Elasticity Solutions Versus Asymptotic Sectional Analysis of Homogeneous, Isotropic, Prismatic Beams

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The original three-dimensional elasticity problem of isotropic prismatic beams has been solved analytically by the variational asymptotic method (VAM). The resulting classical model (Euler-Bernoulli-like) is the same as the superposition of elasticity solutions of extension, Saint-Venant torsion, and pure bending in two orthogonal directions. The resulting refined model (Timoshenko-like) is the same as the superposition of elasticity solutions of extension, Saint-Venant torsion, and both bending and transverse shear in two orthogonal directions. The fact that the VAM can reproduce results from the theory of elasticity proves that two-dimensional finite-element-based cross-sectional analyses using the VAM, such as the variational asymptotic beam sectional analysis (VABS), have a solid mathematical foundation. One is thus able to reproduce numerically with VABS the same results for this problem as one obtains from three-dimensional elasticity, but with orders of magnitude less computational cost relative to three-dimensional finite elements.
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Introduction

The variational asymptotic method (VAM) is a mathematical approach applicable to any problem governed by an energy functional having one or more small parameters. Contrary to the formal asymptotic methods, VAM applies the asymptotic expansion to the energy functional instead of the system of differential equations, [1]. Hence, dropping a small term in the functional is equivalent to neglecting such quantities in several differential equations simultaneously. This implies that, when applicable, VAM is more compact and less cumbersome than standard asymptotic methods. The VAM includes the merits of both variational (systematic) and asymptotic (without ad hoc kinematic assumptions) methods. It allows one to replace a three-dimensional structural model with a reduced-order model in terms of an asymptotic series of certain small parameters inherent to the structure. Although there are different forms of this method, e.g., Ciarlet and Destuynder [2] and Berdichevsky [3], the method used in the present work is more closely aligned with the latter.

The application of the VAM to model beams with general geometry and material has been demonstrated in the theory associated with the computer program VABS (variational asymptotic beam sectional analysis). VABS was first mentioned in [4]. Its development over the past ten years is described in [5–10] and takes the variational asymptotic method (VAM), [3], as the mathematical basis. By means of the VAM, a general three-dimensional nonlinear elasticity problem for a beam-like structure is rigorously split into a two-dimensional linear cross-sectional analysis and a one-dimensional nonlinear beam analysis. It is interesting to know that Trabucho and Viano [11] applied the VAM of Ciarlet and Destuynder [2] to construct mathematical models for rods. Their work is oriented more toward mathematicians than engineers.

In accord with the theory behind it, VABS can perform a classical analysis for initially twisted and curved inhomogeneous, anisotropic beams with arbitrary geometry, material properties, and reference cross sections. It captures both trapeze and Vlasov effects, which are useful for specific beam applications. VABS is also able to calculate the one-dimensional stiffness matrix with transverse shear refinement for initially twisted and curved, inhomogeneous, anisotropic beams with arbitrary geometry and material properties. Finally, the three-dimensional stress and strain fields can be recovered, if required, for finding stress concentrations, interlaminar stresses, etc.

There are a lot of beam theories in the literature. However, almost all published work is of the ad hoc variety, especially in the area of modeling composite structures. Because VABS develops stiffness models that use the same fundamental types of deformation that appear in traditional beam theories (such as those of Euler-Bernoulli, Timoshenko, and Vlasov), some researchers may be tempted to believe that VABS is nothing more than a computerized adaptation of elementary theories. However, VABS is really very different from the traditional beam theories, and the assumptions behind it are far less restrictive. The fact that VABS uses the traditional types of deformation winds up creating a simple and smooth connection to traditional beam theories, so that the one-dimensional beam analyses will remain essentially the same. A large body of additional information regarding three-dimensional behavior of the beam, which need not be considered at all in a one-dimensional beam analysis, is actually taken into account by introducing three-dimensional warping functions that are subsequently calculated.

In view of this, the main purpose of the present work is to take the reader, who is presumed to have a basic understanding of elasticity and calculus of variations, through an analytical derivation and application of the equations used by VABS for a specialized case so that its relationship with traditional theories will be clearer and its mathematical basis (VAM) will appear less arcane. This paper is in essence an *analytical* validation of VABS against the well-established theory of elasticity. Although numerous nu-

