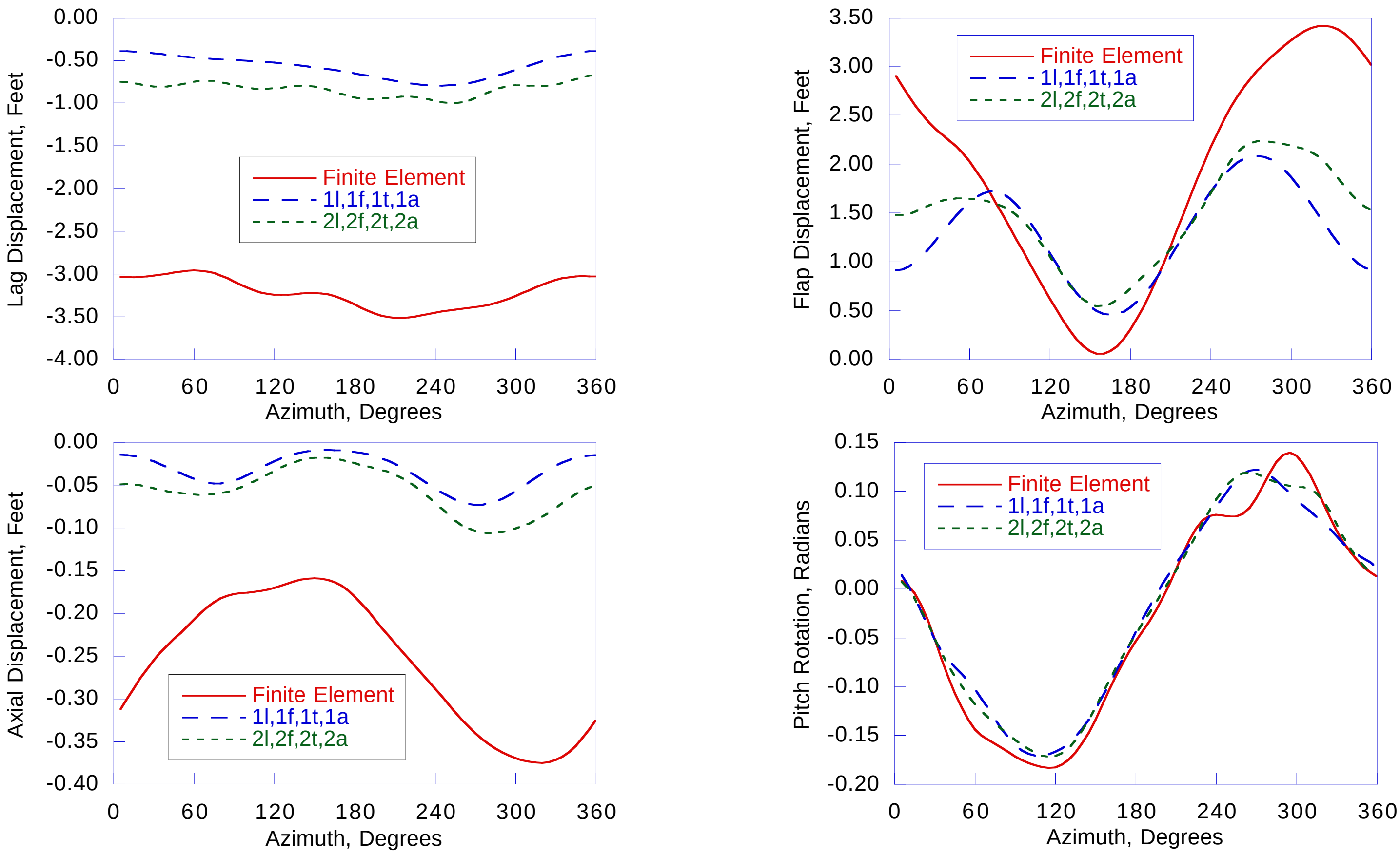


Figure 2-3. Articulated Blade Tip Deflections: Displacement Elements

made in computing those motions, rather than from erroneous aerodynamic loading. Therefore, the approach taken here in focusing on bending-axial response to evaluate blade formulations is fully justified.

2.3 Method 2: The Axial Force Variable

One approach to modifying the analysis formulation to facilitate modal reduction involves parameterizing the axial motion using the axial force rather than the axial displacement. The axial displacement may be written in terms of the axial force using equation (2-4):

$$u' = \frac{V_x}{EA} - \frac{1}{2}w'^2 \tag{2-10}$$

or

$$u = \int_0^x \left(\frac{V_x}{EA} - \frac{1}{2} w'^2 \right) ds \tag{2-11}$$

With this reparameterization, the axial force is now expressed as a single variable, and the numerical conditioning problems associated with the axial displacement variable are absent.

The need for an additional improvement to the analysis procedure becomes evident upon applying the axial force variable to the Hodges-Dowell equations. Substituting equation (2-11) into equations (2-5) and (2-6) gives:

$$m \int_0^x \left(\frac{\ddot{V}_x}{EA} - \dot{w}'^2 - w' \ddot{w}' \right) ds = V_x' + m\Omega^2 x + f_x \tag{2-12}$$

$$m\ddot{w} = \left(V_x w' \right)' - \left(EI_y w'' \right)'' + f_z \tag{2-13}$$

Simplifying equations (2-12) and (2-13) by removing the negligible terms arising from $m\ddot{u}$ leads, once again, to the Hodges-Dowell equations (equations (2-1) and (2-2)). An examination of those equations reveals a quadratic nonlinearity, $(V_x w')'$, which is significant enough to adversely impact modal reduction. Indeed, it was found that a mixed element formulated analogously to equation (2-13) had only modestly improved modal reduction capabilities in comparison with the displacement element. This problem may be removed by expressing the variables as sums of steady state values and dynamic perturbations:

$$V_x = V_{x_0} + \Delta V_x \tag{2-14}$$

$$w = \Delta w \tag{2-15}$$

For simplicity, let the external forces be associated with the perturbations, viz.

$$f_x = \Delta f_x \tag{2-16}$$

$$f_z = \Delta f_z \tag{2-17}$$

Substituting these representations into equations (2-12) and (2-13) and placing linear terms on the left-hand side gives:

$$\Delta V_x' = \Delta f_x \tag{2-18}$$

$$m\Delta\ddot{w} - \left(V_{x_0} \Delta w' \right)' - \left(\Delta V_x w_0' \right)' + \left(EI_y \Delta w'' \right)'' = \left(\Delta V_x \Delta w' \right)' + \Delta f_z \tag{2-19}$$

Inasmuch as the axial force in a rotor blade varies little from its steady-state value, the only nonlinear term appearing in these equations, $\left(\Delta V_x \Delta w' \right)'$, will be quite small.

Although the axial force variable has not been applied to rotor blades in the form described here, its likely effectiveness in modal reduction may be inferred from the success of a similar concept, the axial elongation variable, which is defined as:

$$u_e = \int_0^x \frac{V_x}{EA} ds \tag{2-20}$$

Clearly, the underlying motivations of the axial force and axial elongation variables, insofar as they treat axial forces, are identical and their implementations are quite similar. The effectiveness of the axial elongation variable at modal reduction has been well documented (Smith, 1992). Although the axial elongation variable is the preferred parameterization in rotorcraft practice, the axial force variable is preferred in this paper because it is consistent with the literature on mixed elements.

Although the axial force variable should greatly facilitate modal reduction of the blade equations, severe pitfalls may result if it is not implemented in a suitable manner. For example, if the method is implemented as described in (Smith, 1992), the axial deflection must must be computed by integrating the axial force outward from the spin axis, as implied by equation (2-11). But this requires that an element variable, the *u* displacement, must be computed from nodes other than those to which the element is attached, which necessitates a major modification to the usual finite element assembly process. For example, equation (2-11) implies that the axial displacement of the root of the *n*th element outboard from the spin axis in a single load path blade must be computed as follows:

$$u = \sum_{i=1}^{n-1} \int_0^x \left(\frac{V_x}{EA} - \frac{1}{2} w'^2 \right)_i ds \tag{2-21}$$

where the subscript *i* signifies the *i*th element outboard from the spin axis. Conversely, computing the element's axial forces requires summing the axial forces on any outboard elements in an analogous fashion. A more serious consequence of computing the axial response using equation (2-21) is that that quantity is uniquely defined only when a single load path is present; in other words, it is applicable only to blade models with a *tree topology*. Special procedures are required to handle multiple load paths, which occur in bearingless rotors, but which also can arise in blades of any configuration when that are modeled using two or three dimensional elements. It will be seen later, however, that the mixed finite element method may used to remove these difficulties, and offers a means to apply the axial elongation approach to blades of arbitrary topology.

2.4 Method 3: Axial Force *and* Axial Displacement Variables – A Mixed Element

It has been shown that parameterizing axial motions using the Lagrangian axial displacement gives unrestricted modeling freedom at the expense of ill-conditioned and highly nonlinear axial-bending coupling, while using the axial force as a variable makes modeling more awkward, but simplifies axial-bending coupling. We will now seek to obtain the advantages of both methods – without their limitations – by using both axial displacements *and* axial forces as variables. That may be accomplished by augmenting the Hodges-Dowell equations with an equation that relates the axial force and the axial displacement. Three equations result:

$$\frac{V_x}{EA} = u' + \frac{1}{2} w'^2 \tag{2-22}$$

$$m\ddot{u} = V_x' + m\Omega^2 (x + u) + f_x \tag{2-23}$$

$$m\ddot{w} = \left(V_x w' \right)' + f_z \tag{2-24}$$

Since axial forces *and* displacements are variables in these equations, it is dubbed a *mixed element*.

The advantages of the mixed element equations are more readily appreciated when they are expressed in variational form. That process involves applying finite element interpolations to the three independent variables: $V_x = [H_{V_x}]\{q_{V_x}\}$, $u = [H_u]\{q_u\}$, and $w = [H_w]\{q_w\}$. In what follows, all variables are assumed to have been discretized, but – when possible – they are displayed as continuous for improved readability. The finite element interpolations are substituted into equations (2-22), (2-23), and (2-24), which are then are multiplied by δV_x, δu, and δw, respectively. The results of these operations are then summed and integrated over the length of the element giving:

$$\int_0^l (\{\delta q_{V_x}\}^T \{\bar{Y}_{V_x}\} + \{\delta q_u\}^T \{\bar{Y}_u\} + \{\delta q_w\}^T \{\bar{Y}_w\}) dx = 0 \tag{2-25}$$

where:

$$\{\bar{Y}_{V_x}\} = [H_{V_x}]^T \left(\frac{V_x}{EA} - u' - \frac{1}{2} w'^2 \right) \tag{2-26}$$

$$\{\bar{Y}_u\} = [H_u]^T [m\ddot{u} - V_x' - m\Omega^2(x+u) - f_x] \tag{2-27}$$

$$\{\bar{Y}_w\} = [H_w]^T \left[m\ddot{w} - \left(V_x w'\right)' - f_z\right] \tag{2-28}$$

For future reference, note that it is often convenient to express the discretized form of u' as follows:

$$u' = \left[H_u'\right]\{q_u\} = \left[H_{u'}\right]\{q_{u'}\} \tag{2-29}$$

for some set of unknowns, $\{q_{u'}\}$, which is of size $N_u - 1$.

The usefulness of the mixed element for modal reduction may be inferred from arguments similar to those used in discussing the axial force variable. First, consider the $\{\delta q_u\}$ variation. Setting the coefficient of that variation to zero gives, after removing the small contributions from u, and $\ddot{u}$:

$$\int_0^l [H_u]^T [V_x' - m\Omega^2(x+u) - f_x] dx = 0 \tag{2-30}$$

which is analogous to the first of the Hodges-Dowell equations, and which largely determines V_x. Thus, in contrast to the axial displacement variable method, the axial force, which is a key quantity, is determined directly from what is essentially an equilibrium equation. Using V_x in the $\{\delta q_w\}$ variation gives:

$$\int_0^l [H_w]^T \left[m\ddot{w} - \left(V_x w'\right)' - f_z\right] dx = 0 \tag{2-31}$$

which is analogous to the second of the Hodges-Dowell equations, and which largely determines w. Finally, the $\{\delta q_{V_x}\}$ variation leads to:

$$\int_0^l [H_{V_x}]^T \left(\frac{V_x}{EA} - u' - \frac{1}{2}{w'}^2\right) dx = 0 \tag{2-32}$$

which is analogous to equation (2-11), and which largely determines u. Observe that *weak enforcement* of the governing equations is crucial in allowing the axial force to be computed as a separate variable. Also, weak enforcement of the force-displacement equation (equation (2-32)) permits the implicit determination of u from that equation and eliminates the need to constrain the model's topology in order to be able to calculate u by integrating outward from the spin axis as in equation (2-11).

The mixed element equations just presented (i.e., equations (2-25) – (2-28)), were developed in a rather *ad hoc* manner, and apply only to a blade with coupled axial and flap motions. The Appendix presents the derivation of a complete mixed finite element for rotor blade applications, in which coupled axial, flap, lag, and torsion motions are considered, and the structural terms are developed systematically from a mixed variational principle.

The accuracy of the mixed element developed in the Appendix will now be evaluated using the examples used in evaluating the axial displacement variable. The first example is the flap frequencies of the articulated blade. Plots of the first two flap frequencies versus the number of axial modes are shown in Figure (2-4). In contrast to the corresponding results seen in Figure (2-2) for the displacement element, only a single axial mode – but actually, two generalized coordinates – are required to match the finite element results accurately.

Periodic solutions for finite element and modal bases are compared in Figure (2-5). for blade tip displacements, and in Figure (2-6) for blade root loads. It may be seen that there is a dramatic improvement in how the modal solutions approximate blade tip displacements when compared with the corresponding results for the displacement elements (Figures (2-3). The axial displacement is not approximated quite as well as the other displacements, probably because of the nonlinearity of the axial foreshortening effect. However, the primary interest in the axial displacement is its impact on lag moments through the Coriolis effect, but as may be seen in Figure (2-6), the modal approximations of the root lag moments is quite good, which suggests that the modal approximation of the axial displacement is satisfactory for practical purposes. Still, it is desirable to improve the modal approximation of the axial displacement, and a technique for doing that is the subject of the remainder of this paper.

