Rotorcraft Aeroelasticity

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- Helicopter rotors can be classified according to the root retention mechanism
    - Most older designs are articulated rotors, i.e. hinged at the root with flap and lead-lag hinges and pitch bearings
    - Most newer rotors are hingeless, i.e. clamped but may possess a pitch bearing
    - Hingeless rotors without a pitch bearing are called bearingless rotors
    - Additional rotor types include teetering and gimballed rotors
- Newer designs have lower part count, weight, and possibly maintenance costs
- While older designs are typically free of aeroelastic instabilities, this is not true of the newer designs

Rotorcraft dynamics problems are usually classified as

- aeroelastic stability (includes isolated blade and isolated rotor problems)
- aeromechanical stability (coupled rotor-fuselage)
- vibration (meaning “forced response” or time-dependent forces passed from the rotor to the fuselage)
- loads (meaning blade loads or stresses)

Within the total spectrum of rotorcraft aeroelasticity there are several solutions that are typically found:

- steady-state trim solution in hover (time independent)
- eigensolution for perturbation equations linearized about the steady-state hover solution (constant coefficients)
- steady-state trim solution in forward flight (periodic in time)
- eigensolution for perturbation equations linearized about steady-state forward flight solution (periodic coefficients)
- time marching solution in forward flight

Commonly used methods for deriving governing equations

- Newton-Euler methods
- Lagrange's equations
- Lagrange's form of D'Alembert's principle (a.k.a. Kane's method)
- Hamilton's principle
- principle of virtual work

Commonly used mathematical techniques and solution procedures include

- various time-integration methods
- spatial transition matrix methods
- Ritz and/or Galerkin methods
- discrete element methods (i.e. systems of springs and rigid bodies used to approximate continuous members)
- finite element/multi-flexible-body methods
- Floquet theory
- perturbation methods

Modeling by components

- Aerodynamics
  - Details of flow are generally not needed in order to predict low-frequency aeromechanical or aeroelasticity instabilities
  - The most important modeling aspects to capture are
    - generalized forces (i.e. lift, drag and pitching moments)
    - rotor induced inflow
    - rotor wake (i.e. inflow dynamics)
    - blade-vortex interaction
    - advancing blade compressibility effects (especially in transonic regime)
    - retreating blade stall (static as well as dynamic)

Modeling by components (continued)

- Rotor
  - Individual blades may be represented as rigid bodies, flexible beams, or 3-D finite element models
  - Rigid blade analysis still used in conceptual design, control system design, simulation
  - 3-D finite element analysis requires $10^5 - 10^7$ degrees of freedom
  - Beam analysis is most commonly used
  - Capability of beam models to capture composite material behavior an details of the inner structure is relatively recent development

Modeling by components (continued)

- Rotor (continued)
  - The most important modeling aspects to capture are
    - geometrical nonlinearity
    - cross-sectional warping (chiefly its effect on cross-sectional properties)
    - shear deformation
    - initial twist (and curvature)
    - anisotropic materials
    - arbitrary geometry
- Airframe
  - There are situations in which details of airframe interaction with aerodynamics or rotor are important
  - Usually only its low-frequency modes are considered, possibly only its rigid-body behavior

Warning:

- The analysis of rotorcraft and rotor blades in particular is a minefield fraught with many opportunities to make mistakes
- Some errors have shown up in the literature more than once and continue to do so
- There are a few books that treat the subject in some detail
  - Johnson
  - Bramwell
  - Bielawa
- If one wishes to understand details, however, he must delve into the literature
- Survey papers have appeared in the *Journal of Aircraft* in recent years

- Lagrange's equations for holonomic system in terms of generalized coordinates

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r} = F_r \quad r = 1, 2, \ldots, n$$

where

  - K is the system kinetic energy
  - q_r is the r^{th} generalized coordinate
  - n is the number of generalized coordinates
  - $F_r = \boldsymbol{F} \cdot \frac{\partial {}^I\boldsymbol{v}^P}{\partial \dot{q}_r} + \boldsymbol{M} \cdot \frac{\partial {}^I\boldsymbol{\omega}^B}{\partial \dot{q}_r}$ is the r^{th} generalized force for all external forces acting on the system and where
    - $\boldsymbol{F}$ is the resultant force acting at point P on a rigid body
    - ${}^I\boldsymbol{v}^P$ is the inertial velocity of P
    - $\boldsymbol{M}$ is the resultant moment about P acting on a rigid body
    - ${}^I\boldsymbol{\omega}^B$ is the inertial angular velocity of the rigid body

- Alternative form of Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = Q_r \quad r = 1, 2, \ldots, n$$

where

- $L = K - P$
- P is the potential energy for some or all conservative forces
- Q_r is the r^{th} generalized force for all nonconservative external forces acting on the system and for any conservative forces not represented in P

- Kane's equations

$$F_r + F_r^* = 0$$

where F_r^* is the generalized inertia force

$$F_r^* = -m\boldsymbol{a}^{B^* I} \cdot \frac{\partial^I \boldsymbol{v}^{B^*}}{\partial \dot{q}_r} - \left[\underline{\boldsymbol{I}} \cdot {}^I\boldsymbol{\alpha}^B + {}^I\boldsymbol{\omega}^B \times \left(\underline{\boldsymbol{I}} \cdot {}^I\boldsymbol{\omega}^B\right)\right] \cdot \frac{\partial^I \boldsymbol{\omega}^B}{\partial \dot{q}_r}$$

  - The bracketed quantity is the left-hand side of Euler's dynamical equation for a rigid body (also valid if $B^* \rightarrow O$)
  - The strength of Kane's method, which we are not taking advantage of here, is the possibility of using motion variables that are a linear combination of the $\dot{q}_r$'s

- Hamilton's principle/principle of virtual work
  - Normally applied to continuous systems

$$\delta \int_{t_1}^{t_2} (K - U) dt + \int_{t_1}^{t_2} \overline{\delta W} dt = \sum_{r=1}^{n} \frac{\partial K}{\partial \dot{q}_r} \delta q_r \bigg|_{t_1}^{t_2}$$

where

$$K = \frac{1}{2} \iiint_V \rho \, {}^I\boldsymbol{v}^P \cdot {}^I\boldsymbol{v}^P dV$$

  - U is the strain energy
  - $\overline{\delta W}$ is the virtual work of all forces not accounted for in U
  - Terms involving kinetic energy constitute the virtual work of inertial forces

- Consider a body B moving in a frame A
  - Unit vectors $\boldsymbol{a}_i$, $i = 1, 2, 3$, are fixed in A
  - Unit vectors $\boldsymbol{b}_i$, $i = 1, 2, 3$, are fixed in B and $C_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{a}_j$
  - The angular velocity of B in A is denoted ${}^A\boldsymbol{\omega}^B$ and

$$\omega_i = {}^A\boldsymbol{\omega}^B \cdot \boldsymbol{b}_i$$

with $\omega^T = \lfloor \omega_1 \quad \omega_2 \quad \omega_3 \rfloor$
- The kinematical equations are

$$-\dot{C}C^T = \widetilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- Euler's dynamical equations in matrix form are

$$I\dot{\omega} + \widetilde{\omega} I \omega = T$$

  where
  - T is the column matrix of measures of the moment about B^*, the mass center of B, or about an inertially fixed point O
  - I is the body inertia matrix, measures of the inertia dyadic expressed in body-fixed unit vectors
- Consider a case in which the body B is undergoing constant angular velocity, say $\omega = \Omega = \text{const.}$
- Thus, C is a periodic function of time, denoted by $\overline{C}$ so that $\widetilde{\Omega} = -\dot{\overline{C}}\,\overline{C}^T$ from which follows $\dot{\overline{C}} = -\widetilde{\Omega}\overline{C}$

- Now, we perturb the motion by an infinitesimal amount, so that $C = (\Delta - \widetilde{\theta})\overline{C}$ where Δ is the identity matrix
- It follows that $\dot{C} = -\widetilde{\dot{\theta}}\overline{C} + (\Delta - \widetilde{\theta})\dot{\overline{C}}$
- Thus, one may find that $\widetilde{\omega} = \widetilde{\dot{\theta}} + (\Delta - \widetilde{\theta})\widetilde{\Omega} + \widetilde{\Omega}\widetilde{\theta}$
- Therefore, $\omega = \Omega + \dot{\theta} + \widetilde{\Omega}\theta$
- Recalling that Ω is constant, $\dot{\omega} = \ddot{\theta} + \widetilde{\Omega}\dot{\theta}$
- Euler's dynamical equation then leads to two equations
  - for steady-state motion $\widetilde{\Omega} I \Omega = T$
  - for perturbation motion

$$I\ddot{\theta} + (I\widetilde{\Omega} + \widetilde{\Omega}I - \widetilde{H})\dot{\theta} + (\widetilde{\Omega}I\widetilde{\Omega} - \widetilde{H}\widetilde{\Omega})\theta = 0$$

- Let $\Omega = \Omega_1 e_1$ so that the body is spinning about $\boldsymbol{a}_1 = \boldsymbol{b}_1$
- Thus,

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \Omega_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_1 - I_2 - I_3 \\ 0 & -(I_1 - I_2 - I_3) & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}$$

$$+\Omega_1^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_1 - I_3 & 0 \\ 0 & 0 & I_1 - I_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = 0$$

- Let $I_1 > I_2 > I_3$ (recall spin axis is $\boldsymbol{a}_1 = \boldsymbol{b}_1$, which is the axis corresponding to the maximum moment of inertia)
- Note that $I_2 + I_3 \geq I_1$
- Let $\theta_i = \hat{\theta}_i \exp(i\lambda t)$
- One finds two roots: $\frac{\lambda}{\Omega_1} = 1$ and

$$\left(\frac{\lambda}{\Omega_1}\right)^2 = \left(\frac{I_1}{I_2} - 1\right)\left(\frac{I_1}{I_3} - 1\right) > 0$$

- This implies simple harmonic motion (i.e. not unstable!)