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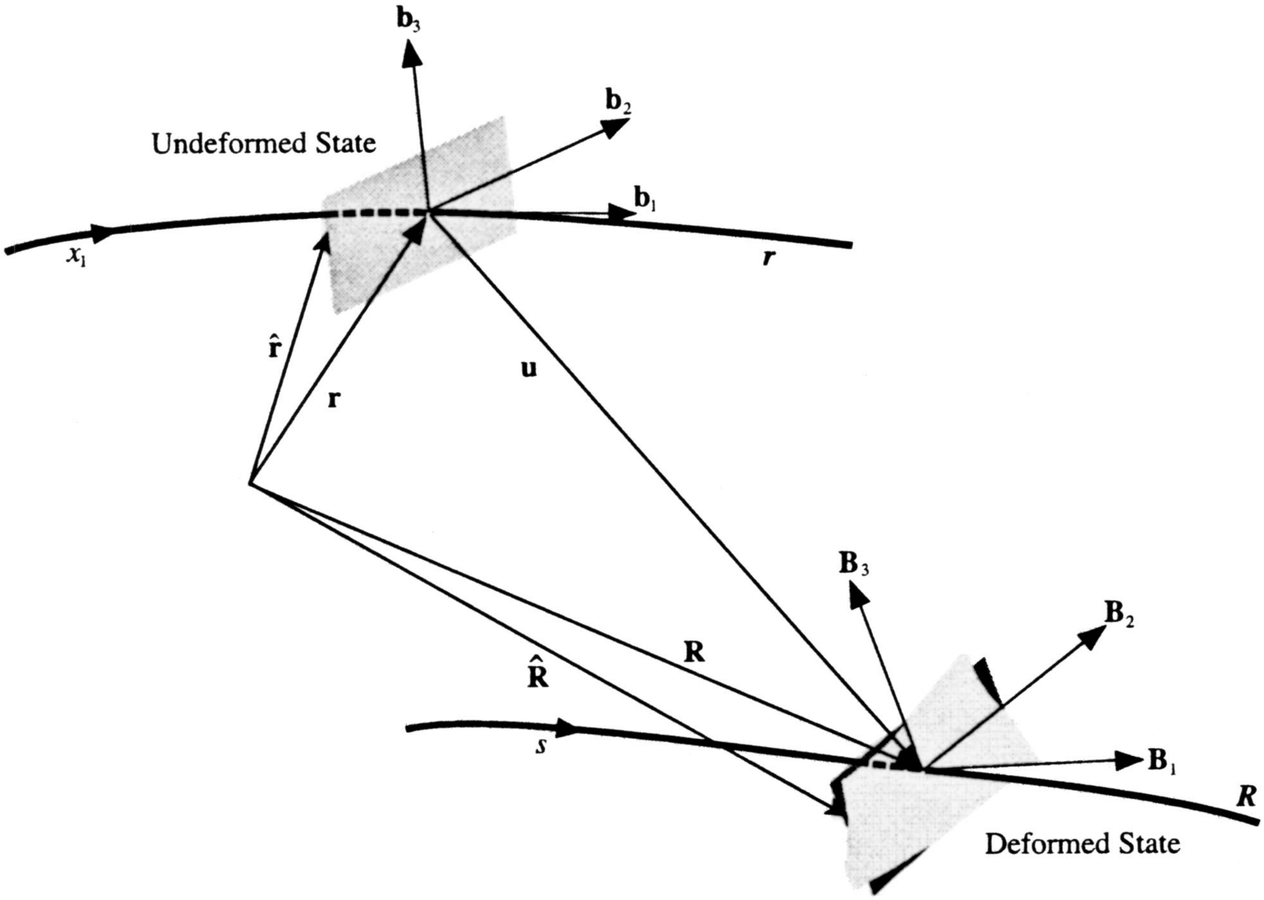

Fig. 1 Schematic of beam deformation

merical validation examples have been provided in [6,7,10,12], the present extensive and rigorous validation is required to demonstrate conclusively its versatility and accuracy. This paper then should increase the reader's confidence in results obtained from VABS.

To accomplish the above, the present work specializes the VABS general formulation for the analysis of isotropic, prismatic beams. Starting with the governing differential equations and associated boundary conditions of elasticity theory, we set out to prove (a) that the results from the classical model of VABS are the same as the superposition of elasticity solutions of extension, Saint-Venant torsion, and pure bending in two orthogonal directions; and (b) that the results from the Timoshenko-like model of VABS are the same as the superposition of the elasticity solutions of extension, Saint-Venant torsion, and both bending and transverse shear in two orthogonal directions.

Three-Dimensional Formulation

As sketched in Fig. 1, a beam can be represented by a reference line r measured by x_1, and a typical cross section s with h as its characteristic dimension and described by cross-sectional Cartesian coordinates x_α. Note that here and throughout the paper, Greek indices assume values 2 and 3 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated. For the convenience of comparing with elasticity solutions, the locus of all cross-sectional centroids along the beam is chosen as the reference line. An orthonormal triad $\mathbf{b}_i$ is chosen for the purpose of resolving tensorial quantities in component form for actual computation. For convenience, $\mathbf{b}_i$ is chosen to be tangent to x_i, respectively.