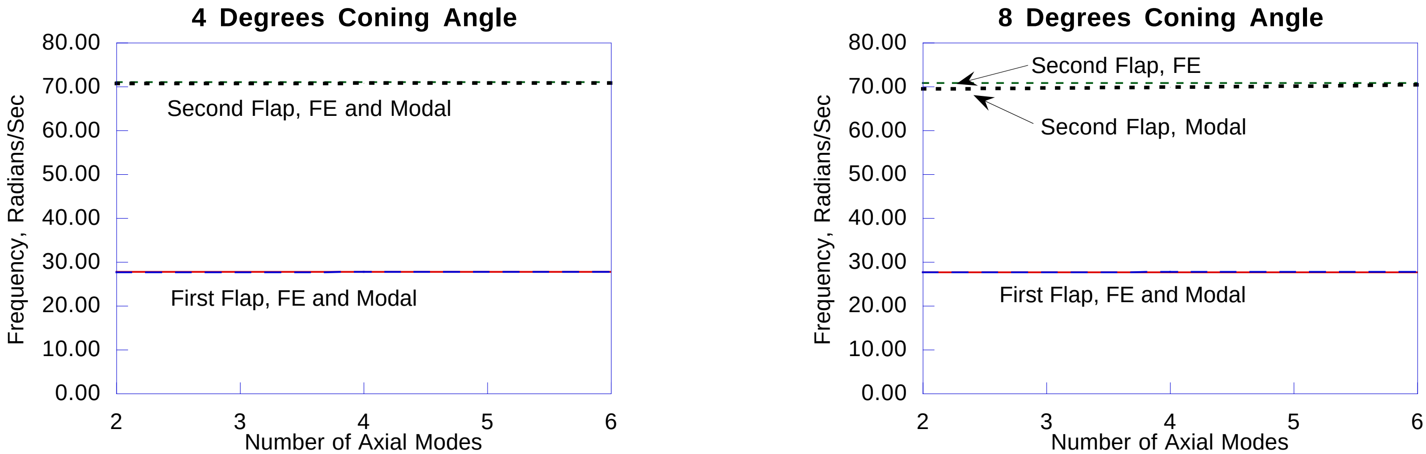


Figure 2-4. Articulated Blade Flap Frequencies: Mixed Elements

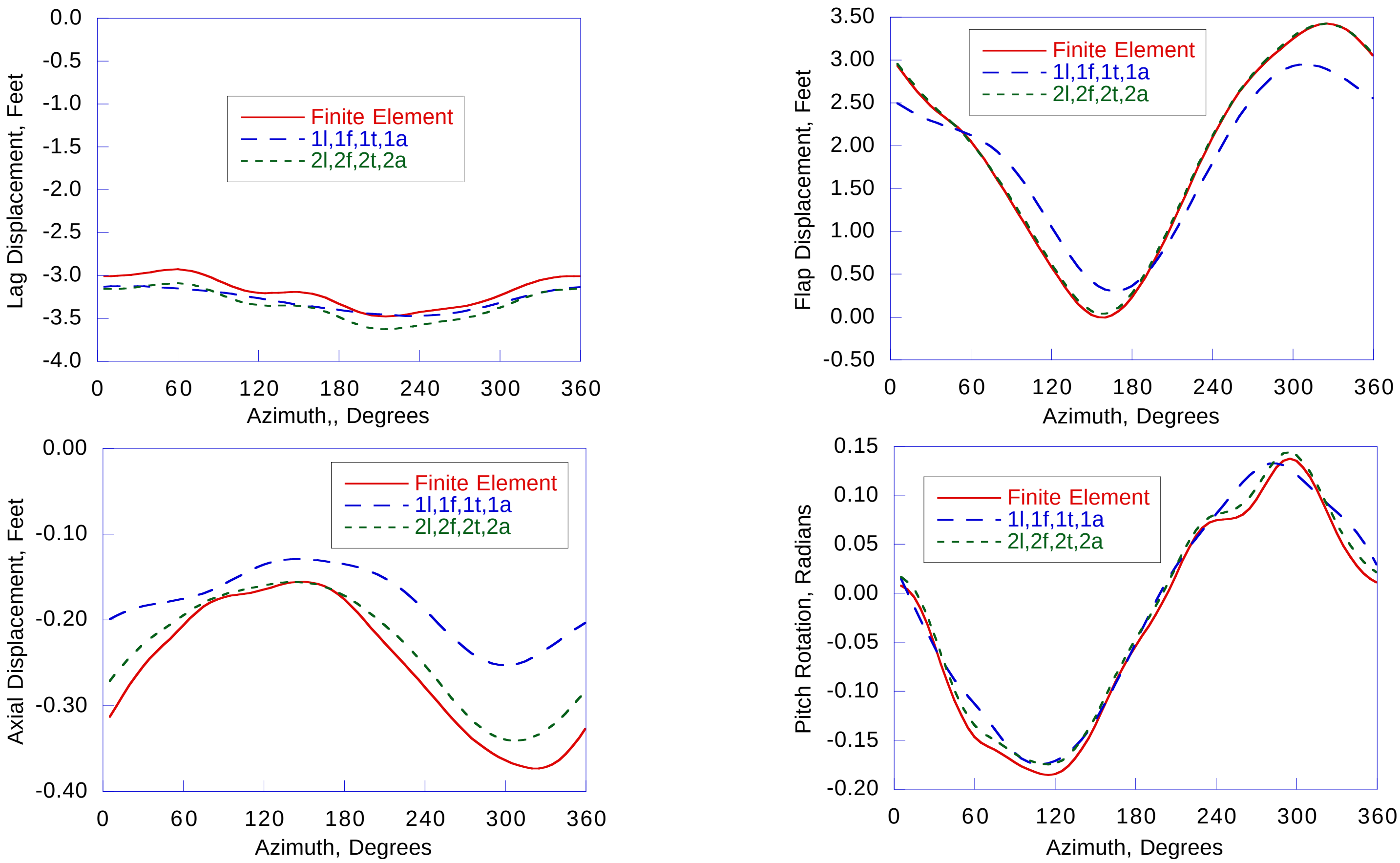


Figure 2-5. Articulated Blade Tip Deflections: Mixed Elements

3 Mixed Finite Element Analysis of Axial Displacement

3.1 Overview

An interesting feature of the axial force variable method becomes apparent upon examination of equation (2-11.) As noted earlier, the axial strain is significantly smaller than the deformation gradient quantities, and if that term is dropped, there results:

$$u = -\int_0^x \frac{1}{2} w'^2 ds \qquad (3\text{-}35)$$

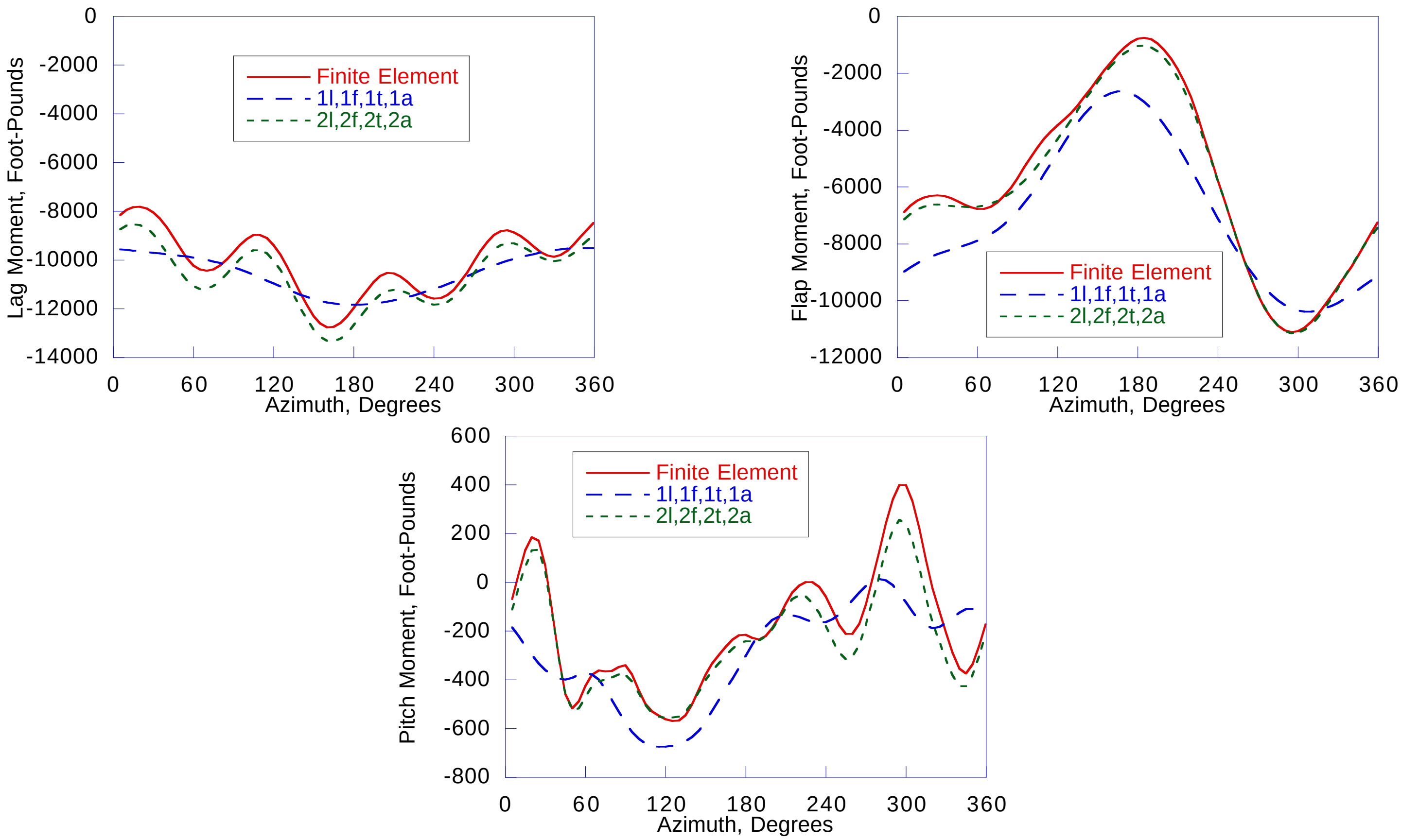


Figure 2-6. Articulated Blade Root Loads: Mixed Elements

which is the axial foreshortening term. In other words, the axial foreshortening effect is embedded within the axial force variable method. The mixed finite element analogue of equation (2-11) is equation (2-32), which cannot approximate the foreshortening effect well in modal space because the axial displacement is obtained by a *linear* superposition of eigenmodes. However, it seems that if – analogous to equation (2-11) – one were to solve equation (2-32) for the axial displacement in terms of the remaining variables, the axial foreshortening effect would be accurately recovered. This is the strategy that will be used in this section to improve the modal representation of the axial displacement.

3.2 Axial Displacement Analysis for a Single Mixed Element

For the case of a single element, the first step in computing the axial displacement involves substituting the finite element discretization for u in equation (2-32). Then, rearranging terms gives:

$$\left(\int_0^l [H_{V_x}]^T [H_{u'}] dx\right) \{q_{u'}\} = \int_0^l [H_{V_x}]^T \left(\frac{V_x}{EA} - \frac{1}{2} w'^2\right) dx \qquad (3\text{-}34)$$

Convergence considerations dictate (Szabo, 1991) that

$$N_u - 1 = N_{V_x} \qquad (3\text{-}35)$$

and recalling that $\{q_{u'}\}$ is of length $N_u - 1$, it may be concluded that the coefficient of $\{q_{u'}\}$ in equation (3-34) is a square matrix. Since that matrix should also be nonsingular in a properly formulated element, equation (3-34) can be solved for $\{q_{u'}\}$ as follows:

$$\{q_{u'}\} = \left(\int_0^l [H_{V_x}]^T [H_{u'}] dx\right)^{-1} \int_0^l [H_{V_x}]^T \left(\frac{V_x}{EA} - \frac{1}{2}w'^2\right) dx \tag{3-36}$$

Upon noting that $u'_E = \frac{V_x}{EZ}$, it becomes evident that equation (3-36) embeds the axial foreshortening effect in the mixed finite element analysis, and since $u'_e = \frac{V_x}{EA}$, that equation is a particularly vivid illustration of the links between the axial force variable, the axial elongation variable, and the mixed finite element methods.

3.3 Axial Displacement Analysis for General Structures

3.3.1 Extensional-Inextensional Decomposition of Displacement Field In order to extend the axial displacement analysis to structures of arbitrary topology, the definition of the quantity must be appropriately generalized. For example, in blades with droop, sweep, or precone, the axial displacement is discontinuous at transition points, and in multiple load path blades, it may be multiply defined where load paths intersect. The axial displacement variable may be generalized based on the notion that it is a displacement field that makes elements stretch. This idea will now be further developed by using the δV_x equations of the mixed finite element, which relate axial forces and axial displacements.