- Let $\Omega = \Omega_3 e_3$ (i.e. spin axis is $\boldsymbol{a}_3 = \boldsymbol{b}_3$, which is the axis corresponding to the minimum moment of inertia)
- One again finds two roots: $\frac{\lambda}{\Omega_3} = 1$ and

$$\left(\frac{\lambda}{\Omega_3}\right)^2 = \left(1 - \frac{I_3}{I_1}\right)\left(1 - \frac{I_3}{I_2}\right) > 0$$

- This also implies simple harmonic motion (i.e. not unstable!)

- Let $\Omega = \Omega_2 e_2$ (i.e. spin axis is $\boldsymbol{a}_2 = \boldsymbol{b}_2$, which is the axis corresponding to the intermediate moment of inertia)
- One again finds two roots: $\frac{\lambda}{\Omega_2} = 1$ and

$$\left(\frac{\lambda}{\Omega_2}\right)^2 = \left(\frac{I_2}{I_1} - 1\right)\left(\frac{I_2}{I_3} - 1\right) < 0$$

- This case yields unstable motion
- This result has important implications for rotor blades
  - They "want" to be spun about the axis of maximum moment of inertia
  - Thus, they "want" to be at flat pitch!

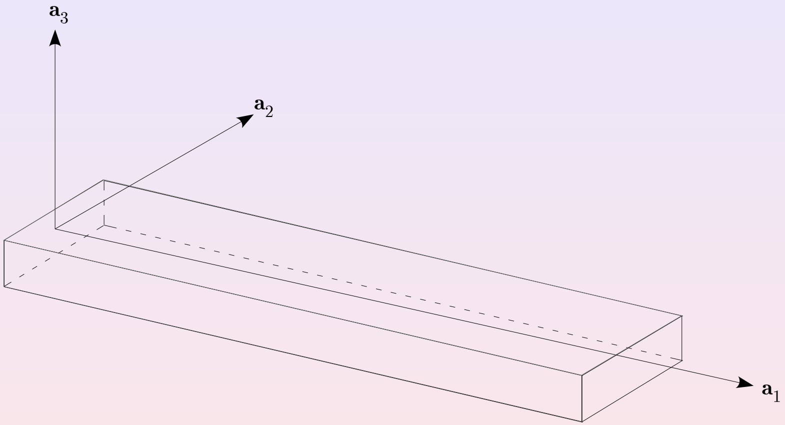


Figure: Blade-like rigid body

- Rotor blades are sometimes represented by rigid blades that are hinged at their root end
- Let $\Omega = \Omega_3 e_3$ (i.e. spin axis is $\boldsymbol{a}_3 = \boldsymbol{b}_3$), so that the "blade" is at zero pitch angle (flat pitch)
- Let $I_3 \approx I_1 + I_2$ (appropriate for bodies of this general shape)
- Let the root end (the left end as pictured) be a fixed point O in the inertial frame
- Let the blade undergo small rotations in any direction about O

- The rotating blade thus has, from the inertial effects, effective stiffnesses!
  - pitch stiffness of $I_1\Omega_3^2$ from inertial terms
  - flap stiffness of $I_2\Omega_3^2$ from inertial terms
  - lead-lag "stiffness" of zero
- Inertial effects thus provide minimum frequencies for some types of motion
  - Minimum pitch frequency is Ω_3 from so-called propeller moment (also called tennis racquet effect)
  - Minimum flap frequency is Ω_3
  - Minimum lead-lag frequency is zero

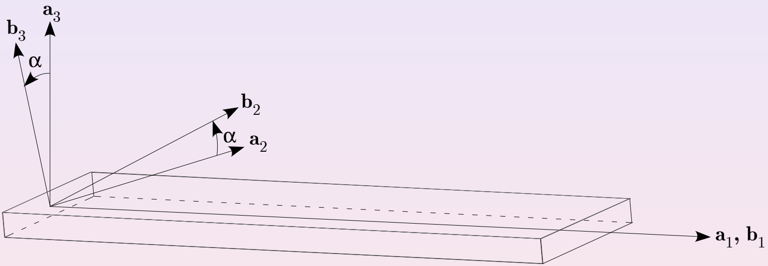
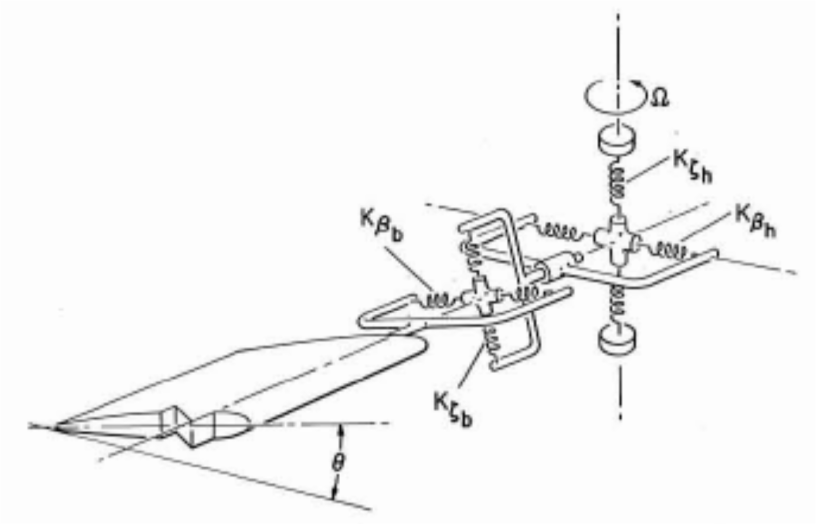


Figure: Blade-like rigid body with pitch angle α

- Now, consider the same blade mounted with a built-in pitch angle α
  - To maintain constant angular velocity ${}^I\boldsymbol{\omega}^B = \Omega_{a3}\boldsymbol{a}_3$ one must exert a constant torque $\boldsymbol{T} = {}^I\boldsymbol{\omega}^B \times (\underline{\boldsymbol{I}} \cdot {}^I\boldsymbol{\omega}^B)$
  - This simplifies to a nose-up torque caused by the propeller moment $\boldsymbol{T} = \boldsymbol{a}_1\Omega_{a3}^2(I_3 - I_2)\sin\alpha\cos\alpha \approx \boldsymbol{a}_1\Omega_{a3}^2 I_1 \sin\alpha\cos\alpha$
- Consider the same blade mounted with a built-in “precone” angle β
  - This gives rise to a blade-tip-down flapping moment $\boldsymbol{T} = \boldsymbol{a}_2\Omega_{a3}^2(I_3 - I_1)\sin\beta\cos\beta \approx \boldsymbol{a}_2\Omega_{a3}^2 I_2 \sin\beta\cos\beta$
  - A positive precone angle can be used to relieve steady-state bending stresses near the root

- Note: Addition of root rotational springs and/or of a hub offset can be used to make the resulting frequencies closer to those of actual blades
- For example, Ormiston and Hodges (1972), modeled the flap-lag dynamics of hingeless rotor blades using a centrally-hinged, spring-restrained rigid-blade model
- To account for flexibility inboard and outboard of the pitch-change bearing, a system of springs was introduced
- One may derive the equations of motion presented by Ormiston and Hodges (1972) using Lagrange's equation


Ω
K_{ζ_h}
K_{β_b}
K_{β_h}
K_{ζ_b}
θ

- The potential energy in terms of the two flap angles β_h and β_b and the two lead-lag angles ζ_h and ζ_b can be written as

$$P = \frac{1}{2}\left(K_{\zeta h}\zeta_h^2 + K_{\beta h}\beta_h^2 + K_{\zeta b}\zeta_b^2 + K_{\beta b}\beta_b^2\right)$$

- Blade motion is expressible in terms of the blade lead-lag and flap orientation angles ζ and β, respectively, given by

$$\begin{aligned}\zeta =& \zeta_h + \zeta_b \cos\theta - \beta_b \sin\theta \\ \beta =& \beta_h + \zeta_b \sin\theta + \beta_b \cos\theta\end{aligned}$$

- These can be solved for ζ_b and β_b and the result subsituted into the potential energy, yielding $P = P(\zeta, \beta, \zeta_h, \beta_h)$