The spatial position vector $\hat{\mathbf{r}}$ of any point in the undeformed beam structure can be written as

$$\hat{\mathbf{r}}(x_1,x_2,x_3)=\mathbf{r}(x_1)+x_\alpha \mathbf{b}_\alpha \tag{1}$$

where $\mathbf{r}$ is the position vector of the points of the reference line. Note for a prismatic beam, the beam axis in the undeformed state is straight. Finally, $\mathbf{r}'=\mathbf{b}_1$ and $(\ )'$ means the partial derivative with respect to x_1.

After deformation, the particle that had position vector $\hat{\mathbf{r}}$ in the undeformed state now has the position vector $\hat{\mathbf{R}}$ in the deformed state. Another orthonormal triad $\mathbf{B}_i$ is introduced to express the deformed configuration, and the $\mathbf{B}_i$ unit vectors are not necessarily tangent to the deformed beam coordinates. However, for the convenience of applying VAM, we choose $\mathbf{B}_i$ to coincide with $\mathbf{b}_i$ in the case of zero deformation, $\mathbf{B}_1$ to be tangent to the deformed beam reference axis, and $\mathbf{B}_\alpha$ determined by a rotation about $\mathbf{B}_1$. Then $\mathbf{B}_i$ can be related to $\mathbf{b}_i$ by a rotation tensor which is called the global rotation tensor, [13], such that

$$\mathbf{C}^{Bb}=\mathbf{B}_i\mathbf{b}_i\,. \tag{2}$$

$\mathbf{C}^{bB}$ is the inverse rotation to bring $\mathbf{B}_i$ back to $\mathbf{b}_i$ which means

$$\mathbf{C}^{Bb}\cdot\mathbf{C}^{bB}=\mathbf{I} \tag{3}$$

where $\mathbf{I}$ is the identity tensor. Please note that we do *not* make any restrictive assumption here by choosing $\mathbf{B}_1$ to be tangent to x_1. Instead, the transverse shear deformation will be included in the warping functions introduced below and will be explicitly brought into evidence when we fit the asymptotic model into an engineering model that can account for this type of deformation, such as a Timoshenko-like model.

The position vector $\hat{\mathbf{R}}$ can be represented as

$$\hat{\mathbf{R}}(x_1,x_2,x_3)=\mathbf{R}(x_1)+x_\alpha \mathbf{B}_\alpha(x_1)+w_i(x_1,x_2,x_3)\mathbf{B}_i(x_1) \quad (4)$$

where $\mathbf{R}$ is the position vector to a point on the reference line of the deformed beam, and w_i are the components of warping, both in and out of the cross-sectional plane. By introducing the unknown three-dimensional warping functions into the formulation, one takes into account all possible deformation.

One should note that Eq. (4) is four times redundant because of the way warping was introduced. One must impose four appropriate constraints on the displacement field to remove the redundancy. The four constraints applied here are

$$\langle w_i \rangle = 0 \quad (5)$$

$$\langle x_2 w_3 - x_3 w_2 \rangle = 0 \quad (6)$$

where the notation $\langle\ \rangle$ means integration over the reference cross section. The implication of Eq. (5) is that warping does not contribute to the rigid-body displacement of the cross section. This leads to one-dimensional displacement variables for extension and bending that have easily identifiable geometric meanings: they correspond to the measure numbers in the $\mathbf{b}_i$ basis of the average displacement of the cross section. Equation (6) implies the torsional rotation variable is the average rotation of the cross section about $\mathbf{B}_1$.

To formulate this problem in an intrinsic form, we need the definition of the one-dimensional generalized Lagrangean strains:

$$\boldsymbol{\gamma} = \mathbf{C}^{bB}\cdot\mathbf{R}' - \mathbf{b}_1 \quad (7)$$

$$\mathbf{B}_i' = \kappa_j \mathbf{B}_j \times \mathbf{B}_i \quad (8)$$

where the column matrices of the "force-strain" measures $\gamma = \lfloor \gamma_{11}\ 0\ 0 \rfloor^T$ and κ_i are the "moment-strain" measures. Based on the concept of decomposition of rotation tensor, [13], if the local rotation is small, which is the case for all the framework of VABS except the trapeze solution (not considered in this paper), the Jaumann-Biot-Cauchy strain components are given by

$$\Gamma_{ij} = \frac{1}{2}(F_{ij}+F_{ji}) - \delta_{ij} \quad (9)$$

where δ_{ij} is the Kronecker symbol, and F_{ij} the mixed-basis component of the deformation gradient tensor such that

$$F_{ij} = \mathbf{B}_i \cdot \mathbf{G}_k \mathbf{g}^k \cdot \mathbf{b}_j . \quad (10)$$

Here $\mathbf{G}_k = \partial \hat{\mathbf{R}}/\partial x_k$ is the covariant basis vector of the deformed configuration and $\mathbf{g}^k = \mathbf{b}_k$ for prismatic beams.