If the $\{\delta q_{V_x}\}$ equations of the model are collected together, they may be arranged in the form:

$$[\bar{U}]\{q_u\} = -[\bar{V}]\{q_{V_x}\} + \{f_{FS}\} \tag{3-37}$$

where $[\bar{U}]$ $(N_V \times N_U)$ and $[\bar{V}]$ $(N_V \times N_V)$ are matrices and $\{f_{FS}\}$ $(N_V \times 1)$ are the nonlinear terms arising from axial foreshortening, in which N_V are the number of force degrees of freedom, and N_U is the number of displacement degrees of freedom. Since the model is assumed to contain only structural elements, $N = N_U + N_V$ where N is the total number of degrees of freedom. The matrix $[\bar{U}]$ embodies the features of the displacement field that are needed to generalize the axial displacement concept to structures of arbitrary configuration. The *null space* of $[\bar{U}]$ is a subspace of displacement vectors such that $[\bar{U}]\{q\} = 0$, and the dimension of this subspace is $nullity(\bar{U})$, which means that is contains $nullity(\bar{U})$ basis vectors. Members of this subspace do not generate axial forces in the model, and can only make elements of the model bend or undergo perturbational rigid motions. Therefore, the null space may also be thought of as the *inextensional subspace* of the model. It can be shown from basic matrix theory that the null space may be expanded into a basis that spans the entire displacement space of the finite element model. The additional basis vectors have the property that $[\bar{U}]\{q\} \neq 0$, which means that they cause the elements to stretch. If these additional basis vectors are made orthogonal to the null space, then the subspace that they span will consist *only* of stretching motions, and will be referred to as the *extensional subspace* of the model. The dimension of the extensional subspace is $rank(\bar{U})$, and from basic matrix theory, we have the important relationship:

$$rank(\bar{U}) + nullity(\bar{U}) = N_U \tag{3-38}$$

Therefore, any displacement vector may be regarded as a combination of extensional and inextensional basis vectors. The measure numbers of the extensional bases are the *extensional coordinates* of a displacement vector, and are generalizations of the axial displacement degrees of freedom of the single element model. Correspondingly, the measure numbers of the inextensional basis vectors are the *inextensional coordinates*, and are generalizations of the bending degrees of freedom of the single element structural model.

Another important result from basic matrix theory is that

$$dimension(range(\bar{U})) = rank(\bar{U}) \tag{3-39}$$

which implies that if $rank(\bar{U}) < N_V$, then not all the axial forces are independent. The model will then have redundant or multiple load paths, and the degree of redundancy will be equivalent to the number of independent relationships among the axial forces. These relationships are generated as byproducts of the solution process to be described shortly.

Generalizing the single element analysis procedure described earlier involves using equation (3-37) to solve for the extensional part of the displacement field, $\{q_{ext}\}$ in terms of the axial forces and the foreshortening terms.

3.3.2 Solution Procedure The most common method for solving for $\{q_{ext}\}$ employs the *singular-value decomposition* of $[\bar{U}]$. Recall that the singular-value decomposition of $[\bar{U}]$ may be written:

$$[\bar{U}] = [A]\Sigma[B]^T \tag{3-40}$$

where $[A]$ $(N_V \times N_V)$ are the left singular vectors, and B^T $(N_U \times N_U)$ are the right singular vectors. The matrix Σ may be written as:

$$\Sigma = [\Sigma_T \quad 0] \tag{3-41}$$

where $[\Sigma_T]$ is a diagonal matrix that lists, in descending order, the positive square roots of the eigenvalues of $[\bar{U}][\bar{U}]^T$. Since $[\bar{U}][\bar{U}]^T$ is generally positive semi-definite, its eigenvalues are either zero or positive. The positive eigenvalues are associated with the extensional displacement field, and will be denoted with the subscript *ext*, while the zero eigenvalues are associated with the inextensional displacement field, and will be denoted with the subscript *inext*. It proves convenient to write Σ in the alternative form:

$$\Sigma = \begin{bmatrix} \Sigma_{ext} & 0 \\ 0 & 0 \end{bmatrix} \tag{3-42}$$

where $[\Sigma_{ext}]$ $(rank(\bar{U}) \times rank(\bar{U}))$ contains the positive part of Σ_T. The matrices $[A]$ $(N_V \times N_V)$ and $[B]$ $(N_u \times N_u)$ appearing in equation (3-40) are orthogonal matrices, in which the columns of $[A]$ are the eigenvectors of $[\bar{U}][\bar{U}]^T$, and the columns of $[B]$ are the eigenvectors of $[\bar{U}]^T[\bar{U}]$. It follows that $[A]$ and $[B]$ may be partitioned as follows:

$$[A] = [A_{ext}|A_{inext}] \tag{3-43}$$

$$[B] = [B_{ext}|B_{inext}] \tag{3-44}$$

The ordering of the columns of $[A]$ and $[B]$ pertaining to nonzero eigenvalues matches the ordering of their corresponding eigenvalues in $[\Sigma]$.

Consider, now, the matrix $[B]$. Since the columns of this matrix fully span the finite element space, the following coordinate transformation may be applied to an arbitrary vector $\{q_u\}$:

$$\{q_u\} = [B]\begin{Bmatrix} q_{ext} \\ q_{inext} \end{Bmatrix} \tag{3-45}$$

Substituting equation (3-45) into equation (3-37), premultiplying by $[A]^T$, and then premultiplying the top partition by Σ_{ext}^{-1} gives:

$$q_{ext} = (\Sigma_{ext})^{-1}[A_{ext}]^T(-[\bar{V}]\{q_V\} + \{f_{FS}\}) \tag{3-46}$$

$$0 = [A_{inext}]^T(-[\bar{V}]\{q_V\} + \{f_{FS}\}) \tag{3-47}$$

Equation (3-46) constitutes the generalization to models of arbitrary topology of the axial displacement equation derived earlier for a single element (see equation (3-36)). Equation (3-47) is the equation mentioned earlier that relates axial forces when the model is redundant. In view of the dependencies among the axial forces, these degrees of freedom may be partitioned into a set of $(N_V - rank(\bar{U}))$ dependent variables, $\{q_{Vdep}\}$, and a set of $rank(\bar{U})$ independent variables, $\{q_{Vindep}\}$. Similarly, $[\bar{V}]$ may be partitioned as $[\bar{V}] = [\bar{V}_{dep}|\bar{V}_{indep}]$. Equation (3-47) may be now solved for $\{q_{Vdep}\}$ as follows:

$$\{q_{Vdep}\} = \left([A_{inex}]^T[V_{dep}]\right)^{-1}\left(-[A_{inex}]^T[V_{indep}]\{q_{Vindep}\} + [A_{inex}]^T\{f_{FS}\}\right) \tag{3-48}$$

The complete dynamical equations of the system are obtained by combining the mixed element axial force equations (equation (3-37)) with the remaining system equations of motion, which are derived by calculating the work associated with virtual displacements of the N_u Lagrangian degrees of freedom. The equations of motion may be written in the form:

$$f_{total} = f_{inertia} + f_{aerodynamic} + f_{structural} = 0 \tag{3-49}$$

The following outlines a strategy for solving these equations. In a preliminary processing step, the matrix $[\bar{U}]$ is formed, and its singular value decomposition is calculated. Then, equations (3-45), (3-46), and (3-48) are applied so that the original N_u equations of motion may be expressed in terms of $N_u - rank(\bar{U})$ generalized inextensional degrees of freedom, $\{q_{inext}\}$, and $rank(\bar{U})$ independent axial forces, $\{q_{Vindep}\}$. Next, the eigenmodes of the system governing equations are computed about some convenient state, whereupon the reduced system degrees of freedom may be expressed as:

$$\begin{Bmatrix} q_{inext} \\ q_{Vindep} \end{Bmatrix} = \begin{bmatrix} \Phi_{inext} \\ \Phi_{Vindep} \end{bmatrix} \{q_{modal}\} \tag{3-50}$$

where $[\Phi]$ is a column matrix of eigenvectors, and $\{q_{modal}\}$ are the corresponding generalized coordinates of the modally reduced model.

In the timestepping phase of the analysis, the N_{modal} degrees of freedom are stepped forward from t^n to t^{n+1}, typically using an implicit integration scheme. Setting $\{f_{total}\} = 0$ for the new time step leads to N_{modal} equations for the degrees of freedom that are usually solved with a Newtonian iteration scheme. Once a trial set of degrees of freedom for the new time step, $\{q^{n+1}_{modal}\}$, are obtained, it is necessary to compute the corresponding generalized forces. To accomplish this, equations (3-45), (3-46), (3-48), and (3-50) are used to calculate the finite element displacements and generalized forces. The same equations may be used to develop expressions for the generalized forces of the modal degrees of freedom in terms of the finite element generalized forces.

3.3.3 Plate Bending Analysis The analytical difficulties associated with rotor blade analysis are consequences of the stiffening effect of the blade's spin. Any flexible structural blade is potentially subject to the same difficulties, regardless of the choice of mathematical paradigm used to model the blade. But a particularly intriguing feature of the analysis just described is that it should be possible to extend it to *all* blade models, and not just those composed of beams. For example, it may be desirable to develop a high fidelity model of a flexbeam by modeling it with plate or shell elements instead of beam elements. This section illustrates how the axial foreshortening analysis just developed may be extended to a blade model containing plate finite elements.

Plate Geometry and Kinematics

The undeformed plate is assumed to be rectangular and to lie in the x-y plane. All deflections are referred to the middle surface of the undeformed plate. The inplane deflections in the x and y directions are denoted u and v, and the out-of-plane deflection, which is in the z direction, is denoted w. A local coordinate basis is established whose origin is located on the middle surface of the undeformed plate, and with basis vectors oriented in the x, y and z directions.

Inplane motions are based on the "Kirchhoff assumption," which is analogous to Bernoulli's hypothesis for beams, and states that lines normal to the middle surface remain straight and normal under the deformation:

$$u = \hat{u} - zw_x \tag{3-51}$$
$$v = \hat{v} - zw_y \tag{3-52}$$
$$w = \hat{w} \tag{3-53}$$

where the hatted quantities are the deflections of the middle surface, and z is the distance to the middle surface.

If classical, "moderate deflection" plate theory (i.e., Von Karman plate theory) is assumed, the strains at a generic point are given by:

$$\varepsilon_{xx} = e_x - zw_{xx} \tag{3-54}$$
$$\varepsilon_{yy} = e_y - zw_{yy} \tag{3-55}$$
$$\gamma_{xy} = e_{xy} - 2zw_{xy} \tag{3-56}$$

where the e's are the strains of the middle surface:

$$e_x = \hat{u}_x + \frac{1}{2}w_x^2 \tag{3-57}$$

$$e_y = \hat{v}_y + \frac{1}{2}w_y^2 \tag{3-58}$$

$$e_{xy} = \hat{v}_x + \hat{u}_y + w_x w_y \tag{3-59}$$

It is assumed that the plate consists of n layers, each composed of an orthotropic material that is assumed to be in a state of plane strain; i.e., $\gamma_{xz} = \gamma_{yz} = \varepsilon_{zz} = 0$. The constitutive relations for layer k are as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{23} & Q_{33} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}^{(k)} \tag{3-60}$$

where the Q_{ij} are the measure numbers of Hooke's tensor for the material in layer k.

The strain energy of the plate is calculated by forming the strain energy density for each material, and integrating the energy density first over the thickness of each layer, and then over the surface of the entire plate. The result is:

$$U = \int \Phi dA \tag{3-61}$$

where Φ is the *plate strain energy density* (i.e., strain energy density per unit area), viz.

$$\Phi = \frac{1}{2} \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}^T \begin{bmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} & B_{21} & B_{22} & B_{23} \\ A_{31} & A_{32} & A_{33} & B_{31} & B_{32} & B_{33} \\ B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\ B_{21} & B_{22} & B_{23} & D_{21} & D_{22} & D_{23} \\ B_{31} & B_{32} & B_{33} & D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3-62}$$

where

$$\kappa_x = \hat{w}_{xx} \tag{3-63}$$

$$\kappa_y = \hat{w}_{yy} \tag{3-64}$$

$$\kappa_{xy} = \hat{w}_{xy} \tag{3-65}$$

are the *generalized strains* of the plate, and A_{ij}, B_{ij} and D_{ij} are computed by integrating the lamina stiffnesses, $Q_{ij}^{(k)}$ over the thickness of the plate:

$$A_{ij} = \sum_{k=1}^{N} Q_{ij}^{(k)}(z_{k+1} - z_k) \tag{3-66}$$

$$B_{ij} = -\frac{1}{2}\sum_{k=1}^{N} Q_{ij}^{(k)}(z_{k+1}^2 - z_k^2) \tag{3-67}$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} Q_{ij}^{(k)}(z_{k+1}^3 - z_k^3) \tag{3-68}$$

In view of the definition of $[B]$ given above, the symmetry of $[B]$ follows from the symmetry of $[Q]$, and this fact has been exploited in equation (3-62). Equation (3-62) may also be written in the more compact form:

$$\Phi = \frac{1}{2}\begin{Bmatrix} e \\ \kappa \end{Bmatrix}^T \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} e \\ \kappa \end{Bmatrix} \tag{3-69}$$

The *generalized stresses* corresponding to the generalized strains are obtained by differentiating the strain energy with respect to the generalized strains. Differentiating Φ with respect to the translational strains gives the *in-plane force resultants*:

$$N_{xx} = \frac{\partial \Phi}{\partial e_x} = A_{11}\varepsilon_x + A_{12}\varepsilon_y + A_{13}\varepsilon_{xy} + B_{11}\kappa_x + B_{12}\kappa_y + B_{13}\kappa_{xy} \tag{3-70}$$

$$N_{yy} = \frac{\partial \Phi}{\partial e_y} = A_{21}\varepsilon_x + A_{22}\varepsilon_y + A_{23}\varepsilon_{xy} + B_{21}\kappa_x + B_{12}\kappa_y + B_{23}\kappa_{xy} \tag{3-71}$$

$$N_{xy} = \frac{\partial \Phi}{\partial e_{xy}} = A_{31}\varepsilon_x + A_{32}\varepsilon_y + A_{33}\varepsilon_{xy} + B_{31}\kappa_x + B_{32}\kappa_y + B_{33}\kappa_{xy} \tag{3-72}$$

and differentiating Φ with respect to the bending curvatures gives the *moment resultants*:

$$M_{xx} = \frac{\partial \Phi}{\partial \kappa_x} = B_{11}\varepsilon_x + B_{12}\varepsilon_y + B_{13}\varepsilon_{xy} + D_{11}\kappa_x + D_{12}\kappa_y + D_{13}\kappa_{xy} \tag{3-73}$$

$$M_{yy} = \frac{\partial \Phi}{\partial \kappa_y} = B_{21}\varepsilon_x + B_{22}\varepsilon_y + B_{23}\varepsilon_{xy} + D_{21}\kappa_x + D_{22}\kappa_y + D_{23}\kappa_{xy} \tag{3-74}$$

$$M_{xy} = \frac{\partial \Phi}{\partial \kappa_{xy}} = B_{31}\varepsilon_x + B_{32}\varepsilon_y + B_{33}\varepsilon_{xy} + D_{31}\kappa_x + D_{32}\kappa_y + D_{33}\kappa_{xy} \tag{3-75}$$