- Since the generalized inertia forces for the blade can be written entirely in terms of ζ and β, one can introduce

$$K_{\zeta_b} = \frac{K_\zeta}{R}; \qquad K_{\zeta_h} = \frac{K_\zeta}{1-R}$$
$$K_{\beta_b} = \frac{K_\beta}{R}; \qquad K_{\beta_h} = \frac{K_\beta}{1-R}$$

and set

$$\frac{\partial P}{\partial \zeta_h} = \frac{\partial P}{\partial \beta_h} = 0$$

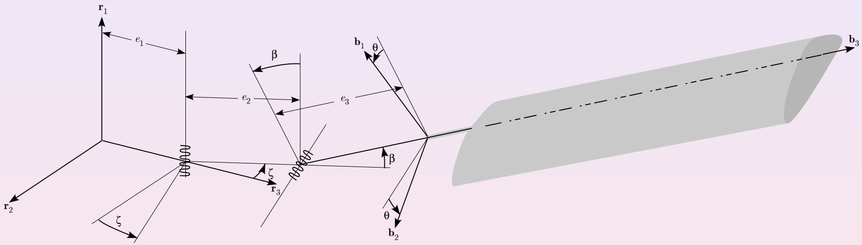
- This yields a potential energy

$$P = \frac{1}{2}\begin{Bmatrix}\zeta\\ \beta\end{Bmatrix}^T \begin{bmatrix} \frac{K_\zeta - R(K_\zeta - K_\beta)\sin^2\theta}{\Delta} & \frac{R(K_\zeta - K_\beta)\sin 2\theta}{2\Delta} \\ \frac{R(K_\zeta - K_\beta)\sin 2\theta}{2\Delta} & \frac{K_\beta + R(K_\zeta - K_\beta)\sin^2\theta}{\Delta} \end{bmatrix} \begin{Bmatrix}\zeta\\ \beta\end{Bmatrix}$$

where

$$\Delta = 1 + \frac{R(1-R)(K_\zeta - K_\beta)^2 \sin^2\theta}{K_\zeta K_\beta}$$

- When $R = 0$ there is no elastic coupling, and when $R = 1$ there is “full” elastic flap-lag coupling


r1
e1
β
b1
θ
b3
e2
e3
β
ζ
r3
r2
ζ
θ
b2

- The kinetic energy is then expressed in terms of unknowns ζ and β, along with the pitch angle θ as a “control parameter”
- The orientation of the blade is expressed as follows
  - Align the blade-fixed unit vectors $\boldsymbol{b}_i$ with those fixed in the rotating coordinate system comprised of $\boldsymbol{r}_i$
  - Rotate the blade about $\boldsymbol{b}_1$ by ζ
  - Rotate the blade about $\boldsymbol{b}_2$ by β
  - Rotate the blade about $\boldsymbol{b}_3$ by θ
- Thus, ${}^I\omega^B = (\Omega + \dot{\zeta})\boldsymbol{r}_1 + \dot{\beta}(\sin\theta\boldsymbol{b}_1 + \cos\theta\boldsymbol{b}_2) + \dot{\theta}\boldsymbol{b}_3$

- One may also write $\omega_i = {}^I\boldsymbol{\omega}^B \cdot \boldsymbol{b}_i$ so that

$$\omega_1 = (\Omega + \dot{\zeta})\cos\beta\cos\theta + \dot{\beta}\sin\theta$$
$$\omega_2 = -(\Omega + \dot{\zeta})\cos\beta\sin\theta + \dot{\beta}\cos\theta$$
$$\omega_3 = (\Omega + \dot{\zeta})\sin\beta + \dot{\theta}$$

- In case the root end is an inertially fixed point O, the kinetic energy is $K = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$
- We may approximate the moments of inertia so that $I_1 = I_2 = I$ and $I_3 << I$ so that $K = \frac{I}{2}\left(\omega_2^2 + \omega_3^2\right)$

- Thus,

$$K = \frac{I}{2}\left[\left(\Omega + \dot{\zeta}\right)^2 (\cos^2\beta \sin^2\theta + \sin^2\beta) + \dot{\beta}^2 \cos^2\theta + \dot{\theta}^2\right]$$

so that

$$\frac{\partial K}{\partial \dot{\zeta}} = I(\Omega + \dot{\zeta})\cos^2\beta \qquad \frac{\partial K}{\partial \zeta} = 0$$

$$\frac{\partial K}{\partial \dot{\beta}} = I\dot{\beta} \qquad \frac{\partial K}{\partial \beta} = -I(\Omega + \dot{\zeta})^2 \cos\beta \sin\beta$$

- As expected, the inertial terms are independent of ζ

- For the elastically uncoupled case where $R = 0$ the potential energy reduces to $P = \frac{1}{2}\left[K_\beta(\beta - \beta_{\text{pc}})^2 + K_\zeta\zeta^2\right]$
- The elastic restoring moments are thus

$$\frac{\partial P}{\partial \zeta} = K_\zeta \zeta \qquad \frac{\partial P}{\partial \beta} = K_\beta(\beta - \beta_{\text{pc}})$$

where effect of the pre-cone angle has been added
- Let $\zeta = \overline{\zeta} + \hat{\zeta}(t)$ and $\beta = \overline{\beta} + \hat{\beta}(t)$ (i.e. a static equilibrium value plus a small perturbation)
- Denote uncoupled lead-lag and flap frequencies of the nonrotating blade as $\omega_\zeta^2 = K_\zeta/(I\Omega^2)$ and $\omega_\beta^2 = K_\beta/(I\Omega^2)$

- Static equilibrium equations with no external forces are

$$\omega_\zeta^2\overline{\zeta} = 0 \qquad \omega_\beta^2(\overline{\beta} - \beta_{\text{pc}}) + \cos\overline{\beta}\sin\overline{\beta} = 0$$

- $\overline{\zeta} = 0$ while $\overline{\beta}$ is governed by the values of ω_β and β_{pc}
- Letting $\hat{q} = \lfloor \hat{\zeta} \quad \hat{\beta} \rfloor$ and primes denote derivatives with respect to non-dimensional time Ωt, one finds the perturbation equations to be of the form

$$M\hat{q}'' + C\hat{q}' + K\hat{q} = 0$$

- This form is a typical way of writing equations that govern aeroelastic/aeromechanical stability problems

- In our case
  - $M = \begin{bmatrix} \cos^2\overline{\beta} & 0 \\ 0 & 1 \end{bmatrix}$, $C = 2\cos\overline{\beta}\sin\overline{\beta}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $K = \begin{bmatrix} \omega_\zeta^2 & 0 \\ 0 & p^2 - 2\sin^2\overline{\beta} \end{bmatrix}$
  - p is the flapping frequency of the rotating blade and $p^2 = 1 + \omega_\beta^2$
  - $\overline{\zeta}$ does not appear in the perturbation equations – why?
- Changes of variable $\hat{\zeta} = \check{\zeta}\exp(i\omega t)$ and $\hat{\beta} = \check{\beta}\exp(i\omega t)$
- Existence of a nontrivial solution requires that

$$\det\begin{bmatrix} \omega_\zeta^2 - \omega^2\cos^2\overline{\beta} & -2i\omega\cos\overline{\beta}\sin\overline{\beta} \\ 2i\omega\cos\overline{\beta}\sin\overline{\beta} & p^2 - \omega^2 - 2\sin^2\overline{\beta} \end{bmatrix} = 0$$

- For $\overline{\beta} = 0$, there are two uncoupled modes with frequencies ω_ζ and p
- For $\overline{\beta} \neq 0$ and $\overline{\beta} << 1$
  - Flapping dominates one mode and leag-lag the other
  - Coupling increases with $\overline{\beta}$
- For $\overline{\beta} = O(1)$ coupling is so large that dominant type of motion is difficult to determine
- For $\overline{\beta} = \frac{\pi}{2}$
  - The system looks like a rotating shaft with stiffness in the spin direction K_ζ
  - Frequencies are infinity and $\sqrt{p^2 - 2}$, the latter of which must be positive showing that $\Omega < \sqrt{K_\beta / I}$, the critical speed

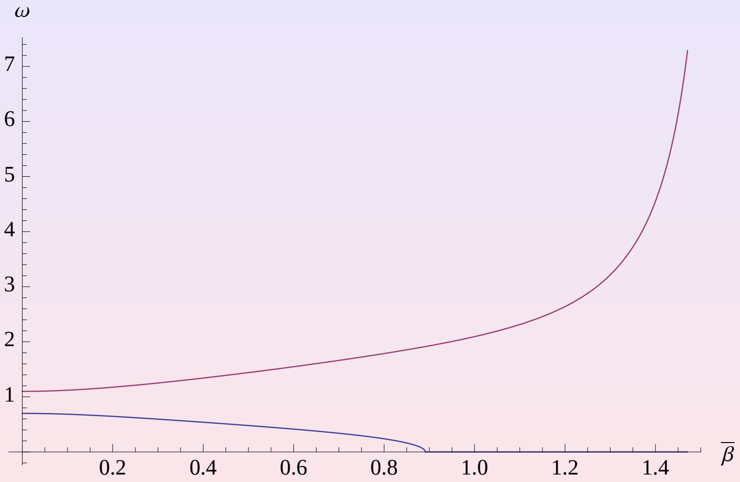


Figure: $p = 1.1$ and $\omega_\zeta = 0.7$

- Point of neutral stability is $\sin\overline{\beta} = p/\sqrt{2}$
- Effects of R and θ are hardly noticeable for this plot
- Gyroscopic/Coriolis matrix
  - Source of the coupling
  - Without these terms
    - the curves cross
    - the results at $\sin\overline{\beta} = p/\sqrt{2}$ are unaffected
    - the point of instability is unaffected