Because of the small strain assumption, which is applicable in the framework of a geometrically nonlinear formulation, we may neglect all terms that are products of the warping and the one-dimensional generalized strains. Thus, one obtains the three-dimensional strain field as

$$\begin{aligned}
\Gamma_{11} &= \gamma_{11} + x_3\kappa_2 - x_2\kappa_3 + w_1' \\
2\Gamma_{12} &= w_{1,2} - x_3\kappa_1 + w_2' \\
2\Gamma_{13} &= w_{1,3} + x_2\kappa_1 + w_3' \\
\Gamma_{22} &= w_{2,2} \\
2\Gamma_{23} &= w_{3,2} + w_{2,3} \\
\Gamma_{33} &= w_{3,3}
\end{aligned} \quad (11)$$

where $(\ )_{,\alpha}$ means the partial derivative with respect to x_α. For an isotropic elastic body with Young's modulus E, shear modulus G and Poisson's ratio ν, twice the three-dimensional strain energy per unit length can be written as, [14],

$$\begin{aligned}
2\Pi &= E\langle \Gamma_{11}^2 \rangle + 4G\langle \Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{23}^2 \rangle \\
&\quad + \frac{E}{(1+\nu)(1-2\nu)} \left\langle \begin{Bmatrix} \nu\Gamma_{11}+\Gamma_{22} \\ \nu\Gamma_{11}+\Gamma_{33} \end{Bmatrix}^T \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \right. \\
&\quad \left. \times \begin{Bmatrix} \nu\Gamma_{11}+\Gamma_{22} \\ \nu\Gamma_{11}+\Gamma_{33} \end{Bmatrix} \right\rangle .
\end{aligned} \quad (12)$$

From energy principles, we know that the exact warping functions satisfying the constraints, Eqs. (5) and (6), should minimize the strain energy in Eq. (12). However, the same difficulties as one finds in solving general three-dimensional elasticity problems will be encountered if one tries to solve this minimization problem directly. Fortunately, as demonstrated in publications related to VABS, the VAM can be used to solve for the unknown warping functions asymptotically to avoid the difficulty of the original three-dimensional formulation. This will be illustrated *analytically* in the following sections.

Classical Model

Before applying the VAM, one must define the small parameters of the problem. It was mentioned above that products of the one-dimensional generalized strains and warping are assumed to be small because of the small-strain assumption. The assumption of small strain is adopted for the purpose of deriving a geometrically nonlinear beam formulation. It will be assumed and subsequently validated from the results that the warping is of the order of $h\varepsilon$ with h as the characteristic dimension of the cross section. The smallness of the one-dimensional generalized strains is taken into account as follows. The stretching of the beam reference line is denoted by γ_{11}; the maximum strain induced by twist is of the order of $h\kappa_1$, while the maximum strain induced by bending is of the order of $h\kappa_\alpha$. This observation is consistent with the small local rotation assumption used to derive Eq. (9). Now, let us denote the order of the maximum strain as $\varepsilon = \max(\gamma_{11}, h\kappa_i)$. This small parameter is then utilized when deriving the three-dimensional strain field, Eq. (11), so that the smallness of ε need not be used in the rest of derivation. Another small parameter is h/l where l is the wavelength of beam axial deformation. This is the only small parameter one needs for prismatic beams for the purpose of solving the unknown warping functions asymptotically and obtaining a strain energy asymptotically correct up to a certain order.

The classical model of a prismatic beam is represented in terms of a strain energy per unit length that is asymptotically correct up to the order of $\mu\varepsilon^2$ where μ is of the order of the maximum material constant. All the prime terms in Eq. (11) are of order h/l higher than the rest and do not contribute to such an energy. Then this energy, which is called the zeroth-order energy, can be obtained from Eq. (12) as