These equations may also be expressed more compactly as:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} e \\ \kappa \end{Bmatrix} \tag{3-76}$$

The presence of axial foreshortening terms in the in-plane strains will likely cause difficulties in a plate element that are similar to those in a beam element. As in the case of a beam element, it is expected that those difficulties may be lessened or eliminated by using the in-plane stress resultants as degrees of freedom in the mathematical model. To effect this modification, first solve for the in-plane strains as functions of the in-plane stress resultants and the bending curvatures:

$$\{e\} = [A]^{-1}\left(\{N\} - [B]\{\kappa\}\right) \tag{3-77}$$

Substituting this expression into the strain energy allows the in-plane stresses to appear explicitly in the equations of motion. However, equation (3-77), with $\{e\}$ expressed in terms of displacements, may also be regarded as a constraint, which may be reintroduced into the equations of motions by appending it to the strain energy via Lagrange multipliers. The augmented strain energy expression may then be written as follows:

$$\Phi = \frac{1}{2}\begin{Bmatrix} [A]^{-1}\left(\{N\} - [B]\{\kappa\}\right) \\ \kappa \end{Bmatrix}^T \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} [A]^{-1}\left(\{N\} - [B]\{\kappa\}\right) \\ \kappa \end{Bmatrix} + \tag{3-78}$$

$$\{\lambda\}^T \left(\begin{Bmatrix} \hat{u} + \frac{1}{2}w_x^2 \\ \hat{v} + \frac{1}{2}w_y^2 \\ \hat{v}_x + \hat{u}_y + w_x w_y \end{Bmatrix} - [A]^{-1}\left(\{N\} - [B]\{\kappa\}\right) \right) \tag{3-79}$$

where $\{\lambda\} = \{\lambda_x, \lambda_y, \lambda_{xy}\}^T$. Setting the first variation of Φ with respect to the in-plane stresses to zero shows that the Lagrange multipliers are equal to the in-plane stresses; viz.

$$\left\{ \begin{array}{c} \frac{\partial \Phi}{\partial N_x} \\ \frac{\partial \Phi}{\partial N_y} \\ \frac{\partial \Phi}{\partial N_{xy}} \end{array} \right\} = 0 \Longrightarrow \left\{ \begin{array}{c} \lambda_x \\ \lambda_y \\ \lambda_{xy} \end{array} \right\} = \left\{ \begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array} \right\} \tag{3-80}$$

The troublesome foreshortening terms appear in the displacement-force constraint equations, which may be recovered by setting the first variation of Φ with respect to the Lagrange multipliers to zero. Discretizing these equations, and then collecting them together gives:

$$\left[\hat{U}\right] \{q_{\hat{U}}\} = \left[\hat{N}\right] \{q_{\hat{N}}\} + \{f_{FS}\} \tag{3-81}$$

in which the bending curvature terms have been absorbed into $\{f_{FS}\}$. Evidently, equation (3-81) has the same form as equation (3-37), and application of the singular value decomposition to $\left[\hat{U}\right]$ allows the extensional part of the displacement field to be expressed in terms of the inextensional part of the field and the in-plane forces and the foreshortening terms. In other words, the foreshortening terms in plate models may be treated in a manner entirely analogous to their treatment in beam models. Moreover, it is evident from the foregoing analysis that the same general approach will apply to models composed of plates *and* beams.

4 Generalized Axial Foreshortening and Nonlinear Normal Modes

It has been noted that a weakness of classical modal superposition is that it is *linear* and that this makes it difficult to approximate *nonlinear* phenomena, such as the axial foreshortening effect. The generalized axial foreshortening analysis developed in the previous section sought to overcome this limitation by embedding the axial foreshortening effect directly in the axial displacement analysis. Recently, a general procedure for embedding nonlinear effects in basis reduction procedures, the *nonlinear normal mode method*, has been introduced by Shaw and Pierre (Shaw, 1993). In this section, a relationship will be derived between the nonlinear mode method and the generalized axial foreshortening method, but first, a brief description of the nonlinear normal method is given. The reader is encouraged to consult the reference just cited and the numerous references given there for a more complete description of the method.

In the classical linear normal method, the relationship of the degrees of freedom are constant relative to each other, and the eigenmode is unique except for an arbitrary scaling factor. For example, the response of a dynamic system in the *i*th normal mode is:

$$\{q\} = [\Phi_i]\,\mu_i \tag{4-82}$$

where the $[\Phi_i]$ is the eigenvector, and μ_i is a generalized coordinate that represents the mode. Classical modal superposition seeks to combine these modes in a linear fashion to represent system response, viz.:

$$\{q\} = [\Phi]\,\{\mu\} \tag{4-83}$$

where $[\Phi]$ is a modal matrix of some subset of eigenmodes, and $\{\mu\}$ is a vector of generalized coordinates. The fidelity with which modal basis reduction can model dynamic phenomena can be improved by replacing the fixed, linear relationships given in equations (4-82) and (4-83) with a set of *nonlinear* relationships termed an *invariant manifold*:

$$\begin{cases} q_i = X_i(\mu_m) \\ \dot{q}_i = Y_i(\mu_m) \end{cases} \qquad m \in S(m) \tag{4-84}$$

where $S(m)$ is the set of indices of the eigenmodes that comprise the reduced basis, and the X_i and Y_i are nonlinear functions. The manifold is termed *invariant* because as in the case of linear eigenmodes, the motion of the system always remains on the manifold once the manifold is activated.

A numerical procedure for determining the manifold equations has been given in (Pescheck, 2001), but the generalized axial elongation process readily yields an approximate analytical solution for the manifold, and it is this solution that establishes the link between the generalized axial elongation method and the nonlinear normal mode method. If the dependent axial force degrees of freedom are expressed in terms of independent quantities using equation (3-48), and the results substituted into equation (3-46), the extensional coordinates may be expressed as follows:

$$q_{ext} = [A_q]\{q_{V_{indep}}\} + [A_f]\{f_{FS}\} \tag{4-85}$$

where $[A_q]$ and $[A_f]$ are matrices. An approximation for an invariant manifold may be obtained simply by applying a modal transformation to the independent coordinates in equation (4-85). Since the foreshortening term is quadratic in the primal Lagrangian displacements (see equation (2-9)), the modal expansion of the *i*th may be written:

$$\{f_{FS}\}_i = a_{ijk} q_{modal\,j} q_{modal\,k} \tag{4-86}$$

in which the a_{ijk} are constants, the $q_{modal\,j,k}$ are modal generalized coordinates, and the summation convention for indices is assumed. Finally, substituting equation (4-86) into equation (4-85) and then applying a modal expansion to the force terms leads to the following:

$$q_{ext_i} = \alpha_{ij} q_{modal\,j} + \beta_{ijk} q_{modal\,j} q_{modal\,k} \tag{4-87}$$

Equation (4-87) is a nonlinear relationship between the displacement and nodal degrees of freedom. It therefore embodies the key concept of an "invariant manifold" although differing in several respects from the manifold relationships presented in (Shaw, 1993), which were originally developed for a broader range of applications. First, it is expressed entirely in terms of displacement variables, while the manifolds developed in (Shaw, 1993) are obtained from equations of motion cast in first-order form, and are therefore velocity-dependent. For the nonlinear constraint considered here, which is dominated by the axial foreshortening effect, velocity dependencies are weak, and neglecting them in the manifold definition is generally a good approximation. Another difference with the manifolds developed in (Shaw, 1993) is that equation (4-87) considers only nonlinear coupling between extensional and inextensional degrees of freedom, while the development in (Shaw, 1993) considers coupling among all modal coordinates. In rotorcraft applications, however, it is generally observed that while the nonlinear couplings among inextensional modes (i.e., bending and torsion modes) are certainly present, they do not appreciably distort the character of those modes; indeed, such couplings have been traditionally ignored when defining generalized coordinates in rotorcraft analyses, without producing adverse effects. Finally, the manifolds developed in (Shaw, 1993) are expressed in the form of relationships between "master" and "slave" modes, while equation (4-87) is a relationship between modal coordinates and other generalized coordinates. This difference, however, is largely cosmetic and may be eliminated simply by expressing the q_{ext_i} in terms of modal coordinates, and choosing suitable modes as independent or "master" degrees of freedom. In summary, the generalized axial elongation method is formally different than the nonlinear normal mode method, but it treats the most crucial aspect of rotor blade basis reduction, the axial foreshortening effect, in a manner quite similar to the nonlinear normal method, and it may be viewed as an approximation to that method.

5 Conclusions

A unified development of three blade analysis methods that have good basis reduction attributes has been presented. As a prelude, the analysis issues associated with various blade formulations were examined by applying these formulations to the Hodges-Dowell blade equations specialized to flap-axial motions. It was shown that classical displacement-based finite elements, while permitting full topological generality, are problematic owing to the near-inextensibility of the blade, which makes the small, but critical, axial strain difficult to approximate accurately in modal space. The mixed finite element method was then extensively described because it plays a central role in understanding how the analysis problems may be resolved, and it may also be used a starting point from which the other two effective blade analysis methods may be developed. The mixed finite element method was introduced in a two-step process: First, an analysis method was proposed in which the axial force replaces the axial displacement as an analysis variable, and while this approach is likely to be more effective in approximating the axial force, it imposes topological restrictions on the models that can be analyzed. Then, the mixed finite element method was obtained from the axial force method by retaining the axial force as a solution variable, and then appending the force-axial displacement equation as an additional equation. It was shown that while this method has good modal reduction accuracy without imposing topological restrictions on the model, it achieves these advantages at the expense of adding additional degrees of freedom to the analysis. In order to reduce the number of degrees of freedom, an alternative method was developed that employs the mixed finite element equations to solve for the extensible portion of the displacement field in terms of the axial forces and the inextensible displacements; effectively, this method generalizes the axial elongation method to blade models of arbitrary topology, and its derivation reveals that it may be regarded as a variant of the mixed finite element method. Finally, the generalized elongation method was shown to be an approximation for an invariant manifold that may be used as a generalized basis function in a nonlinear normal mode analysis.

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A Development of a Mixed Element for Rotor Blades

A.1 Overview

A mixed finite element for rotor blade analysis has been developed and implemented in an experimental version of the Second Comprehensive Generation Helicopter Analysis System (2GCHAS) (1). In essence, the finite element already used in 2GCHAS for rotor blade analysis, the so-called "Nonlinear Beam Element," was converted into a mixed element. In contrast to other mixed elements that have been used for rotor blades, e.g., (Hodges, 1990) and (Bauchau, 1993), the mixed treatment is limited to the axial direction. This limitation reduces the number of degrees of freedom that are needed and simplifies both the derivation and the programming of the element. Only the structural terms of the 2GCHAS element are impacted by the conversion to a mixed element, but for the sake of completeness, the element's inertia terms are also derived here. The derivation of this element has already been presented in the literature (see (Ruzicka, 2001),) but it's included here for the reader's convenience.

The conversion process proved to be quite fast and straightforward, as the reader may infer by comparing the theory of the mixed element with the theory of the original 2GCHAS element and noting the strong parallelism between the two. The ease of conversion is stressed here because it should be duplicatable for *any* displacement-based finite element code, not just 2GCHAS, thus underscoring the utility of mixed elements. The derivation presented here employs the notations and conventions of the 2GCHAS, and the reader should consult (1) for the necessary background material.

A.2 Element Geometry and Kinematics

A.2.1 Element Geometry The geometry of the undeformed element is shown in Figure (A-7). The element is assumed to move in a non-inertial reference frame denoted E, whose motion relative to the inertial frame I is prescribed. The E frame absorbs the large rigid-body motions of the rotorcraft, while the smaller motions of the deforming blade relative to the frame are analyzed using a "moderate deformation" blade theory familiar to rotorcraft analysts. The geometry and motion of the element are defined with the aid of a coordinate system within the E frame with basis vectors $\{\mathbf{b}_1^E, \mathbf{b}_2^E, \mathbf{b}_3^E\}$. The beam element reference axis is assumed to be initially straight and parallel to the unit vector $\mathbf{b}_1^E$. A structural reference frame S is located on the reference axis a distance x from the origin of the E frame. Its orientation differs from E by a rotation about $\mathbf{b}_1^E$ by the built-in twist angle θ_t, rotating the E frame coordinates y and z into alignment with S frame cross-sectional coordinates η and ζ (see Figure (A-7)). Thus, the position and orientation of S with respect to E are

$$\{r_E^{SE}\} = \lfloor x, 0, 0 \rfloor^T$$

$$\left[T^{SE}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_t & \sin\theta_t \\ 0 & -\sin\theta_t & \cos\theta_t \end{bmatrix} \tag{A-88}$$

A.2.2 Element Kinematics In this section the displacement of a generic material point A within the undeformed beam is developed; in the deformed beam it is denoted by A'. In order to carry out this development, consider first the motion of E in I. Then, the 3-D displacement field of the beam is represented in terms of 1-D variables and cross-sectional position coordinates. An intermediate frame S' is introduced and used to relate the motion of A' to I.

The deformed beam cross-sectional frame S', which becomes coincident with S when the beam is in its undeformed state, is specified in the following way: The material points in the undeformed beam which lie along the reference line move when the beam deforms, deforming into a curved line which is not, in general, the same length as the original reference line because of the possibility of stretching. Similarly, the material points in the plane perpendicular to the reference line are denoted as the reference cross section. This plane of points, determined by the η and ζ coordinate directions, also moves when the beam deforms. The points remain contiguous in the deformed beam so that they make up a surface which is very close to a plane.