- Now consider $\overline{\beta} = 0$ and behavior versus θ and ω_ζ at fixed p and $R = 0, 1$
- Just a reminder
  - $R = 0$ corresponds to blades in which all the flap and lag flexibility is inboard of the pitch-change bearing
  - $R = 1$ corresponds to blades in which all the flap and lag flexibility is outboard of the pitch-change bearing

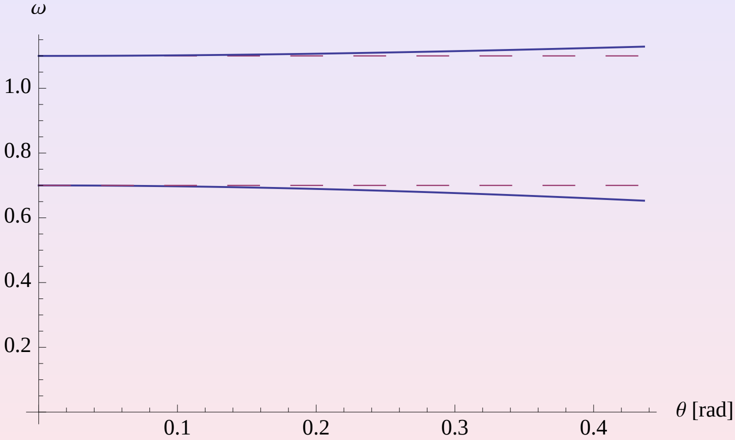


Figure: $p = 1.1$ and $\omega_\zeta = 0.7$; $R = 0$ dashed, $R = 1$ solid

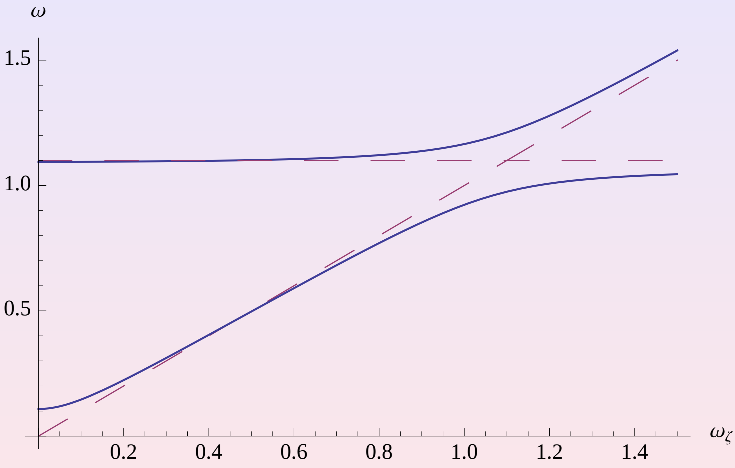


Figure: $p = 1.1$ and $\theta = 15°$; $R = 0$ dashed, $R = 1$ solid

- We start with the virtual work of aerodynamic forces

$$\overline{\delta W} = \int_0^\ell \boldsymbol{F} \cdot \delta \boldsymbol{r} dx + \int_0^\ell \boldsymbol{M} \cdot \overline{\delta \psi} dx$$

  where
  - $\boldsymbol{F}$ is the distributed aerodynamic force along the blade (lift and drag) applied at the section aerodynamic center Q
  - $\boldsymbol{M}$ is the distributed aerodynamic pitching moment along the blade about Q
  - $\delta \boldsymbol{r}$ is virtual displacement at Q at an arbitrary x
  - $\overline{\delta \psi}$ is virtual rotation of the blade at an arbitrary x
- Following Ormiston and Hodges (1972), we ignore pitching moment

- $\boldsymbol{F} = L\boldsymbol{w}_1 + D\boldsymbol{w}_2$ where
  - L and D are sectional lift and drag
  - $\boldsymbol{w}_1$ is the unit vector along which lift acts
  - $\boldsymbol{w}_2$ is the unit vector along which drag acts (opposite of the relative wind vector $W\boldsymbol{w}_2$)
- Note that

$$\begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{Bmatrix}$$

- According to strip theory

$$L = \frac{\rho W^2 c c_\ell(\alpha)}{2} \qquad D = \frac{\rho W^2 c c_d(\alpha)}{2}$$

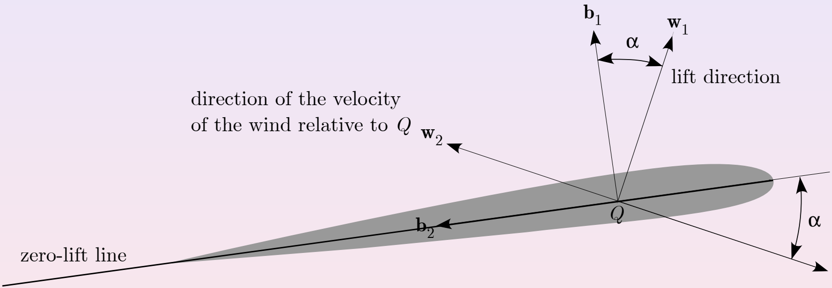


Figure: Schematic of rotor blade airfoil showing unit vectors

- To further simplify we take $c_d(\alpha) = c_{d0}$ and $c_\ell(\alpha) = a \sin\alpha$
- Thus,

$$\boldsymbol{F} = \frac{1}{2}\rho c W^2 (a \sin\alpha \boldsymbol{w}_1 + c_{d0}\boldsymbol{w}_2)$$

- We write the relative wind as

$$W\boldsymbol{w}_2 = {}^I\boldsymbol{u}^Q - {}^I\boldsymbol{v}^Q$$

- Read as the air velocity at Q minus the inertial velocity of the blade at Q
- Assuming the point Q coincident with the x-axis of the blade, we write ${}^I\boldsymbol{v}^Q = x(\omega_2\boldsymbol{b}_1 - \omega_1\boldsymbol{b}_2)$

- The airflow for the hovering flight condition is determined by the induced inflow velocity
- A reasonable approximation for this quantity is ${}^{I}\boldsymbol{u}^{Q} = -x\nu\cos\beta\,\boldsymbol{r}_1$ where $\nu(\theta)$ can be found from blade-element/momentum theory (see Gessow and Myers)
- The relative wind vector can be written

$$W\boldsymbol{w}_2 = W(\sin\alpha\boldsymbol{b}_1 + \cos\alpha\boldsymbol{b}_2)$$

so that

$$\tan\alpha = \frac{W\boldsymbol{w}_2\cdot\boldsymbol{b}_1}{W\boldsymbol{w}_2\cdot\boldsymbol{b}_2}$$

- The relative wind vector components are

$$W\boldsymbol{w}_2 \cdot \boldsymbol{b}_1 = x(\Omega + \dot{\zeta}) \cos\beta \sin\theta - x\dot{\beta}\cos\theta - x\nu\cos^2\beta\cos\theta$$
$$W\boldsymbol{w}_2 \cdot \boldsymbol{b}_2 = x(\Omega + \dot{\zeta}) \cos\beta \cos\theta + x\dot{\beta}\sin\theta + x\nu\cos^2\beta\sin\theta$$

- Clearly
  - W^2 is the sum of the squares of the right-hand sides
  - $\tan\alpha$ is the ratio of the right-hand sides
- Letting $\alpha = \theta - \phi$ with ϕ as the inflow angle, one can approximate ϕ as $\phi = \overline{\phi} + \hat{\phi}(t)$ so that
  - $\tan\overline{\phi} = \frac{\nu\cos\overline{\beta}}{\Omega}$ where $\nu = \frac{\pi\Omega\sigma}{6}\left(\sqrt{1 + \frac{12|\theta|}{\pi\sigma}} - 1\right)$ and $\sigma = \frac{bc}{\pi\ell}$
  - $\hat{\phi} = -\hat{\alpha} = \frac{\dot{\hat{\beta}}\cos^2\overline{\phi}}{\Omega\cos\overline{\beta}} - \frac{\hat{\beta}\sin\overline{\beta}\cos\overline{\phi}\sin\overline{\phi}}{\cos\overline{\beta}} - \frac{\dot{\hat{\zeta}}\cos\overline{\phi}\sin\overline{\phi}}{\Omega}$