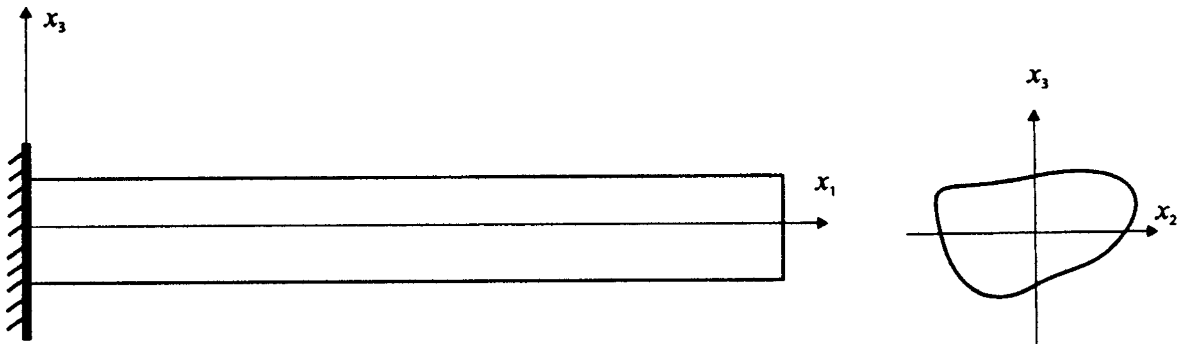

Fig. 2 Sketch of a clamped prism

$$2\Pi_0=ES\gamma_{11}^2+EI_\alpha\kappa_\alpha^2+G\langle(w_{1,2}-x_3\kappa_1)^2+(w_{1,3}+x_2\kappa_1)^2+(w_{3,2}+w_{2,3})^2\rangle$$
$$+\frac{E}{(1+\nu)(1-2\nu)}\left\langle\begin{Bmatrix}\nu(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)+w_{2,2}\\ \nu(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)+w_{3,3}\end{Bmatrix}^T\begin{bmatrix}1-\nu & \nu\\ \nu & 1-\nu\end{bmatrix}\begin{Bmatrix}\nu(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)+w_{2,2}\\ \nu(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)+w_{3,3}\end{Bmatrix}\right\rangle \quad (13)$$

with

$$S=\langle 1\rangle \quad I_2=\langle x_3^2\rangle \quad I_3=\langle x_2^2\rangle \quad (14)$$

where S is the cross-sectional area and I_α are the principal area moments of inertia about x_α. The warping functions that minimize the above energy are governed by the Euler-Lagrange equations of this energy functional, given by

$$w_{1,22}+w_{1,33}=0 \quad (15)$$

$$2(1-\nu)w_{2,22}+(1-2\nu)w_{2,33}+w_{3,23}-2\nu\kappa_3=0 \quad (16)$$

$$2(1-\nu)w_{3,33}+(1-2\nu)w_{3,22}+w_{2,23}+2\nu\kappa_2=0 \quad (17)$$

and the associated boundary conditions

$$n_3(x_2\kappa_1+w_{1,3})+n_2(w_{1,2}-x_3\kappa_1)=0 \quad (18)$$

$$n_3(w_{2,3}+w_{3,2})+\frac{2n_2}{1-2\nu}[\nu(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)+\nu w_{3,3}+(1-\nu)w_{2,2}]=0 \quad (19)$$

$$n_2(w_{2,3}+w_{3,2})+\frac{2n_3}{1-2\nu}[\nu(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)+\nu w_{2,2}+(1-\nu)w_{3,3}]=0 \quad (20)$$

where n_α is the direction cosine of outward normal with respect to x_α. Here, to maintain a simpler derivation, we do not use Lagrange multipliers to enforce the constraints of Eqs. (5) and (6). Instead, we keep these constraints in mind and check whether they can be satisfied by the solution. It can be observed that Eqs. (15) and (18) are just the equations of Saint-Venant warping $\psi(x_2,x_3)$ in elasticity textbooks such as, [15], except

$$w_1(x_1,x_2,x_3)=\hat{w}_1(x_1,x_2,x_3)=\psi(x_2,x_3)\kappa_1(x_1). \quad (21)$$

Hence the first approximation of the out-of-plane warping $\hat{w}_1$ can be solved by the methods given in elasticity books. According to the theory of elasticity, ψ can be determined up to a constant, and one can choose the constant so that the constraint $\langle\hat{w}_1\rangle=0$ is satisfied. The following functions of w_α satisfy the other constraints as well as Eqs. (16), (17), (19), and (20):

$$w_2=\hat{w}_2=-\nu(x_2\gamma_{11}+x_2x_3\kappa_2)+\frac{\nu\kappa_3}{2}\left(x_2^2-x_3^2+\frac{I_2-I_3}{S}\right)$$
$$w_3=\hat{w}_3=-\nu(x_3\gamma_{11}-x_2x_3\kappa_3)-\frac{\nu\kappa_2}{2}\left(x_2^2-x_3^2+\frac{I_2-I_3}{S}\right). \quad (22)$$