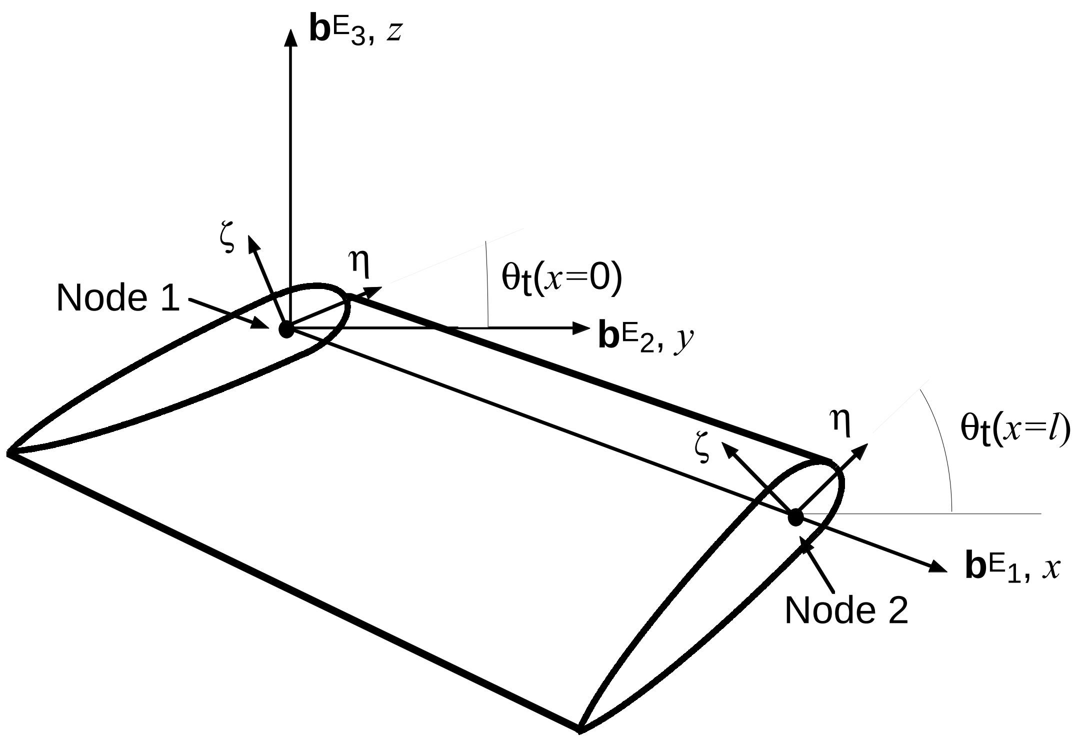

Figure A-7. Geometry of Undeformed Element

In accordance with Euler-Bernoulli beam theory, it is assumed that the frame S' rigidly transports the η and ζ coordinate directions at any particular value of x to a new orientation, perpendicular to the reference line of the deformed beam at the same material point (which is associated with the same value of x in the undeformed beam). Any deviation of the surface of points from a plane is, in accordance with Euler-Bernoulli theory, small. The orientation of S' in E can be expressed in terms of a set of 1-D variables which govern the position of and the rotation of S' about the deformed beam reference line.

As expected, three rotations are sufficient to express the direction cosines of S' in E, denoted by $\left[T^{S'E}\right]$. A set of body 3: 3-2-1 orientation angles (Kane *et al*, 1983) is used. The direction cosine matrix $\left[T^{S'E}\right]$ may be expressed in terms of these angles as:

$$\left[T^{S'E}\right] = \begin{bmatrix} c_3c_2 & c_2s_3 & -s_2 \\ -s_3c_1+s_1s_2c_3 & c_1c_3+s_1s_2s_3 & s_1c_2 \\ s_1s_3+c_1c_3s_2 & -c_3s_1+c_1s_3s_2 & c_1c_2 \end{bmatrix} \tag{A-89}$$

where $s_1 = \sin\theta_1$, $c_1 = \cos\theta_1$, etc. The orientation of S' with respect to E is now expressible as

$$\begin{Bmatrix} \mathbf{b}_1^{S'} \\ \mathbf{b}_2^{S'} \\ \mathbf{b}_3^{S'} \end{Bmatrix} = \left[T^{S'E}\right] \begin{Bmatrix} \mathbf{b}_1^{E} \\ \mathbf{b}_2^{E} \\ \mathbf{b}_3^{E} \end{Bmatrix} \tag{A-90}$$

Upon expressing the direction cosines in equation (A-89) in terms of the 1-D variables v', w', and ϕ, and retaining terms to $O(\varepsilon^2)$, the transformation matrix $\left[T^{S'E}\right]$ becomes (see (Hodges, 1980)):

$$\left[T^{S'E}\right] = \begin{bmatrix} 1-\frac{v'^2}{2}-\frac{w'^2}{2} & v' & w' \\ -(v'c_1+w's_1) & c_1(1-\frac{v'^2}{2})-v'w's_1 & s_1(1-\frac{w'^2}{2}) \\ v's_1-w'c_1 & -s_1(1-\frac{v'^2}{2})-v'w'c_1 & c_1(1-\frac{w'^2}{2}) \end{bmatrix} \tag{A-91}$$

The position of the generic point A' on the deformed beam cross section is then defined as

$$\begin{aligned} \mathbf{r}^{A'I} &= \mathbf{r}^{EI}+\mathbf{r}^{A'E} \\ &= \mathbf{r}^{EI}+(x+u)\mathbf{b}_1^E+v\mathbf{b}_2^E+w\mathbf{b}_3^E+\Psi\kappa_x\mathbf{b}_1^{S'}+\eta\mathbf{b}_2^{S'}+\zeta\mathbf{b}_3^{S'} \end{aligned} \tag{A-92}$$

This equation can be written in column matrix form if each vector is expressed in terms of its measure numbers in the E basis, viz.,

$$\begin{aligned} \left\{r_E^{A'I}\right\} &= \left\{r_E^{EI}\right\}+\left\{r_E^{A'E}\right\} \\ &= \left\{r_E^{EI}\right\}+\begin{Bmatrix} x+u \\ v \\ w \end{Bmatrix}+\left[T^{ES'}\right]\begin{Bmatrix} \Psi\kappa_x \\ \eta \\ \zeta \end{Bmatrix} \end{aligned} \tag{A-93}$$

Here u, v, and w are the displacement measures of the beam reference line due to elastic deformations in the E frame, $x\mathbf{b}_1^E$ is the vector from the origin of the element frame E to the origin of the structural frame S (see Figure (A-8)), and η and ζ are the cross-sectional position coordinates. The last term is the position of A' within the deformed beam cross-sectional frame S'. (Recall that these coordinates have been convected and thus they correspond to the original cross-sectional coordinates of A in the undeformed beam.) Lastly, an out-of-plane warping is assumed of the form $\kappa_x(x)\Psi(\eta,\zeta)$, where Ψ is the St. Venant torsion warping function and $\kappa_x(x)$ represents the amplitude of the warping (Hodges, 1980a). Subsequently κ_x is defined as the elastic part of twist per unit length or just "elastic twist."

A.3 Stress-Strain Relationship

The stress-strain relationship for a linearly elastic anisotropic solid is characterized by 21 constants which form a fourth-order tensor. The 21 material constants can be obtained in the local Cartesian system along unit vectors $\mathbf{b}_i^S$, associated with the curvilinear coordinates x, η, and ζ. These material constants can be formed into a 6×6 symmetric matrix, the elements of which may vary as functions of x, η, and ζ. This matrix linearly relates the stress components σ_{xx}, $\sigma_{\eta\eta}$, $\sigma_{\zeta\zeta}$, $\sigma_{\eta\zeta}$, $\sigma_{x\zeta}$, and $\sigma_{x\eta}$, with the strain components γ_{xx}, $\gamma_{\eta\eta}$, $\gamma_{\zeta\zeta}$, $\gamma_{\eta\zeta}$, $\gamma_{x\zeta}$, and $\gamma_{x\eta}$, where the order of the components corresponds to that of anisotropic elasticity.

For slender, isotropic beams, the Bernoulli hypothesis approximately holds, according to asymptotic analyses (Berdichevsky, 1976). This hypothesis is that the transverse normal stresses $\sigma_{\eta\eta}$ and $\sigma_{\zeta\zeta}$, along with the distortion shear stress $\sigma_{\eta\zeta}$, are much smaller than the other three stress components. Thus, one solves for the three strain components $\gamma_{\eta\eta}$, $\gamma_{\eta\zeta}$, and $\gamma_{\zeta\zeta}$ in terms of the others. Substituting the results into the original equations for σ_{xx}, $\sigma_{x\eta}$, and $\sigma_{x\zeta}$, one obtains a "reduced" 3-D stress-strain law which can be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{16} & Q_{15} \\ Q_{16} & Q_{66} & Q_{56} \\ Q_{15} & Q_{56} & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{xx} \\ \gamma_{x\eta} \\ \gamma_{x\zeta} \end{Bmatrix} = [Q]\{\Gamma\} \tag{A-94}$$

This reduced 3-D stress-strain law is the basis of most elementary beam theories. Some published treatments confuse this with setting the *strain* components $\gamma_{\eta\eta}$, $\gamma_{\eta\zeta}$, and $\gamma_{\zeta\zeta}$ equal to zero. These are not zero, but may be calculated from the above reduction process.

Note that, in the special case of the beam having laminated construction with the ζ direction perpendicular to the plane of the laminate, this numbering scheme corresponds to that of common treatments of lamination theory. Note that for the case of an isotropic

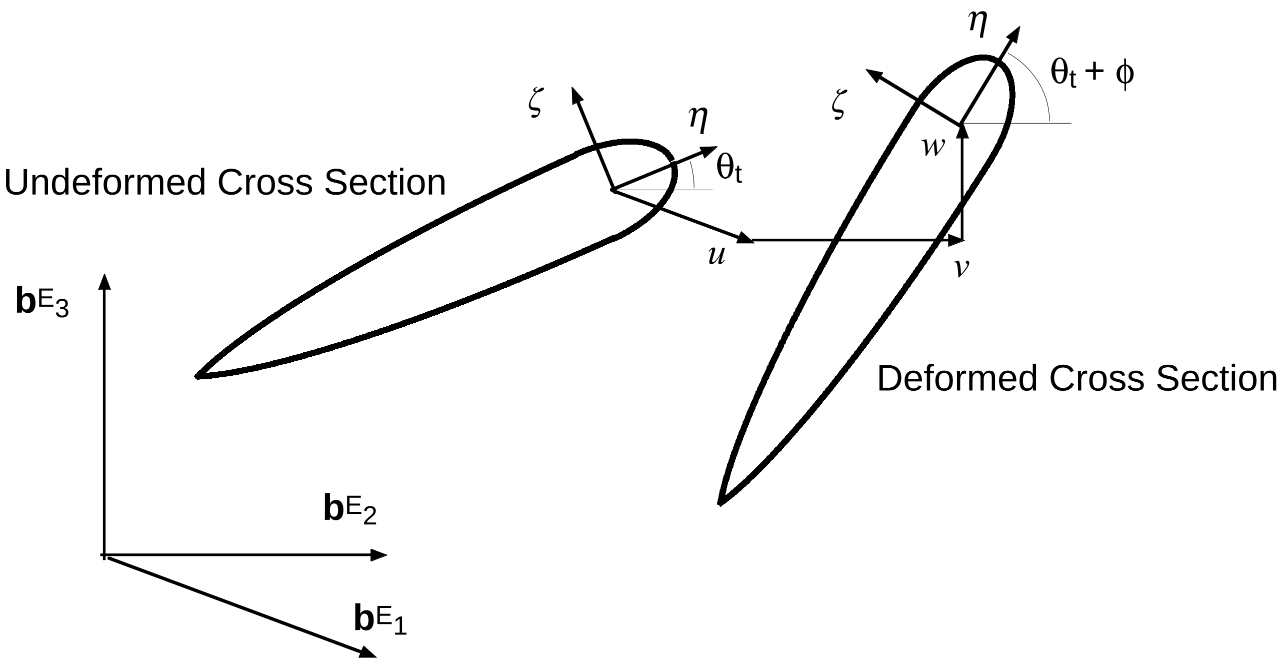

Figure A-8. Undeformed and Deformed Element Cross Section

material

$$Q_{11} = E$$

$$Q_{55} = Q_{66} = G \tag{A-95}$$

$$Q_{15} = Q_{16} = Q_{56} = 0$$

With the above 3-D stress-strain law, one can write the strain energy per unit length (or 1-D strain energy function)

$$\Phi = \frac{1}{2}\left\langle \{\Gamma\}^T [Q] \{\Gamma\} \right\rangle \tag{A-96}$$

where $\langle(\bullet)\rangle$ refers to an integral of $(\bullet)$ over the cross section, which forms the basis for beam theory. In order to determine the 1-D function Φ, the strain field must be expressed in such a way that this cross-sectional integral can be evaluated. Note that the strain energy of the element is simply

$$U = \int_0^{\ell} \Phi \, dx \tag{A-97}$$

A.4 1-D Strain Energy Function

From axiomatic, Euler-Bernoulli rod theory, the strain energy density of the beam element (i.e., strain energy per unit length) may be written in the form:

$$\Phi = \frac{1}{2}\begin{Bmatrix}\varepsilon_x\\ \kappa_x\\ \kappa_\eta\\ \kappa_\zeta\end{Bmatrix}^T \begin{bmatrix} E_0A & E_0D_0 & E_0Ae_{t_\zeta} & -E_0Ae_{t_\eta}\\ E_0D_0 & G_0J & E_0D_3 & -E_0D_2\\ E_0Ae_{t_\zeta} & E_0D_3 & E_0I_\eta & E_0I_{\eta\zeta}\\ -E_0Ae_{t_\eta} & -E_0D_2 & E_0I_{\eta\zeta} & E_0I_\zeta \end{bmatrix} \begin{Bmatrix}\varepsilon_x\\ \kappa_x\\ \kappa_\eta\\ \kappa_\zeta\end{Bmatrix} \tag{A-98}$$

where ε_x, κ_x, κ_η, and κ_ζ are generalized beam strains and curvatures defined by:

$$\varepsilon_x = s' - 1$$

$$\kappa_x = \left(\mathbf{b}_2^{S'}\right)' \cdot \mathbf{b}_3^{S'} - \theta_t'$$

$$\kappa_\eta = -\left(\mathbf{b}_1^{S'}\right)' \cdot \mathbf{b}_3^{S'}$$

$$\kappa_\zeta = \left(\mathbf{b}_1^{S'}\right)' \cdot \mathbf{b}_2^{S'} \tag{A-99}$$

Equation (1) is adequate for the stiff, articulated blade considered in this paper, but note that for bearingless blades, it must be augmented by an additional nonlinear strain term that gives rise to the so-called “trapeze effect” (see (Borri, 1986)).