- One can approximate $W^2 = \overline{W^2} + \widehat{W^2}$, so that
  - $\overline{W^2} = x^2\Omega^2 \frac{\cos^2\overline{\beta}}{\cos^2\overline{\phi}}$
  - $\widehat{W^2} = 2\Omega x^2 \left[\dot{\hat{\zeta}}\cos^2\overline{\beta} + \dot{\hat{\beta}}\frac{\cos\overline{\beta}\sin\overline{\phi}}{\cos\overline{\phi}} - \Omega\hat{\beta}\frac{\cos\overline{\beta}\sin\overline{\beta}(1+\sin^2\overline{\phi})}{\cos^2\overline{\phi}}\right]$
- Recalling the definitions of generalized force, we can now write

$$Q_\zeta = \frac{1}{2}\int_0^\ell \rho c W^2 (a\sin\alpha\,\boldsymbol{w}_1 + c_{d0}\boldsymbol{w}_2)\cdot\frac{\partial^I \boldsymbol{v}^Q}{\partial\dot{\zeta}}dx$$

$$Q_\beta = \frac{1}{2}\int_0^\ell \rho c W^2 (a\sin\alpha\,\boldsymbol{w}_1 + c_{d0}\boldsymbol{w}_2)\cdot\frac{\partial^I \boldsymbol{v}^Q}{\partial\dot{\beta}}dx$$

- The partials of the velocities are $\frac{\partial^I \boldsymbol{v}^Q}{\partial \dot{\zeta}} = -x \cos\beta(\boldsymbol{b}_1 \sin\theta + \boldsymbol{b}_2 \cos\theta)$ and $\frac{\partial^I \boldsymbol{v}^Q}{\partial \dot{\beta}} = -x(\boldsymbol{b}_1 \cos\theta - \boldsymbol{b}_2 \sin\theta)$
- The generalized forces can be written as

$$\begin{Bmatrix} Q_\zeta \\ Q_\beta \end{Bmatrix} = \begin{Bmatrix} \overline{Q}_\zeta \\ \overline{Q}_\beta \end{Bmatrix} + \begin{Bmatrix} \hat{Q}_\zeta \\ \hat{Q}_\beta \end{Bmatrix}$$

- Letting $d = c_{d0}/a$, we find steady-state generalized forces

$$\begin{Bmatrix} \overline{Q}_\zeta \\ \overline{Q}_\beta \end{Bmatrix} = \Omega^2 \frac{\rho a c \ell^4}{8} \begin{Bmatrix} -\frac{\cos^3 \overline{\beta}}{\cos \overline{\phi}}(\sin \overline{\alpha} \tan \overline{\phi} + d) \\ \frac{\cos^2 \overline{\beta}}{\cos^2 \overline{\phi}}(\sin \overline{\alpha} \cos \overline{\phi} - d \sin \overline{\phi}) \end{Bmatrix}$$

- The important quantity $\Gamma = \frac{\rho a c \ell^4}{I}$ is called the Lock number
- Note that typical values for the Lock number are 3 – 8
- Γ is roughly a measure of the importance of aerodynamic forces to inertial forces

- The perturbed generalized forces

$$\begin{Bmatrix} \hat{Q}_\zeta \\ \hat{Q}_\beta \end{Bmatrix}$$

  can be written as additional terms in the C and K matrices
- The unknowns $\hat{\zeta} = \check{\zeta} \exp(st)$ and $\hat{\beta} = \check{\beta} \exp(st)$
- The eigenvalues $s = \sigma \pm i\omega$ are complex conjugate pairs where σ and ω
  - reflect modal damping and modal frequencies
  - depend on configuration parameters and flight condition

- We expect aerodynamic damping to contribute
  - large flap damping because of large $\dot{\hat{\beta}}$ term in the flapping moment due to lift, dominated by

$$\frac{\Gamma}{8}\Omega\cos\overline{\beta}\cos\theta\;\dot{\hat{\beta}}$$

  - very small lead-lag damping because of a very small $\dot{\hat{\zeta}}$ term in the lead-lag moment, partially due to profile drag and equal to

$$\frac{\Gamma}{8}2\Omega\cos^3\overline{\beta}(\sin\overline{\phi}\sin\overline{\alpha}+d\cos\overline{\phi})\dot{\hat{\zeta}}$$

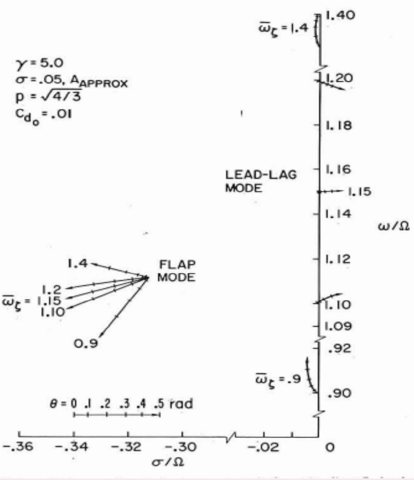


Figure: Root locus for $R = 0$

- Clearly, the flap mode is highly damped from the aerodynamics
- The lead-lag mode damping is very near neutral stability, the profile drag being the main driver for its damping
- Lead-lag mode goes unstable for values of ω_ζ just above unity
- Note: A confusing choice of notation in Ormiston and Hodges (1972) has two meanings for σ
    - as a configuration parameter, it is the solidity $= \frac{bc}{\pi\ell}$
    - as a result, it is the real part of the eigenvalue

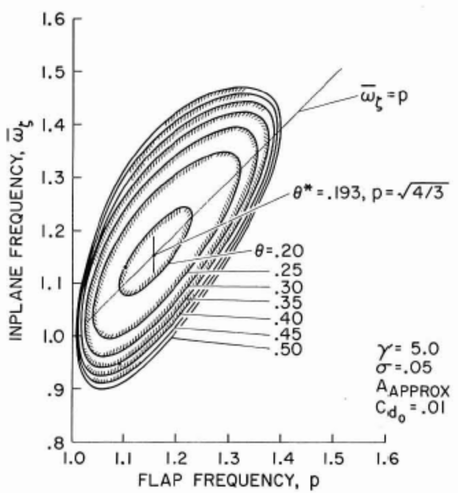


Figure: Stability boundary for $R = 0$

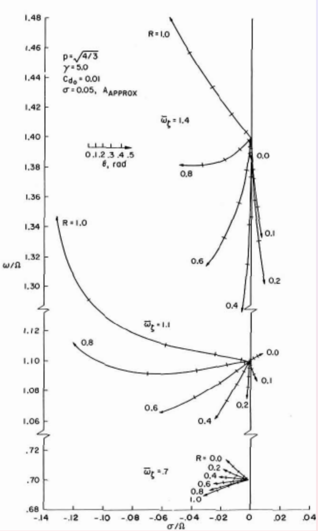


Figure: Root locus for variable R

- With varying R, the flap mode is still highly damped from the aerodynamics (not shown)
- The lead-lag mode exhibits much more damping for larger values of R
- Lead-lag mode goes unstable for values of ω_ζ just above unity and small values of R
- Ormiston and co-workers also looked at the influence of
  - pitch-lag coupling with the lag hinge tilted so the blade was forced to pitch as it underwent lead-lag motion
  - large flap-lag coupling obtained from orienting the principal axes of bending at large angles, up to 90° with the greatest effect coming around 45°

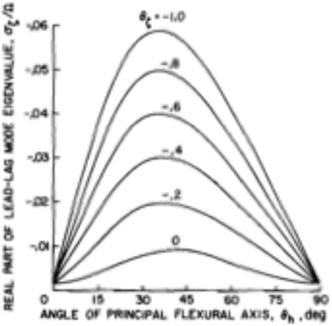


Figure 1. Effect of inclination of principal flexural axes on lead-lag mode damping at zero pitch angle (θ = 0°), p = 1.1, $\omega_{\zeta 0}/\Omega$ = 0.7, γ = 8, $2cd_p/a$ = 0.01/π, R = 1.0.

Figure: Lead-lag damping increase in presence of pitch-lag coupling and large flap-lag coupling

- Pitch-lag coupling alone leads to larger margins of stability
- Flap-lag coupling alone leads to larger margins of stability
- When combined, the damping gained is huge
- Findings were experimentally confirmed
- Unfortunately, the increase in damping does not carry over to the coupled rotor-fuselage problem
- Problem continues to worked on by Ghandi (RPI) and Venkatesan (IIS)

- Introduce coordinate systems and displacement variables sufficient to define the position of every point in both undeformed and deformed states
- Obtain the inertial velocity and acceleration of an arbitrary material point
- Obtain the virtual work done by inertial forces
- Obtain strain-displacement relations
- Obtain the strain energy and its variation
- Obtain the virtual work of applied forces
- Given geometric boundary conditions, find natural boundary conditions and partial differential equations of motion

- Consider a cantilevered beam undergoing planar deformation
  - length ℓ
  - uniform bending stiffness *EI* and mass per unit length *m*
  - root of the beam is a distance *e* from the center of rotation

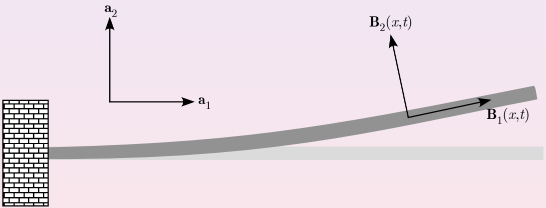


Figure: Schematic of cantilevered beam rotating about an axis fixed in inertial space and parallel to $\boldsymbol{a}_2$