Having obtained all the warping functions, the three-dimensional strain field can be recovered by Eq. (11) up to the zeroth order as

$$\Gamma_{11}=\gamma_{11}+x_3\kappa_2-x_2\kappa_3$$
$$2\Gamma_{12}=w_{1,2}-x_3\kappa_1$$
$$2\Gamma_{13}=w_{1,3}+x_2\kappa_1$$
$$\Gamma_{22}=-\nu(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)$$
$$2\Gamma_{23}=0$$
$$\Gamma_{33}=-\nu(\gamma_{11}+x_3\kappa_2-x_2\kappa_3). \quad (23)$$

If one takes the definition of torsional rigidity from elasticity texts, which is

$$GJ=G\langle x_2^2+x_3^2+x_2\psi_{1,3}-x_3\psi_{1,2}\rangle \quad (24)$$

where J is the Saint-Venant torsion constant, then the asymptotically correct three-dimensional energy, up to the order of $\mu\varepsilon^2$, can be written as

$$2\Pi_0=ES\gamma_{11}^2+GJ\kappa_1^2+EI_2\kappa_2^2+EI_3\kappa_3^2. \quad (25)$$

This energy coincides with the result of classical beam theory; however, it is obtained without any ad hoc kinematic assumptions whatsoever. Such ad hoc assumptions as assuming the cross section to be rigid in its own plane or setting $\nu=0$ are common in the development of traditional beam theories in the literature.

For a straight beam clamped at $x_1=0$ and under the tip load F_1, M_i at $x_1=L$ (see Fig. 2), the one-dimensional strain measures can be solved with the help of the strain energy Eq. (25) as

$$\gamma_{11}=\frac{F_1}{ES} \quad \kappa_1=\frac{M_1}{GJ} \quad \kappa_2=\frac{M_2}{EI_2} \quad \kappa_3=\frac{M_3}{EI_3}. \quad (26)$$

If a linear beam theory is used, the three-dimensional displacement field can be recovered as

$$u_1 = \frac{F_1}{ES}x_1 - \frac{M_3}{EI_3}x_2x_1 + \frac{M_2}{EI_2}x_3x_1 + \psi\frac{M_1}{GJ}$$

$$u_2 = -\nu x_2\frac{F_1}{ES} - \nu x_2x_3\frac{M_2}{EI_2} + \frac{\nu M_3}{2EI_3}\left(x_2^2 - x_3^2 + \frac{I_2 - I_3}{S}\right) + \frac{M_3}{EI_3}\frac{x_1^2}{2}$$

$$u_3 = -\nu x_3\frac{F_1}{ES} + \nu x_2x_3\frac{M_3}{EI_3} + \frac{\nu M_2}{2EI_2}\left(x_2^2 - x_3^2 + \frac{I_2 - I_3}{S}\right) - \frac{M_2}{EI_2}\frac{x_1^2}{2} \tag{27}$$

which is essentially the superposition of elasticity solutions for extension, pure bending in two directions and torsion. The only exception to this statement is that there is a difference of a constant from the elasticity solutions for u_α due to differences in the way the clamped boundary condition is handled. Published elasticity solutions enforce the clamped condition at the beam reference line. In our case, however, since the one-dimensional variables implied by the VAM solution are averages of the three-dimensional displacement, the most straightforward solution in our framework constrains the average displacement to be zero. Clearly, by enforcing a modified boundary condition in the one-dimensional beam theory, so as to mimic the clamped condition used in the elasticity solutions, the two solutions will become identical; in particular, the constant terms in u_2 and u_3 will simply drop out.

From the above, it is clearly shown that the above classical model stores the complete three-dimensional energy of prismatic beams due to uniform extension, uniform torsion, and pure bending in two directions obtained by elasticity theory. The linearized three-dimensional displacement field recovered by VABS is the same as that obtained from elasticity theory.

Timoshenko-Like Model

Elasticity theory has another set of equations to solve for the so-called flexure problem, which involves both bending and transverse shear. For this VABS provides a Timoshenko-like model. Because a Timoshenko-like model can at most approximate the original three-dimensional energy up to the order of $\mu\varepsilon^2(h/l)^2$, a strain energy that is asymptotically correct to the second order of h/l is sought first

$$2U_1 = \epsilon^T A\epsilon + 2\epsilon^T B\epsilon' + \epsilon'^T C\epsilon' + 2\epsilon^T D\epsilon'' \tag{28}$$