For elements composed of isotropic materials, the cross sectional constants in equation (A-99) may be computed as follows. Substitute the displacement obtained from the element’s kinematical relations (equations (A-91) and (A-93)) into small strain, moderate rotation strain-displacement relations that can adequately capture rotor blade kinematics (see (Danielson, 1988).) Next, substitute the strains into equation (A-94) to compute the stresses, and then integrate the stresses over the cross section to obtain forces and moments. This process gives:

$$A = \frac{1}{E_0}\langle Q_{11}\rangle \qquad k_A^2 = \frac{1}{A}(I_\eta + I_\zeta)$$

$$e_{t_\eta} = \frac{1}{E_0A}\langle Q_{11}\eta\rangle \qquad e_{t_\zeta} = \frac{1}{E_0A}\langle Q_{11}\zeta\rangle$$

$$I_\zeta = \frac{1}{E_0}\left\langle Q_{11}\eta^2\right\rangle \qquad I_\eta = \frac{1}{E_0}\left\langle Q_{11}\zeta^2\right\rangle$$

$$I_{\eta\zeta} = -\frac{1}{E_0}\langle Q_{11}\eta\zeta\rangle \qquad B_1 = \frac{1}{E_0}\left\langle Q_{11}(\eta^2+\zeta^2)^2\right\rangle$$

$$B_2 = \frac{1}{E_0}\left\langle Q_{11}\eta(\eta^2+\zeta^2)\right\rangle \qquad B_3 = \frac{1}{E_0}\left\langle Q_{11}\zeta(\eta^2+\zeta^2)\right\rangle$$

$$D_0 = \frac{1}{E_0}\left\langle Q_{1s}\right\rangle + \frac{1}{E_0}\theta_t'\left\langle Q_{11}(\zeta\Psi_{,\eta} - \eta\Psi_{,\zeta})\right\rangle \qquad D_1 = \frac{1}{E_0}\left\langle(\eta^2+\zeta^2)Q_{1s}\right\rangle$$

$$+\frac{1}{E_0}\theta_t'\left\langle(\eta^2+\zeta^2)Q_{11}(\zeta\Psi_{,\eta} - \eta\Psi_{,\zeta})\right\rangle$$

$$D_2 = \frac{1}{E_0}\left\langle Q_{1s}\eta\right\rangle + \frac{1}{E_0}\theta_t'\left\langle \eta Q_{11}(\zeta\Psi_{,\eta} - \eta\Psi_{,\zeta})\right\rangle \qquad D_3 = \frac{1}{E_0}\left\langle Q_{1s}\zeta\right\rangle + \frac{1}{E_0}\theta_t'\left\langle \zeta Q_{11}(\zeta\Psi_{,\eta} - \eta\Psi_{,\zeta})\right\rangle$$

$$J = \frac{1}{G_0}\left\langle Q_{55}(\Psi_{,\zeta}+\eta)^2 + 2Q_{56}(\Psi_{,\eta}-\zeta)(\Psi_{,\zeta}+\eta) + Q_{66}(\Psi_{,\eta}-\zeta)^2\right\rangle + \frac{1}{G_0}\theta_t'^2\left\langle Q_{11}(\zeta\Psi_{,\eta}-\eta\Psi_{,\zeta})^2\right\rangle$$
$$+2\frac{1}{G_0}\theta_t'\left\langle(\zeta\Psi_{,\eta}-\eta\Psi_{,\zeta})[Q_{15}(\Psi_{,\zeta}+\eta)+Q_{16}(\Psi_{,\eta}-\zeta)]\right\rangle$$

where $Q_{1s} = Q_{15}(\Psi_{,\zeta}+\eta)+Q_{16}(\Psi_{,\eta}-\zeta)$. For an element composed of anisotropic materials, three-dimensional effects such as inplane warping become significant, and elementary beam theory is inadequate for computing the cross sectional constants. Furthermore, calculations based on asymptotic theories such as VABS (Cesnik, 1997) show that the Bernoulli hypothesis is not as accurate for composite beams as it is for isotropic beams (Yu, 2001). However, even in this more complex case, the formal, asymptotic analysis on which VABS is based shows that the *form* of equation (A-98) remains correct, and the 4×4 matrix in equation (A-98) can be calculated using a code such as VABS, which may be applied to beams with arbitrary cross-sectional geometry and material properties. In general these properties depend on the cross-sectional geometry, the material anisotropy, and the local initial twist. One is far better off using a cross-sectional analysis tool such as VABS to determine the cross-sectional elastic constants without ad hoc assumptions.

It is well-known that the displacement-based element behaves poorly in modal reduction, and that this problem may be remedied by formulating the axial strain in terms of the axial force rather than the axial displacement. This may be accomplished by differentiating the strain energy with respect to the axial strain to obtain an expression for the axial force, and then using that expression to solve for the axial strain in terms of the axial force and the other generalized strains; viz.

$$\varepsilon_x = \frac{1}{A}\left(\frac{V_x}{E_0} - D_0\kappa_x - Ae_{t_\zeta}\kappa_\eta + Ae_{t_\eta}\kappa_\zeta\right) \tag{A-100}$$

Substituting equation (A-100) into equation (A-98) gives an expression for the strain energy in terms of the axial force, V_x. It is important, however, that the axial strain somehow be re-introduced explicitly into the strain energy to exhibit its coupling with the finite element axial degrees of freedom, but this must be done in a way that allows all the variables to be treated as independent, despite the redundancy between ε_x and V_x. This goal may be accomplished by appending equation (A-100) to the strain energy via a Lagrange multiplier, whereupon the strain energy becomes:

$$\Phi = \frac{1}{2}\{\gamma^*\}^T[K_0]\{\gamma^*\} + \lambda\left[\varepsilon_x - \frac{1}{A}\left(\frac{V_x}{E_0} - D_0\kappa_x - Ae_{t\zeta}\kappa_\eta + Ae_{t\eta}\kappa_\zeta\right)\right] \tag{A-101}$$

where λ is a Lagrange multiplier, $\{\gamma^*\}$ is the re-parameterized column matrix of generalized strains:

$$\{\gamma^*\} = \begin{Bmatrix} \frac{1}{A}\left(\frac{V_x}{E_0} - D_0\kappa_x - Ae_{t\zeta}\kappa_\eta + Ae_{t\eta}\kappa_\zeta\right) \\ \kappa_x \\ \kappa_\eta \\ \kappa_\zeta \end{Bmatrix} \tag{A-102}$$

and $[K_0]$ is the coefficient matrix in equation (A-98):

$$[K_0] = \begin{bmatrix} E_0A & E_0D_0 & E_0Ae_{t_\zeta} & -E_0Ae_{t_\eta} \\ E_0D_0 & G_0J & E_0D_3 & -E_0D_2 \\ E_0Ae_{t_\zeta} & E_0D_3 & E_0I_\eta & E_0I_{\eta\zeta} \\ -E_0Ae_{t_\eta} & -E_0D_2 & E_0I_{\eta\zeta} & E_0I_\zeta \end{bmatrix} \tag{A-103}$$

By taking the variation of Φ with respect to V_x, λ may be identified as the axial force, and therefore, the strain energy may be written:

$$\Phi = \frac{1}{2}\{\gamma^*\}^T[K_0]\{\gamma^*\} + V_x\left[\varepsilon_x - \frac{1}{A}\left(\frac{V_x}{E_0} - D_0\kappa_x - Ae_{t\zeta}\kappa_\eta + Ae_{t\eta}\kappa_\zeta\right)\right] \tag{A-104}$$

where ε_x is expressed in terms of the primal displacement variables u, v, w, and ϕ.

If the strain energy is used as shown in equation (5), it will be found that the coefficients of the equations pertaining to V_x will differ greatly from the other coefficients for typical rotor blades, and this raises the specter of ill-conditioning of the system equations. The coefficients of the element's equations may be made more uniform in magnitude by replacing V_x with a strain-like quantity defined as:

$$\bar{\varepsilon} \equiv \frac{V_x}{E_0A} \tag{A-105}$$

whereupon the strain energy becomes:

$$\Phi = \frac{1}{2}\{\gamma^*\}^T[K_0]\{\gamma^*\} + \bar{\varepsilon}\Big[E_0A(\varepsilon_x - \bar{\varepsilon}) + E_0D_0\kappa_x + E_0Ae_{t\zeta}\kappa_\eta - E_0Ae_{t\eta}\kappa_\zeta\Big] \tag{A-106}$$

A.5 Contributions from Strain Energy Function

In this section the contributions to the equations of motion from strain energy are formulated. For this, the strain energy must be expressed in terms of the displacement and force variables of the analysis. The generalized internal loads are then obtained from the first variation of the strain energy function, and the contribution to the Jacobian is obtained from the second variation. Like the previous rendition of this element, which employed only displacement variables, the variables in the "strain" column matrix all have dimensions of strain; but the last of these variables, $\bar{\varepsilon}$, is actually the axial force scaled by a factor. Differentiating the strain energy with respect to this variable does not produce a "force," but instead yields an implicit relation for the axial strain, a phenomenon which is characteristic of the Reissner-Hellinger variational principle. The ordering scheme is applied to the energy prior to undertaking these operations, thus guaranteeing the appropriate preservation of symmetry. Although formally negligible, the terms of cubic and higher degree in the 1-D strains are retained to improve modeling of bearingless rotors. These terms are simplified in that κ_x in those terms can be taken as ϕ'. This results in a simple expression for both the internal loads and the Jacobian.

A.5.1 Generalized Internal Forces In order to define the generalized internal forces, first note that the 1-D generalized strain measures have the convenient property that derivatives of the 1-D strain energy function with respect to these measures gives the 1-D stress resultants which correspond to axial section force V_x, twisting moment M_x, and bending moments M_η and M_ζ. The derivative with respect to the "strain" $\bar{\varepsilon}$ gives a quantity dubbed α, which has the dimensions of force, but may be regarded more properly as the residual of the inverse constitutive relation for the axial strain, and the subsequent application of a weighting function (i.e., $\delta\varepsilon$) to this quantity will constitute the weak enforcement of the relation. The internal forces are therefore:

$$E_0A\bar{\varepsilon} = \frac{\partial\Phi}{\partial\varepsilon_x}$$
$$M_x = \frac{\partial\Phi}{\partial\kappa_x} = E_0D_0\bar{\varepsilon} + (G_0J - \frac{E_0D_0^2}{A})\kappa_x + (E_0D_3 - E_0D_0e_{t\zeta})\kappa_\eta + (-E_0D_2 + E_0D_0e_{t\eta})\kappa_\zeta$$

$$M_\eta = \frac{\partial\Phi}{\partial\kappa_\eta} = E_0Ae_{t_\zeta}\bar{\varepsilon} + (E_0D_3 - E_0D_0e_{t_\zeta})\kappa_x + (E_0I_\eta - E_0Ae_{t_\zeta}^2)\kappa_\eta + (E_0I_{\eta\zeta} + AE_0e_{t_\zeta}e_{t_\eta})\kappa_\zeta$$

$$M_\zeta = \frac{\partial\Phi}{\partial\kappa_\zeta} = -E_0Ae_{t_\eta}\bar{\varepsilon} + (-E_0D_2 + E_0D_0e_{t_\eta})\kappa_x + (E_0I_\zeta - E_0Ae_{t_\eta}^2)\kappa_\zeta + (E_0I_{\eta\zeta} + E_0Ae_{t_\zeta}e_{t_\eta})\kappa_\eta$$

$$\alpha \equiv \frac{\partial\Phi}{\partial\bar{\varepsilon}} = E_0A(\varepsilon_x - \bar{\varepsilon}) + E_0D_0\kappa_x + E_0Ae_{t_\zeta}\kappa_\eta - E_0Ae_{t_\eta}\kappa_\zeta \tag{A-107}$$

These quantities correspond to the axial force, twisting moment, and bending moments in the deformed beam basis. They can be arranged in a column matrix $\{F\}$ so that

$$F_i = \frac{\partial\Phi}{\partial\gamma_i} \tag{A-108}$$

where the subscript $i = 1,2,3,4,5$ refers to the set ε_x, κ_x, κ_η, κ_ζ, and $\bar{\varepsilon}$. Note that γ is now defined as: $\gamma = \{\varepsilon_x, \kappa_x, \kappa_\eta, \kappa_\zeta, \bar{\varepsilon}\}^T$.

With anticipation of forming the Jacobian, a symmetric matrix $[K]$ is defined so that

$$K_{ij} = \frac{\partial^2\Phi}{\partial\gamma_i\partial\gamma_j} \tag{A-109}$$

Thus, $[K]$ is given by

$$[K] = \begin{bmatrix} 0 & 0 & 0 & 0 & E_0A \\ 0 & G_0J - \frac{E_0D_0^2}{A} & E_0D_3 - E_0D_0e_{t_\zeta} & -E_0D_2 + E_0D_0e_{t_\eta} & E_0D_0 \\ 0 & E_0D_3 - E_0D_0e_{t_\zeta} & E_0I_\eta - E_0Ae_{t_\zeta}^2 & E_0I_{\eta\zeta} + AE_0e_{t_\zeta}e_{t_\eta} & E_0Ae_{t_\zeta} \\ 0 & -E_0D_2 + E_0D_0e_{t_\eta} & E_0I_{\eta\zeta} + AE_0e_{t_\zeta}e_{t_\eta} & E_0I_\zeta - E_0Ae_{t_\eta}^2 & -E_0Ae_{t_\eta} \\ E_0A & E_0D_0 & E_0Ae_{t_\zeta} & -E_0Ae_{t_\eta} & -E_0A \end{bmatrix} \tag{A-110}$$

Both $\{F\}$ and $[K]$ are used below to obtain the generalized internal forces and the associated Jacobian.