- The coordinate x is along the locus of centroids (which is the undeformed beam reference line)
- The longitudinal displacement is $u(x,t)$
- The transverse displacement is $v(x,t)$
- Thus, the position vector from its root to any point along the deformed beam reference line is

$$\boldsymbol{R} = (x+u)\boldsymbol{a}_1 + v\boldsymbol{a}_2$$

- Introduce $(\ )' = \frac{\partial(\ )}{\partial x}$ and $(\dot{\ }) = \frac{\partial(\ )}{\partial t}$

- Let $\boldsymbol{B}_1$ be the normal to the cross-section, rotated by β relative to $\boldsymbol{a}_1$ so that

$$\begin{Bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{Bmatrix}$$

- The deformed beam curvature vector $\boldsymbol{K}$ such that $\boldsymbol{B}_i' = \boldsymbol{K} \times \boldsymbol{B}_i$ is clearly $\boldsymbol{K} = \beta' \boldsymbol{a}_3 = \kappa \boldsymbol{a}_3$ for planar deformation
- Later we will show that the stretch and transverse shear measures are given by

$$\begin{aligned} \gamma_x &= (1 + u') \cos\beta + v' \sin\beta \\ \gamma_y &= -(1 + u') \sin\beta + v' \cos\beta \end{aligned}$$

- The strain energy per unit length is then

$$\Psi = \frac{1}{2}\left(EA\gamma_x^2 + GK\gamma_y^2 + EI\kappa^2\right)$$

- To ignore transverse shear, set $\gamma_y = 0$ to get $\tan\beta = \frac{v'}{1+u'}$
  - Wait a minute! I thought v' ought to be $\tan\beta$ (?)
  - We'll explore further below
- The unit vector tangent to the deformed beam reference line is

$$\frac{\partial \boldsymbol{R}}{\partial s} = \frac{\boldsymbol{R}'}{s'} = \frac{(1+u')\,\boldsymbol{a}_1 + v'\boldsymbol{a}_2}{s'} = \boldsymbol{B}_1$$

where $s' = 1 + \gamma_x$ and s is the arc-length along the deformed beam

- We now have two expressions for $\boldsymbol{B}_1$

$$\begin{aligned}\boldsymbol{B}_1 &= \cos\beta \boldsymbol{a}_1 + \sin\beta \boldsymbol{a}_2 \\ &= \frac{(1+u')\,\boldsymbol{a}_1 + v'\boldsymbol{a}_2}{s'}\end{aligned}$$

- It thus follows that $\sin\beta = \frac{v'}{s'}$ and $\cos\beta = \frac{1+u'}{s'}$
- Our result is right because of the foreshortening effect
  - Because the stretching strain is very small, points on the beam axis during bending move axially (toward the root)
  - The arc-length along the deformed beam $ds \approx dx$!
  - Thus, $\tan\beta = \frac{dv}{dx+du}$

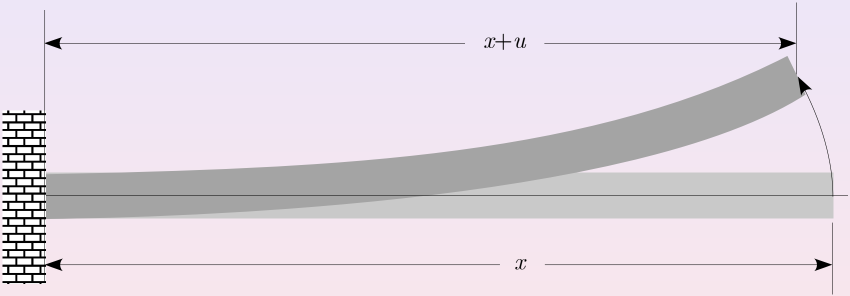


Figure: The foreshortening effect: $ds \approx dx$

- Because $\sin^2\beta + \cos^2\beta = 1$, one finds $s'^2 = (1+u')^2 + v'^2$
- Because $\gamma_x = s' - 1$ we now get

$$\gamma_x = \sqrt{(1+u')^2 + v'^2} - 1$$

- Differentiating the expression for $\sin\beta$ one finds $\beta' \cos\beta = \left(\frac{v'}{s'}\right)'$
- Using expressions for $\cos\beta$, s' and s'' one finally gets

$$\kappa = \beta' = \frac{v''(1+u') - v'u''}{s'^2}$$

- For small strain, we may ignore the denominator of κ yielding
$$\kappa = v''(1 + u') - v'u''$$
- Note that β is now gone from the analysis
- To further simplify the analysis, consider the beam inextensible
- Solve the equation for γ_x for u' to obtain

$$u' = \sqrt{1 - v'^2} - 1 + \frac{\gamma_x}{\sqrt{1 - v'^2}} + O(\gamma_x^2)$$

- For $\gamma_x \approx 0$ we may now eliminate u' from the analysis
- This leads to only one generalized strain, κ, in terms of only one displacement variable v
- Subsituting for u' and u'', one finds

$$\kappa = \frac{v''}{\sqrt{1 - v'^2}} \left[1 + O\left(\gamma_x, \gamma_x'\right)\right]$$

- Thus, for small strain we have simply

$$\kappa = \frac{v''}{\sqrt{1 - v'^2}}$$

- Wait a minute! Curvature is *not*

$$\kappa = \frac{v''}{\sqrt{1 - v'^2}}$$

but instead is

$$\kappa^* = \frac{v''}{\left(1 + v'^2\right)^{\frac{3}{2}}}$$

Right?

- No, not for the case with foreshortening
- If one introduces $\xi = x + u$ and lets $(\ )'$ be the partial with respect to ξ, then one gets κ^* as the curvature
- This is completely wrong for use in a beam theory!
- If x is the length along the undeformed beam, the appropriate curvature is

$$\kappa = \frac{v''}{\sqrt{1 - v'^2}}$$

- For free-vibration problems, we may make one final simplification restricting rotations to be “moderate” so that $v'^2 << 1$ and $u' \approx -\frac{1}{2}v'^2$ or

$$u(x,t) = -\frac{1}{2}\int_0^x \left[\frac{\partial v}{\partial \xi}(\xi,t)\right]^2 d\xi$$

- Now the position vector to an arbitrary point on the beam reference line may be written as

$$\boldsymbol{R} = \left\{x - \frac{1}{2}\int_0^x \left[\frac{\partial v}{\partial \xi}(\xi,t)\right]^2 d\xi\right\}\boldsymbol{a}_1 + v\boldsymbol{a}_2$$

- Finally, for moderate rotations $\kappa \approx v''$

- The strain energy of a uniform, isotropic beam, ignoring transverse shear deformation and stretching of the reference line, is

$$U = \frac{1}{2}\int_0^\ell EI\, v''^2 dx$$

- Thus, one can write

$$\delta U = \int_0^\ell EI\, v''\, \delta v'' dx$$

where δv is the virtual displacement along the beam

- The velocity of an arbitrary point along the beam reference line can be written as

$$\boldsymbol{v} = \dot{u}\boldsymbol{a}_1 + \dot{v}\boldsymbol{a}_2 + \Omega(x+u)\boldsymbol{a}_3$$

- The virtual displacement of an arbitrary point along the beam reference line can be written as

$$\delta\boldsymbol{R} = \delta u\boldsymbol{a}_1 + \delta v\boldsymbol{a}_2$$

- The acceleration of an arbitrary point along the beam reference line can be written as

$$\boldsymbol{a} = \left[\ddot{u} - \Omega^2(x+u)\right]\boldsymbol{a}_1 + \ddot{v}\boldsymbol{a}_2 - 2\Omega\dot{u}\boldsymbol{a}_3$$

- Ignoring the cross-sectional rotary inertia, one may write the virtual work of inertial forces as

$$\overline{\delta W}_{\text{inertial}} = -\int_0^\ell m\boldsymbol{a} \cdot \delta \boldsymbol{R}\, dx$$

- Ignoring higher-order terms in v', this simplifies to

$$\overline{\delta W}_{\text{inertial}} = -\int_0^\ell \left(m\ddot{v}\,\delta v + Tv'\,\delta v' \right) dx$$

where $T = \Omega^2 \int_x^\ell m(\xi)\xi\, d\xi$ is the axial force in the beam generated by spin

- It must be noted that the term involving T is a fundamentally nonlinear effect
- It comes from a leading term in $-\int_0^\ell m\boldsymbol{a}\cdot\delta\boldsymbol{R}dx$

$$
\begin{aligned}
-\int_0^\ell m\boldsymbol{a}\cdot\delta\boldsymbol{R}dx &= -\int_0^\ell m\Omega^2 x\delta u\, dx + \dots \\
&= \int_0^\ell m\Omega^2 x \int_0^x \frac{\partial v}{\partial \xi}(\xi,t)\frac{\partial \delta v}{\partial \xi}(\xi,t)\, d\xi\, dx + \dots \\
&= \int_0^\ell T v' \delta v' dx + \dots
\end{aligned}
$$

- Applying Hamilton's principle, one obtains

$$\int_{t_1}^{t_2}\int_0^\ell \left(EIv''\delta v'' + m\ddot{v}\,\delta v + Tv'\delta v'\right) dx\, dt = 0$$

or

$$\int_0^\ell \left(EIv''\delta v'' + m\ddot{v}\,\delta v + Tv'\delta v'\right) dx = 0$$