where A, B, C, and D are matrices carrying the geometry and material information of the cross section, elements of $\epsilon = \lfloor \gamma_{11}\ \kappa_1\ \kappa_2\ \kappa_3 \rfloor^T$ are the generalized one-dimensional strain measures of Euler-Bernoulli beam theory. For isotropic prismatic beams, in which the locus of cross-sectional centroids is taken as the reference line and cross-sectional principal axes are along x_α, A becomes a diagonal matrix with diagonal terms given by the extensional stiffness ES, the torsional stiffness GJ, and bending stiffnesses EI_2 and EI_3. A Timoshenko-like model is then created out of the energy, Eq. (28), as

$$2U = \epsilon_t^T X\epsilon_t + 2\epsilon_t^T F\gamma + \gamma^T G\gamma \tag{29}$$

where ϵ_t are the classical strain measures (but defined slightly differently because of the framework of Timoshenko-like model), and $\gamma = \lfloor 2\gamma_{13}\ 2\gamma_{23} \rfloor^T$ transverse shear strains. The stiffness matrices X, F, and G can be found by (see [9])

$$G = (Q^T A^{-1} C A^{-1} Q)^{-1}$$

$$F = B^T A^{-1} Q G$$

$$X = A + F G^{-1} F^T \tag{30}$$

where

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}^T. \tag{31}$$

ϵ_t and ϵ are related by $\epsilon_t = \epsilon - Q\gamma$.

By a simple algebraic derivation, F and G can be expressed in terms of the components of A, B, and C as

$$G = \frac{E^2}{C_{33}C_{44} - C_{34}^2}\begin{bmatrix} C_{33}I_3^2 & C_{34}I_2I_3 \\ C_{34}I_2I_3 & C_{44}I_2^2 \end{bmatrix}$$

$$F = \frac{E}{C_{33}C_{44} - C_{34}^2}\begin{bmatrix} (B_{41}C_{33} - B_{31}C_{34})I_3 & (B_{31}C_{44} - B_{41}C_{34})I_2 \\ (B_{42}C_{33} - B_{32}C_{34})I_3 & (B_{32}C_{44} - B_{42}C_{34})I_2 \\ (B_{43}C_{33} - B_{33}C_{34})I_3 & (B_{33}C_{44} - B_{43}C_{34})I_2 \\ (B_{44}C_{33} - B_{34}C_{34})I_3 & (B_{34}C_{44} - B_{44}C_{34})I_2 \end{bmatrix} \tag{32}$$

where the subscripted terms involving B and C are specified elements in those matrices. Here, one can conclude that G is determined by the coefficients associated with $\kappa'_\alpha\kappa'_\beta$ in the asymptotic energy, and F is determined by the coefficients associated with $\kappa_\alpha\gamma'_{11}$ and $\kappa_\alpha\kappa'_i$. This observation is very important because it leads to our finding a closed-form solution for the Timoshenko-like model for isotropic, prismatic beams. To obtain the second-order energy, we perturb the warping functions as

$$w_i = \hat{w}_i + V_i \tag{33}$$

where V_i is of the order $\varepsilon h/l$. Substituting the perturbed warping functions back into Eq. (11), one obtains

$$\Gamma_{11} = \gamma_{11} + x_3\kappa_2 - x_2\kappa_3 + \underline{\hat{w}'_1} + \underline{\underline{V'_1}}$$

$$2\Gamma_{12} = \hat{w}_{1,2} - x_3\kappa_1 + \underline{V_{1,2} + \hat{w}'_2} + \underline{\underline{V'_2}}$$

$$2\Gamma_{13} = \hat{w}_{1,3} + x_2\kappa_1 + \underline{V_{1,3} + \hat{w}'_3} + \underline{\underline{V'_3}}$$

$$\Gamma_{22} = \hat{w}_{2,2} + \underline{V_{2,2}}$$

$$2\Gamma_{23} = \hat{w}_{3,2} + \hat{w}_{2,3} + \underline{V_{3,2} + V_{2,3}}$$

$$\Gamma_{33} = \hat{w}_{3,3} + \underline{V_{3,3}} \tag{34}$$

where the underlined terms are of the order $\varepsilon h/l$, the double underlined terms are of the order $\varepsilon h^2/l^2$, and the rest of the terms are of the order ε. Substituting this perturbed strain field into the energy functional Eq. (12) and neglecting all the terms of order higher than $\mu(h/l)^2\varepsilon^2$, one obtains

$$2\Pi = 2\Pi_0 + 2\Pi_1 + 2\Pi_2 \tag{35}$$

where Π_0 is the energy obtained for the classical model, Eq. (25) and

$$\Pi_1 = E\langle \hat{w}'_1(\underline{\underline{\gamma_{11}}} + x_3\kappa_2 - x_2\kappa_3)\rangle$$

$$+ G[\underline{\langle(\hat{w}_{1,2} - x_3\kappa_1)V_{1,2} + (\hat{w}_{1,3} + x_2\kappa_1)V_{1,3}\rangle}$$

$$+ \underline{\underline{\langle(\hat{w}_{1,2} - x_3\kappa_1)\hat{w}'_2 + (\hat{w}_{1,3} + x_2\kappa_1)\hat{w}'_3\rangle}}]. \tag{36}$$