To proceed with this development, it is necessary to carry out the variation of the strain energy with the 1-D strain measures written in terms of the variables of the analysis u, v, w, ϕ, $\bar{\varepsilon}$ and their derivatives. First, the strain measures are written in terms of the displacement variables. Applying the ordering scheme and retaining terms to $O(\varepsilon^2)$, one obtains

$$\varepsilon_x = \sqrt{(1+u')^2 + v'^2 + w'^2} - 1$$

$$\kappa_x = \phi' + v''w'$$

$$\kappa_\eta = v''s_1 - w''c_1$$

$$\kappa_\zeta = v''c_1 + w''s_1 \tag{A-111}$$

The variation is now most easily done using index notation and the chain rule. Introducing the column matrix

$$\{z\} = \{u \quad v \quad w \quad \phi \quad u' \quad v' \quad w' \quad \phi' \quad u'' \quad v'' \quad w'' \quad \phi'' \quad \bar{\varepsilon}\}^T \tag{A-112}$$

with indices which vary from 1 – 13, and letting all repeated indices be summed over their range, one can express the first variation of Φ as

$$\delta\Phi = \frac{\partial\Phi}{\partial\gamma_i}\frac{\partial\gamma_i}{\partial z_j}\delta z_j = F_iR_{ij}\delta z_j = \delta\{z\}^T[R]^T\{F\} \tag{A-113}$$

where $[R]$ and $\{F\}$ are given by

$$[R] = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{(1+u')}{\varepsilon_x+1} & \frac{v'}{\varepsilon_x+1} & \frac{w'}{\varepsilon_x+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v'' & 1 & 0 & w' & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_\zeta & 0 & 0 & 0 & 0 & 0 & s_1 & -c_1 & 0 & 0 \\ 0 & 0 & 0 & -\kappa_\eta & 0 & 0 & 0 & 0 & 0 & c_1 & s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\{F\} = \begin{Bmatrix} E_0 A \bar{\varepsilon} \\ M_x \\ M_\eta \\ M_\zeta \\ \alpha \end{Bmatrix} \tag{A-114}$$

It is now easily seen that the column matrix of generalized forces corresponding to $\{z\}$ as defined above is

$$\{f\} = [R]^T \{F\} \tag{A-115}$$

The column matrix $\{f\}$ can be written explicitly as

$$\{f\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_\eta \kappa_\zeta - M_\zeta \kappa_\eta \\ \frac{E_0 A \bar{\varepsilon}(1+u')}{\varepsilon_x+1} \\ \frac{E_0 A \bar{\varepsilon} v'}{\varepsilon_x+1} \\ \frac{E_0 A \bar{\varepsilon} w'}{\varepsilon_x+1} + v'' M_x \\ M_x \\ 0 \\ w' M_x + M_\zeta c_1 + M_\eta s_1 \\ M_\zeta s_1 - M_\eta c_1 \\ 0 \\ E_0 A(\varepsilon_x - \bar{\varepsilon}) + E_0 D_0 \kappa_x + E_0 A e_{t_\zeta} \kappa_\eta - E_0 A e_{t_\eta} \kappa_\zeta \end{Bmatrix} \tag{A-116}$$

The contribution of the internal forces is then expressible as

$$\delta U = \int_0^\ell \delta \{z\}^T f \, dx \tag{A-117}$$

A.5.2 Jacobian

The contribution of these terms to the perturbation equations is obtained from forming the Jacobian. This is like taking one more variation, resulting in

$$\Delta\delta\Phi = \delta z_i \left(R_{ki} K_{kl} R_{lj} + \frac{\partial^2 \gamma_k}{\partial z_i \partial z_j} F_k \right) \Delta z_j \tag{A-118}$$

Thus, the contribution can be written in matrix form as

$$\Delta\delta U = \int_0^\ell \delta\{z\}^T([K_1]+[K_2])\Delta\{z\}\,dx \tag{A-119}$$

where $[K_1]$ is given by

$$[K_1] = [R]^T[K][R] \tag{A-120}$$

and $[K_2]$ is

$$[K_2] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -(M_\eta\kappa_\eta + M_\zeta\kappa_\zeta) & 0 & 0 & 0 & 0 & 0 & M_\eta c_1 - M_\zeta s_1 & M_\eta s_1 + M_\zeta c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E_0A\bar{\varepsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E_0A\bar{\varepsilon} & 0 & 0 & M_x & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_\eta c_1 - M_\zeta s_1 & 0 & 0 & M_x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_\eta s_1 + M_\zeta c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{A-121}$$

Thus, the strain energy contribution to the Jacobian is given *explicitly* in terms of $[K_1]+[K_2]$. Only the nonzero elements of $[K_1]$ should be calculated.

A.6 Mixed Element Inertia Terms

The inertia terms of the mixed element are obtained by forming the integral of the virtual work due to inertial loads from the time integral of the variation of the kinetic energy, and then integrating by parts. First, expressions are written for the velocity of an arbitrary point on the beam reference line and the angular velocity of the deformed beam cross section, assuming that the warping of the section is ignored for the purpose of determination of the kinetic energy. Next, the kinetic energy is written based on the geometrically exact formulation of reference (Hodges, 1990), using the cross-sectional integrals defined therein. Finally, the contribution to the final equations of motion from inertial and gravitational forces is determined by a series of operations which include variation with respect to all unknown coordinate functions, integration by parts in the time domain, and application of the ordering scheme. The ordering scheme need not be invoked until the last step.

A.6.1 Kinematics For the purpose of writing the kinetic energy in compact form, the column matrix $\left\{v_E^{A'I}\right\}$ is written as

$$\left\{v_E^{A'I}\right\} = \left\{v_E^{EI}\right\} + \left\{\dot{r}_E^{A'E}\right\} + \left[\widetilde{\omega_E^{EI}}\right]\left\{r_E^{A'E}\right\} \tag{A-122}$$

where $\left\{v_E^{EI}\right\}$ is the column matrix, the elements of which are measure numbers of $\mathbf{v}^{EI}$; $\left\{\omega_E^{EI}\right\}$ is the column matrix, the elements of which are measure numbers of ω^{EI}; and (neglecting warping effects)

$$\left\{r_E^{A'E}\right\} = \begin{Bmatrix} x+u \\ v \\ w \end{Bmatrix} \tag{A-123}$$

The column matrix $\left\{\omega_{S'}^{S'I}\right\}$ contains the measure numbers of the angular velocity of the deformed beam cross sectional frame $\omega^{S'I}$ in the deformed beam basis. An approximation consistent with the ordering scheme is given by

$$\left\{\omega_{S'}^{S'I}\right\} = \begin{bmatrix} 1 & 0 & w' \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ -\dot{w}' \\ \dot{v}' \end{Bmatrix} + \left[T^{S'E}\right]\left\{\omega_E^{EI}\right\} \tag{A-124}$$

Finally, consider the virtual rotation of S' in I. Since the motion of E is prescribed in I, this is the same as the virtual rotation of S' in E, denoted by $\overline{\delta\theta}^{S'E}$. From reference (Hodges, 1980) this vector can be written as

$$\overline{\delta\theta}^{S'E} = \overline{\delta\theta}_{S'_i}^{S'E}\mathbf{b}_i^{S'} \tag{A-125}$$

where, within the accuracy of the ordering scheme, the column matrix containing these elements can be written as

$$\left\{\overline{\delta\theta}_{S'}^{S'E}\right\} = \begin{bmatrix} 1 & 0 & w' \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix} \begin{Bmatrix} \delta\phi \\ -\delta w' \\ \delta v' \end{Bmatrix} \tag{A-126}$$

A.6.2 Kinetic Energy Expression The kinetic energy can now be written as

$$K = \frac{1}{2}\int_0^{\ell}\left[m\left\{v_E^{A'I}\right\}^T\left\{v_E^{A'I}\right\} + 2m\left\{\omega_{S'}^{S'I}\right\}^T[\tilde{e}]\left[T^{S'E}\right]\left\{v_E^{A'I}\right\} + \left\{\omega_{S'}^{S'I}\right\}^T[I]\left\{\omega_{S'}^{S'I}\right\}\right]dx \tag{A-127}$$

where $\{e\}$ is given by

$$\{e\} = \begin{Bmatrix} 0 \\ e_{m_\eta} \\ e_{m_\zeta} \end{Bmatrix} \tag{A-128}$$

$e_{m_\eta} = \langle\rho\eta\rangle$, $e_{m_\zeta} = \langle\rho\zeta\rangle$, $[I]$ is the sectional inertia matrix given by

$$[I] = \begin{bmatrix} i_\eta + i_\zeta & 0 & 0 \\ 0 & i_\eta & i_{\eta\zeta} \\ 0 & i_{\eta\zeta} & i_\zeta \end{bmatrix} \tag{A-129}$$

and $i_\eta = \left\langle\rho\zeta^2\right\rangle$, $i_\zeta = \left\langle\rho\eta^2\right\rangle$, and $i_{\eta\zeta} = -\langle\rho\zeta\eta\rangle$. This is a geometrically-exact expression for the kinetic energy, provided that all vector quantities are written exactly. (Note that the radii of gyration can be obtained from $k_{m\eta} = \sqrt{\frac{i_\zeta}{m}}$ and $k_{m\zeta} = \sqrt{\frac{i_\eta}{m}}$.) The result for the virtual work of body forces is of the form

$$\delta W_b = \delta W_{b0} + \delta W_{b1} + \delta W_{b2} \tag{A-130}$$

where the subscript b refers to these as body force terms, and the subscripts 0, 1, and 2 refer to the zeroth, first, and second moment terms respectively. Below, the first term (the zeroth moment term of K) is written exactly. The second term (the first moment) and the third term (the second moment) are approximated and have been carried out via computer-aided symbolic manipulation based on the above approximation of $\left\{\omega_{S'}^{S'I}\right\}$.

A.6.3 Exact Term When δK is substituted into Hamilton's principle, the first term must be integrated from fixed times t_1 and t_2. After doing this, integrating by parts in time, and setting $\delta\left\{r_E^{A'E}\right\}$ equal to zero at $t = t_1$ and $t = t_2$, one obtains the contribution of this term to the virtual work of the inertial forces

$$\delta W_{b0} = -\int_0^\ell m\delta\left\{r_E^{A'E}\right\}^T \left\{\left\{a_E^{EI}\right\} + \left\{\ddot{r}_E^{A'E}\right\} + 2\left[\widetilde{\omega_E^{EI}}\right]\left\{\dot{r}_E^{A'E}\right\} + \left[\left[\widetilde{\omega_E^{EI}}\right]\left[\widetilde{\omega_E^{EI}}\right] + \left[\dot{\widetilde{\omega_E^{EI}}}\right]\right]\left\{r_E^{A'E}\right\}\right\} dx \qquad \text{(A-131)}$$

This can be written as four terms, three on the left-hand side of the equations of motion (the signs are changed on these terms to refect the change from right to left), and one on the right-hand side. The left-hand-side contributions are to the mass matrix

$$\int_0^\ell m \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^T \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} dx \qquad \text{(A-132)}$$

the gyroscopic matrix (which is non-dissipative)

$$2\int_0^\ell m \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^T \left[\widetilde{\omega_E^{EI}}\right] \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} dx \qquad \text{(A-133)}$$

and the stiffness matrix

$$\int_0^\ell m \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^T \left[\left[\widetilde{\omega_E^{EI}}\right]\left[\widetilde{\omega_E^{EI}}\right] + \left[\dot{\widetilde{\omega_E^{EI}}}\right]\right] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} dx \qquad \text{(A-134)}$$

where the associated rotation matrices are

$$\left[\widetilde{\omega_E^{EI}}\right]\left[\widetilde{\omega_E^{EI}}\right] = \begin{bmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{bmatrix} \qquad \text{(A-135)}$$

and

$$\left[\dot{\widetilde{\omega_E^{EI}}}\right] = \begin{bmatrix} 0 & -\dot{\omega}_3 & \dot{\omega}_2 \\ \dot{\omega}_3 & 0 & -\dot{\omega}_1 \\ -\dot{\omega}_2 & \dot{\omega}_1 & 0 \end{bmatrix} \qquad \text{(A-136)}$$

All the remaining terms contribute to the residual on the right-hand side of the equations. This contribution is

$$-\int_0^\ell m \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^T \left\{\left\{a_E^{EI}\right\} + x\left[\left[\widetilde{\omega_E^{EI}}\right]\left[\widetilde{\omega_E^{EI}}\right] + \left[\dot{\widetilde{\omega_E^{EI}}}\right]\right] \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\right\} dx \qquad \text{(A-137)}$$

Gravity can be treated with the inertial forces just by replacing $\left\{a_E^{EI}\right\}$ with $\left\{a_E^{EI}\right\} - \{g_E\}$ where $\{g_E\}$ is the column matrix which contains the measure numbers of the gravity vector in E basis.

A.6.4 First Moment Term The second term in the kinetic energy contains the contributions from the offset of the beam reference axis and the sectional mass centroid. If sectional analysis codes are used, the reference line can be input as the mass centroid which removes all these terms from the equations.