- Integrating by parts in x, we obtain

$$\int_0^\ell \left[(EIv'')'' + m\ddot{v} - (Tv')'\right]\delta v\, dx + \left\{EIv''\delta v' + \left[Tv' - (EIv'')'\right]\delta v\right\}\Big|_0^\ell = 0$$

- Virtual displacement and rotation are arbitrary everywhere except at the beam root where $\delta v(0,t) = \delta v'(0,t) = 0$
- For the expression to vanish, the integrand must vanish
- Euler-Lagrange partial differential equation of motion is

$$\left(EIv''\right)'' + m\ddot{v} - \left(Tv'\right)' = 0$$

with boundary conditions

$$v(0,t) = v'(0,t) = EI(\ell)v''(\ell,t) = T(\ell)v'(\ell,t) - \left(EIv''\right)'(\ell,t) = 0$$

- For constant EI and $T(\ell) = 0$, one may simplify these to

$$v(0,t) = v'(0,t) = v''(\ell,t) = v'''(\ell,t) = 0$$

- Modal methods (e.g. Ritz or Galerkin methods)
  - Commonly presented in textbooks
  - Assume $v = \sum_{i=1}^{n} q_i(t)\psi_i(x)$ where
    - n is the number of assumed modes
    - q_i are generalized coordinates
    - ψ_i are admissible functions (i.e. satisfying geometric boundary conditions)
    - For free-vibration analysis, one may set $q_i(t) = \hat{q}_i \exp(i\omega t)$
- Finite element methods
  - special case of assumed modes methods
  - generalized coordinates are displacements or rotations at node points

- Transfer matrix methods
  - are powerful and efficient
  - provide excellent accuracy for modal bending moment and shear force distributions
  - are based on a first-order mixed formulation including displacement, rotation, and stress resultants
  - often go by other names, such as associated matrix method or transition matrix method
  - are related to Myklestad method, obtained by
    - lumping the mass at “nodes”
    - assuming stiffness constant along axial segments between nodes

- For example, consider free vibration of a rotating beam
- Governing equation is the ordinary differential equation

$$(EIv'')'' - (Tv')' - \omega^2 mv = 0$$

where ω is the frequency of free vibration

- We now write the equation in first order form with deflection, slope, moment, and shear as state variables:

$$\begin{aligned} v' &= \beta \\ \beta' &= M/EI \\ M' &= -V + T\beta \\ V' &= -\omega^2 mv \end{aligned}$$

- This can be put into matrix form as

$$\begin{Bmatrix} v \\ \beta \\ M \\ V \end{Bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{EI} & 0 \\ 0 & T & 0 & -1 \\ -m\omega^2 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v \\ \beta \\ M \\ V \end{Bmatrix}$$

or $z' = Az$

- By defining the transfer matrix τ such that $z(x) = \tau(x)z(\ell)$, one can verify that τ satisfies the equation

$$\tau' = A\tau \quad \text{with starting value } \tau(\ell) = \Delta$$

where Δ is a 4×4 identity matrix

- To solve for the natural frequencies
  - Integrate this equation numerically (starting at $x = \ell$)
  - Iterate to find values of frequency that satisfy the boundary conditions
- For example, consider a cantilever beam where $v(0) = \beta(0) = M(\ell) = V(\ell) = 0$
- Thus,

$$\begin{Bmatrix} v(0) \\ \beta(0) \end{Bmatrix} = \begin{bmatrix} \tau_{11}(0) & \tau_{12}(0) \\ \tau_{21}(0) & \tau_{22}(0) \end{bmatrix} \begin{Bmatrix} v(\ell) \\ \beta(\ell) \end{Bmatrix}$$

- Standard integration schemes can be used to calculate $\tau(0)$ for any value of ω^2

- For the left hand side of this set of homogeneous equations to vanish, the determinant must vanish yielding

$$D = \tau_{11}(0)\tau_{22}(0) - \tau_{12}(0)\tau_{21}(0) = 0$$

- This determinant D may be regarded as a function of the unknown eigenvalues, i.e. $D(\omega^2)$
- A variety of single-equation root solvers will find as many values of ω^2 as desired

- Once ω^2 is known, one may obtain the mode shapes for v, β, M, and V
  - Substitute ω^2 into $z' = Az$
  - The starting values $M(\ell)$ and $V(\ell)$ are zero
  - Specify $v(\ell) = 1$
  - Since the determinant is zero, $\beta(\ell)$ is determined to be

$$\beta(\ell) = -\frac{\tau_{21}(0)v(\ell)}{\tau_{22}(0)} = -\frac{\tau_{11}(0)v(\ell)}{\tau_{12}(0)}$$

- Accuracy of results depends on accuracy of numerical integration scheme and tolerance in root finding algorithm

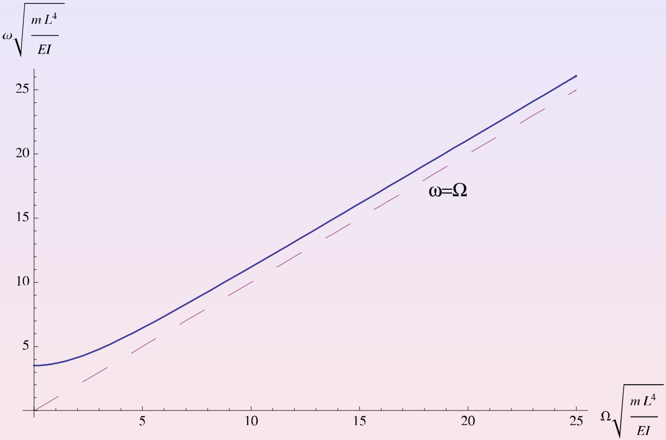


Figure: Fundamental natural flapping frequency of a rotating clamped-free beam with zero hub offset versus angular speed

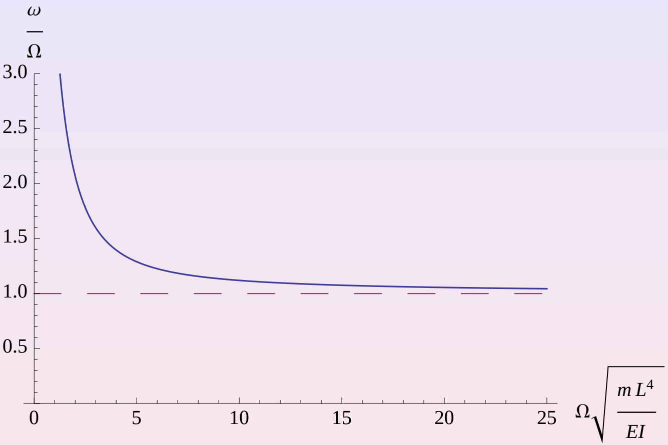


Figure: Fundamental natural flapping frequency (per rev) of a rotating clamped-free beam with zero hub offset versus angular speed

- Flapping frequency en vacuo remains above the rotor angular speed
- Flapping frequency monotonically increases with rotor angular speed
- Flapping frequency normalized with Ω monotonically and asymptotically decreases with rotor angular speed to unity (i.e. once per revolution)

- Governing equation for lead-lag motion is very similar

$$(EIv'')'' - (Tv')' - (\Omega^2 + \omega^2)mv = 0$$

  where ω is the frequency of free vibration and EI is now the lead-lag stiffness
- The eigenvalue is $\Omega^2 + \omega^2$, the value of which for a given EI, m and Ω is unchanged
- Thus, ω^2 is the eigenvalue minus Ω^2 making it possible for the lead-lag frequency to go lower than Ω

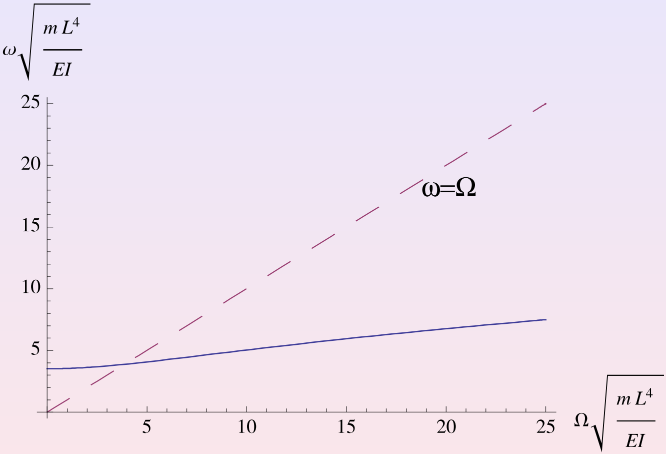


Figure: Fundamental natural lead-lag frequency of a rotating clamped-free beam with zero hub offset versus angular speed

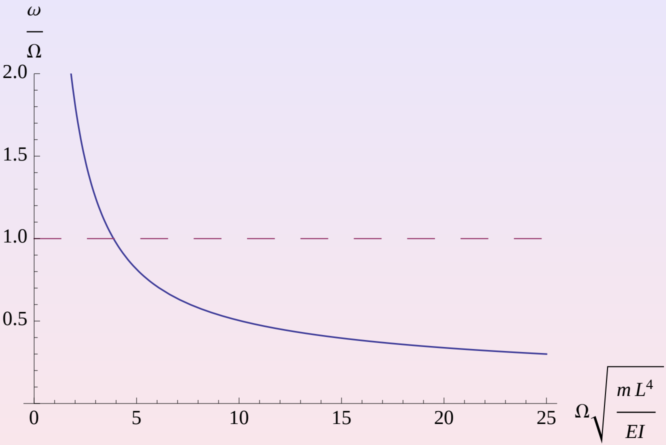


Figure: Fundamental natural flapping frequency (per rev) of a rotating clamped-free beam with zero hub offset versus angular speed

- Lead-lag frequency increases with rotor angular speed but much more slowly than does flapping frequency
- The lead-lag frequency does indeed go lower than Ω
- The lead-lag frequency normalized with Ω crosses the value of unity at which point the rotor transitions from stiff-inplane to soft-inplane

- A governing equation for torsion may be derived similarly:

$$\rho I_p \ddot{\phi} - (GJ\phi')' - (Tk_a^2\phi')' + \rho\Omega^2(I_c - I_f)\phi\cos 2\theta = 0$$

  where term number
  1. is the torsional inertia ($I_p = I_c + I_f$ with I_c and I_f being lag and flap area moments of inertia)
  2. is the Saint-Venant torsion stiffness term
  3. is the “trapeze effect” (k_a is the area radius of gyration)
  4. is the propeller moment (or tennis-racquet effect)
- Both trapeze and tennis-racquet effects increase torsional frequency as a function of rotor angular speed
- Torsional frequencies are usually much larger than flap or lead-lag frequencies

- Now we consider the mode shapes for flapping of a non-rotating beam and of a rotating beam
- While doing so we also will compare the modal bending moments

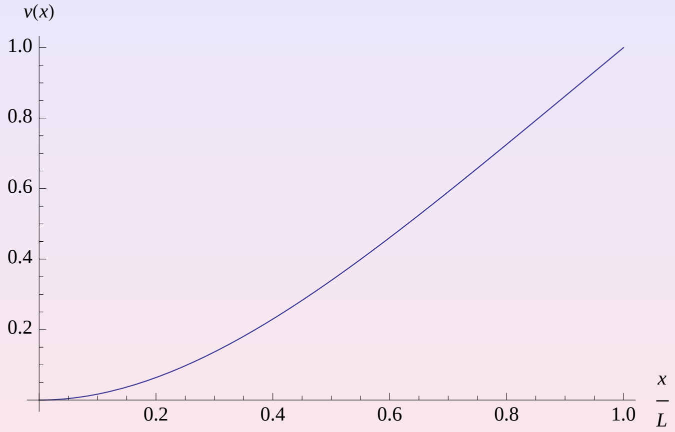


Figure: Fundamental flapping displacement mode shape of a nonrotating clamped-free beam with zero hub offset

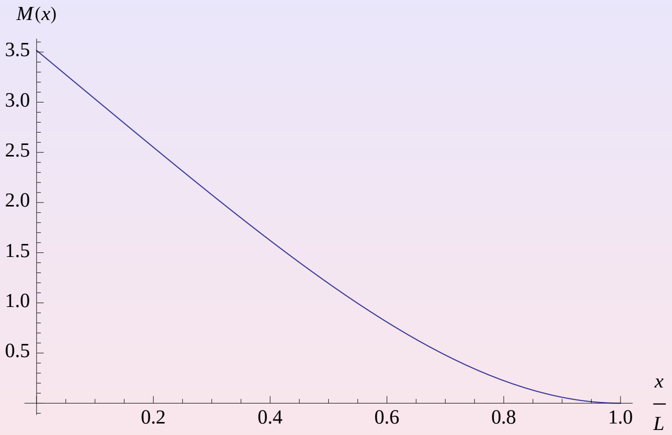


Figure: Fundamental flapping moment mode shape of a nonrotating clamped-free beam with zero hub offset

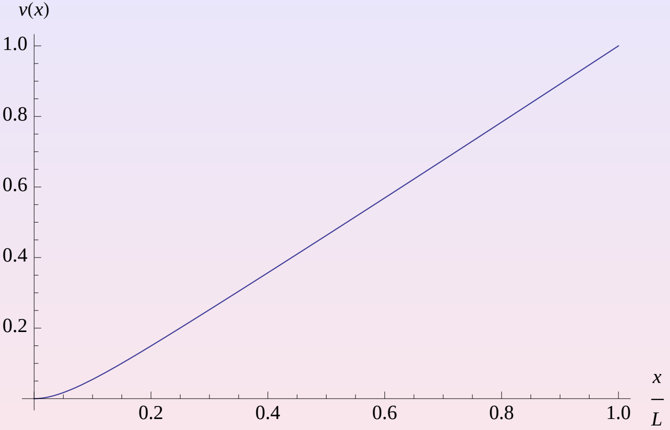


Figure: Fundamental flapping displacement mode shape of a rotating clamped-free beam with zero hub offset for $\Omega = 25\sqrt{\frac{EI}{mL^4}}$

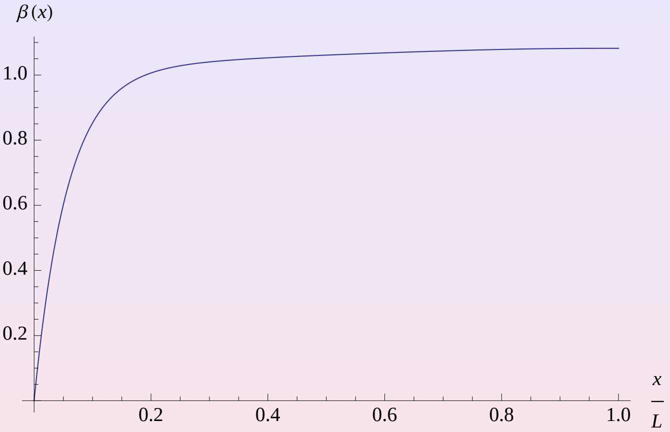


Figure: Fundamental flapping slope mode shape of a rotating clamped-free beam with zero hub offset for $\Omega = 25\sqrt{\frac{EI}{mL^4}}$

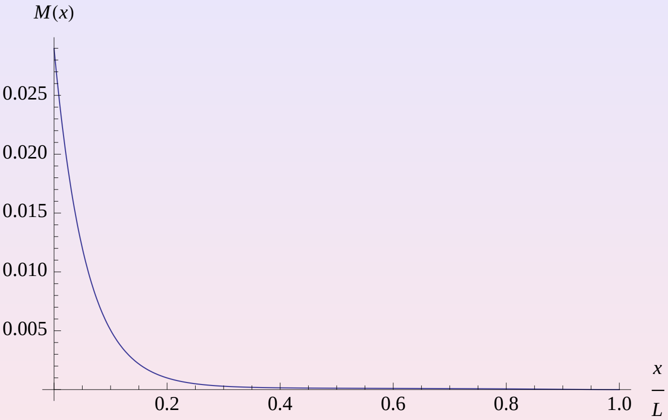


Figure: Fundamental flapping moment mode shape of a rotating clamped-free beam with zero hub offset for $\Omega = 25\sqrt{\frac{EI}{mL^4}}$

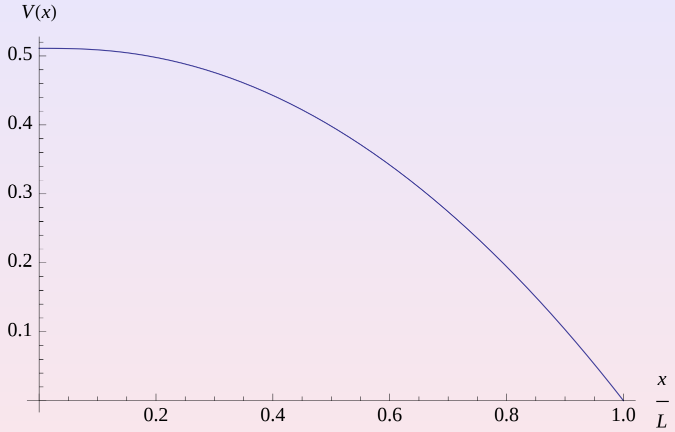


Figure: Fundamental flapping shear mode shape of a rotating clamped-free beam with zero hub offset for $\Omega = 25\sqrt{\frac{EI}{mL^4}}$

- Further observations:
    - Flapping mode shape has most of its curvature near the root
    - The higher the rotor angular speed, the more pronounced the concentration of curvature at the root
    - This effect makes it harder to capture the mode shape at higher and higher rotor angular speeds

- Rotor aeroelasticity involves nonlinearities in flap, lag, and torsion
    - This makes the subject inherently nonlinear
    - Nonlinearities in the torsion equation include a product of bending moments (or curvatures) named after Mil, a Russian rotor dynamicist
    - For rotor blade aeroelasticity, one must first find a trim solution
        - In the hovering flight condition, a static equilibrium state such that thrust or collective pitch is at a desired level
        - In forward flight, a periodic steady-state solution that satisfies time-averaged force and moment equilibrium
    - For aeroelastic stability one linearizes about the trim solution
    - The nonlinear terms produce important coupling in the linearized equations