Taking the variation and integrating over time, one obtains

$$\int_{t_1}^{t_2} \delta K dt = \int_{t_1}^{t_2} \int_0^{\ell} \left(m\delta \left\{ \omega_{S'}^{S'I} \right\}^T [\tilde{e}] \left[T^{S'E} \right] \left\{ v_E^{A'I} \right\} + m \left\{ \omega_{S'}^{S'I} \right\}^T [\tilde{e}] \delta \left[T^{S'E} \right] \left\{ v_E^{A'I} \right\} + m \left\{ \omega_{S'}^{S'I} \right\}^T [\tilde{e}] \left[T^{S'E} \right] \delta \left\{ v_E^{A'I} \right\} \right) dx\, dt + \ldots \tag{A-138}$$

The variation of the angular velocity can be written in terms of the virtual rotation $\left\{ \overline{\delta\theta}_{S'}^{S'E} \right\}$ according to reference (Hodges, 1990) as

$$\delta \left\{ \omega_{S'}^{S'I} \right\} = \left\{ \dot{\overline{\delta\theta}}_{S'}^{S'E} \right\} + \left[\widetilde{\omega_{S'}^{S'I}} \right] \left\{ \overline{\delta\theta}_{S'}^{S'E} \right\} \tag{A-139}$$

Also, the variation of $\left[T^{S'E} \right]$ can be expressed in terms of the virtual rotation as

$$\delta \left[T^{S'E} \right] = - \left[\widetilde{\overline{\delta\theta}_{S'}^{S'E}} \right] \left[T^{S'E} \right] \tag{A-140}$$

With these substitutions, the contribution to the equations of motion can be shown to be in these two terms

$$\delta K = - \int_0^{\ell} m \left(\left\{ \overline{\delta\theta}_{S'}^{S'E} \right\}^T [\tilde{e}] \left[T^{S'E} \right] \left\{ \left\{ a_E^{EI} \right\} + \left\{ \ddot{r}_E^{A'E} \right\} + 2 \left[\widetilde{\omega_E^{EI}} \right] \left\{ \dot{r}_E^{A'E} \right\} + \left[\left[\widetilde{\omega_E^{EI}} \right] \left[\widetilde{\omega_E^{EI}} \right] + \left[\dot{\widetilde{\omega_E^{EI}}} \right] \right] \left\{ r_E^{A'E} \right\} \right\} + \delta \left\{ r_E^{A'E} \right\}^T \left[T^{S'E} \right]^T \left[\left[\widetilde{\omega_{S'}^{S'I}} \right] \left[\widetilde{\omega_{S'}^{S'I}} \right] + \left[\dot{\widetilde{\omega_{S'}^{S'I}}} \right] \right] \{e\} \right) dx + \ldots \tag{A-141}$$

This expression is geometrically exact. However, in the present theory $\left[T^{S'E} \right]$, $\left\{ \omega_{S'}^{S'I} \right\}$, and $\left\{ \overline{\delta\theta}_{S'}^{S'E} \right\}$ are approximated as in equations A-91, A-124, and A-126, respectively. The offset quantities are assumed to be $O(\varepsilon^2)$. Thus, the terms multiplied by the offsets need only be retained to $O(\varepsilon)$.

The above operations result in the contribution of the first moment terms to the virtual work of the inertial force terms on the right-hand side. These can be simplified by introducing

$$\begin{aligned} e_{m_y} &= c_1 e_{m_\eta} - s_1 e_{m_\zeta} \\ e_{m_z} &= s_1 e_{m_\eta} + c_1 e_{m_\zeta} \end{aligned} \tag{A-142}$$

so that the generalized body force contributions from the first moment terms are

$$\delta W_{b1} = \int_0^{\ell} \delta v \left[e_{m_y} \left({\omega_1}^2 + {\omega_3}^2 + 2\omega_1 \dot{\phi} + \dot{\phi}^2 \right) - e_{m_z} \left(\omega_2 \omega_3 - \dot{\omega}_1 - \ddot{\phi} \right) \right] dx$$

$$
\begin{aligned}
&-\int_0^\ell \delta w \left[e_{m_y}\left(\omega_2\omega_3+\dot\omega_1+\ddot\phi\right)-e_{m_z}\left(\omega_1{}^2+\omega_2{}^2+2\omega_1\dot\phi+\dot\phi^2\right)\right]dx\\
&-\int_0^\ell \delta\phi\Big\{e_{m_y}\Big[A_3-\left(\omega_1{}^2+\omega_2{}^2\right)w-\left(A_1-x\omega_2{}^2-x\omega_3{}^2\right)w'\\
&+v\left(\omega_2\omega_3+\dot\omega_1\right)+x\left(\omega_1\omega_3-\dot\omega_2\right)+2\omega_1\dot v+\ddot w\Big]\\
&\quad+e_{m_z}\Big[-A_2+\left(\omega_1{}^2+\omega_3{}^2\right)v+\left(A_1-x\omega_2{}^2-x\omega_3{}^2\right)v'\\
&+w\left(-\omega_2\omega_3+\dot\omega_1\right)-x\left(\omega_1\omega_2+\dot\omega_3\right)+2\omega_1\dot w-\ddot v\Big]\Big\}dx\\
&+\int_0^\ell \delta w' e_{m_z}\left[A_1-x\left(\omega_2{}^2+\omega_3{}^2\right)\right]dx\\
&+\int_0^\ell \delta v' e_{m_y}\left[A_1-x\left(\omega_2{}^2+\omega_3{}^2\right)\right]dx
\end{aligned}
\qquad\text{(A-143)}
$$

Note that in this expression there are no linear terms.

The first moment term yields a total of 17 terms in the perturbed equations of motion, the signs of which are changed to reflect their being on the left-hand side of the equations of motion. In virtual work form, the four “mass matrix” terms are

$$
\begin{aligned}
-\Delta\delta W_{b1} = &-\int_0^\ell \delta v\, m e_{m_z}\Delta\ddot\phi\, dx\\
&+\int_0^\ell \delta w\, m e_{m_y}\Delta\ddot\phi\, dx\\
&-\int_0^\ell \delta\phi\, m e_{m_z}\Delta\ddot v\, dx\\
&+\int_0^\ell \delta\phi\, m e_{m_y}\Delta\ddot w\, dx+\ldots
\end{aligned}
\qquad\text{(A-144)}
$$

The four “gyroscopic matrix” terms are

$$
\begin{aligned}
-\Delta\delta W_{b1} = &-2\int_0^\ell \delta v\, m e_{m_y}(\omega_1+\dot\phi)\Delta\dot\phi\, dx\\
&-2\int_0^\ell \delta w\, m e_{m_z}(\omega_1+\dot\phi)\Delta\dot\phi\, dx\\
&+2\omega_1\int_0^\ell \delta\phi\, m e_{m_y}\Delta\dot v\, dx\\
&+2\omega_1\int_0^\ell \delta\phi\, m e_{m_z}\Delta\dot w\, dx+\ldots
\end{aligned}
\qquad\text{(A-145)}
$$

The nine terms contributing to the Jacobian are

$$
\begin{aligned}
-\Delta\delta W_{b1} = &\int_0^\ell \delta v\, m\left[e_{m_y}\left(\omega_2\omega_3-\dot\omega_1-\ddot\phi\right)+e_{m_z}\left(\omega_1{}^2+\omega_3{}^2+2\omega_1\dot\phi+\dot\phi^2\right)\right]\Delta\phi\, dx\\
&-\int_0^\ell \delta w\, m\left[e_{m_y}\left(\omega_1{}^2+\omega_2{}^2+2\omega_1\dot\phi+\dot\phi^2\right)+e_{m_z}\left(\omega_2\omega_3+\dot\omega_1+\ddot\phi\right)\right]\Delta\phi\, dx\\
&+\int_0^\ell \delta\phi\, m\left[e_{m_y}\left(\omega_2\omega_3+\dot\omega_1\right)+e_{m_z}\left(\omega_1{}^2+\omega_3{}^2\right)\right]\Delta v\, dx\\
&-\int_0^\ell \delta\phi\, m\left[e_{m_y}\left(\omega_1{}^2+\omega_2{}^2\right)+e_{m_z}\left(\omega_2\,\omega_3-\dot\omega_1\right)\right]\Delta w\, dx\\
&+\int_0^\ell \delta\phi\, m\Big\{e_{m_y}\Big[-A_2-x\omega_1\omega_2+\omega_1{}^2 v+\omega_3{}^2 v-\omega_2\omega_3 w\\
&\qquad+w\dot\omega_1-x\dot\omega_3+2\omega_1\dot w-\ddot v+\left(A_1-x\omega_2{}^2-x\omega_3{}^2\right)v'\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+e_{m_z}\Big[-A_3 - x\omega_1\omega_3 - \omega_2\omega_3 v + \omega_1{}^2 w + \omega_2{}^2 w \\
&\qquad -v\dot{\omega}_1 + x\dot{\omega}_2 - 2\omega_1\dot{v} - \ddot{w} + \left(A_1 - x\omega_2{}^2 - x\omega_3{}^2\right) w'\Big]\Big\}\Delta\phi\, dx \\
&- \int_0^\ell \delta w'\, me_{m_y}\left[A_1 - x\left(\omega_2{}^2 + \omega_3{}^2\right)\right]\Delta\phi\, dx \\
&+ \int_0^\ell \delta v'\, me_{m_z}\left[A_1 - x\left(\omega_2{}^2 + \omega_3{}^2\right)\right]\Delta\phi\, dx \\
&- \int_0^\ell \delta\phi\, me_{m_y}\left[A_1 - x\left(\omega_2{}^2 + \omega_3{}^2\right)\right]\Delta w'\, dx \\
&+ \int_0^\ell \delta\phi\, me_{m_z}\left[A_1 - x\left(\omega_2{}^2 + \omega_3{}^2\right)\right]\Delta v'\, dx + \ldots
\end{aligned}
\tag{A-146}
$$

A.6.5 Second Moment Term The third term in the kinetic energy contains the contributions from the sectional inertia matrix $[I]$. Taking the variation of this term and integrating over time yields

$$
\int_{t_1}^{t_2} \delta K dt = \int_{t_1}^{t_2}\int_0^\ell \delta\left\{\omega_{S'}^{S'I}\right\}^T [I]\left\{\omega_{S'}^{S'I}\right\} dx\, dt + \ldots \tag{A-147}
$$

Substituting equation A-139, integrating by parts, and setting $\left\{\overline{\delta\theta}_{S'}^{S'E}\right\}$ equal to zero at $t = t_1$ and $t = t_2$, one obtains the contribution of these terms to the equations of motion

$$
\delta K = -\int_0^\ell \left\{\overline{\delta\theta}_{S'}^{S'E}\right\}^T \left[[I]\left\{\dot{\omega}_{S'}^{S'I}\right\} + \left[\widetilde{\omega_{S'}^{S'I}}\right][I]\left\{\omega_{S'}^{S'I}\right\}\right] dx + \ldots \tag{A-148}
$$

The sectional mass moments of inertia can be regarded as $O(mR^2\varepsilon^3)$; thus, the terms multiplied by them need only be retained to $O(1)$. To this order, there are only two second moment terms. The first is a linear term in the mass matrix, the sign of which is changed to reflect its being on the left-hand side

$$
-\delta W_{b2} = \int_o^\ell \delta\phi (i_\eta + i_\zeta)\ddot{\phi}\, dx + \ldots \tag{A-149}
$$

The second is the residual, in which there is only a nonlinear torsion moment term, given by

$$
\begin{aligned}
\delta W_{b2} = \int_o^\ell \delta\phi \Big\{ &\left(i_\eta - i_\zeta\right)\left[\left(c_1{}^2 - s_1{}^2\right)\omega_2\,\omega_3 - \left(\omega_2{}^2 - \omega_3{}^2\right) s_1\, c_1\right] \\
&- i_{\eta\zeta}\left[4 c_1 s_1\,\omega_2\,\omega_3 + \left(c_1{}^2 - s_1{}^2\right)\left(\omega_2{}^2 - \omega_3{}^2\right)\right] - \left(i_\eta + i_\zeta\right)\dot{\omega}_1 \Big\} dx + \ldots
\end{aligned}
\tag{A-150}
$$

Note that there are no linear terms in this expression. (The one linear term which would have been present is in the mass matrix and not repeated in the residual.) The Jacobian of this expression only contains the one term

$$
\begin{aligned}
-\Delta\delta W_{b2} = \int_o^\ell \delta\phi \Big\{ &\left[(c_1{}^2 - s_1{}^2)(i_\eta - i_\zeta) - 4 s_1 c_1\, i_{\eta\zeta}\right](\omega_2^2 - \omega_3^2) \\
&+ 4\left[(c_1{}^2 - s_1{}^2) i_{\eta\zeta} + s_1 c_1 (i_\eta - i_\zeta)\right]\omega_2\,\omega_3 \Big\}\Delta\phi\, dx
\end{aligned}
\tag{A-151}
$$

This part of the formulation can be extended, if necessary, to the case in which the sectional mass moments of inertia are $O(mR^2\varepsilon^2)$. The contribution of these neglected terms, however, is believed to be negligible for most rotor blades.

B Modal Reduction with Mixed Elements

If mixed elements are employed, it may seem straightforward to recover the finite element displacements from modal coordinates by employing the usual modal reduction relation:

$$\{q\} = [\Phi]\{\alpha\} \tag{B-152}$$

where $[\Phi]$ is the modal matrix, α is a vector of modal coordinates, and $\{q\}$ are the finite element degrees of freedom, in which $\{q\}$ includes the axial force degrees of freedom. However, using this approach implicitly retains a linear relation between the axial forces and displacements, which is not generally correct. This problem may be dealt with by retaining the axial forces as *independent* degrees of freedom. To accomplish this, each eigenvector is written as follows:

$$\{\phi_i\} = \begin{Bmatrix} \phi_d \\ \phi_f \end{Bmatrix} \tag{B-153}$$

where $\{\phi_i\}$ is the ith modal vector, $\{\phi_d\}$ is the displacement part of the eigenmode and $\{\phi_f\}$ is the force part of the eigenmode. When modal reduction is performed, $\{\phi_i\}$ is split into two generalized coordinates containing the force and displacement parts separately:

$$\{\phi_i\} \mapsto [\phi_1|\phi_2] = \begin{bmatrix} \phi_d & 0 \\ 0 & \phi_f \end{bmatrix} \tag{B-154}$$

This approach permits sufficient decoupling of the axial displacements and forces to enable the accurate calculation of each. It has the drawback of requiring that additional generalized coordinates be added to the analysis, but in practice, only one or two of these are needed.