It is easy to prove that the underlined terms vanish for arbitrary V_1. According to Eq. (32), the double underlined terms will not affect the Timoshenko-like model constructed from the second-order energy. Thus, the only terms of interest are

$$\Pi_1 = E\langle \hat{w}'_1(x_3\kappa_2 - x_2\kappa_3)\rangle. \tag{37}$$

Note these terms will not necessarily vanish unless the Saint-Venant warping, for which we have already solved, possesses some kind of symmetry. Thus, the second-order energy becomes

$$
\begin{aligned}
2\Pi_2 = {} & 2E\langle \underline{V_1'(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)}+\hat{w}_1'^2\rangle + G\langle (V_{1,2}+\hat{w}_2')^2 \\
& + 2V_2'(\hat{w}_{1,2}-x_3\kappa_1)\rangle + G\langle (V_{1,3}+\hat{w}_3')^2 + 2V_3'(\hat{w}_{1,3}+x_2\kappa_1) \\
& + (V_{2,3}+V_{3,2})^2\rangle \\
& + \frac{E}{(1+\nu)(1-2\nu)}\left\langle \begin{Bmatrix} \nu\hat{w}_1'+V_{2,2} \\ \nu\hat{w}_1'+V_{3,3} \end{Bmatrix}^T \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \right. \\
& \left. \times \begin{Bmatrix} \nu\hat{w}_1'+V_{2,2} \\ \nu\hat{w}_1'+V_{3,3} \end{Bmatrix} \right\rangle . \qquad (38)
\end{aligned}
$$

The underlined terms create difficulty to solve this problem in the two-dimensional cross-sectional domain. However, our goal is to find an interior solution without consideration of boundary effects at the ends of the beam. Hence, integration by parts with respect to x_1 can be used, and the residual terms at the ends can be considered as having no effect on the interior solution. This argument can be illustrated mathematically if one constrains the warping V_1 in such a way that

$$
\langle V_1(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)|_{x_1=0}\rangle = \langle V_1(\gamma_{11}+x_3\kappa_2-x_2\kappa_3)|_{x_1=L}\rangle . \qquad (39)
$$

The effect of such a constraint will die out after a small distance from the ends according to the Saint-Venant principle. Then, the Euler-Lagrange equations for the functional Π_2 are

$$
V_{1,22}+V_{1,33}+2(\gamma_{11}'+x_3\kappa_2'-x_2\kappa_3')=0 \qquad (40)
$$

$$
2(1-\nu)V_{2,22}+(1-2\nu)V_{2,33}+V_{3,23}+(2\nu-1)x_3\kappa_1'+\hat{w}_{1,2}'=0 \qquad (41)
$$

$$
\begin{aligned}
2(1-\nu)V_{3,33}+(1-2\nu)V_{3,22}+V_{2,23}+(1-2\nu)x_2\kappa_1' \\
+(1-2\nu)\hat{w}_{1,3}'=0 \qquad (42)
\end{aligned}
$$

and the associated boundary conditions given by

$$
n_3(V_{1,3}+\hat{w}_3')+n_2(V_{1,2}+\hat{w}_2')=0 \qquad (43)
$$

$$
n_3(\hat{w}_{2,3}+\hat{w}_{3,2})+\frac{2n_2}{1-2\nu}[\nu V_{3,3}+(1-\nu)V_{2,2}+\nu\hat{w}_1']=0 \qquad (44)
$$

$$
n_2(\hat{w}_{2,3}+\hat{w}_{3,2})+\frac{2n_3}{1-2\nu}[\nu V_{2,2}+(1-\nu)V_{3,3}+\nu\hat{w}_1']=0. \qquad (45)
$$

It is observed that V_α is decoupled from V_1; V_α should be some function multiplying κ_1' and V_1 will be a linear combination of γ_{11}', and κ_α'. The terms associated with V_α will not affect the Timoshenko-like model as shown in Eq. (32). (In fact these terms are related with the Vlasov theory and will be studied in later paper.) Hence, one can set V_α to be zero and drop all terms that have no effect on the Timoshenko-like model. Then, after recalculating Π_2, one finds

$$
\begin{aligned}
2\Pi_2 = {} & 2E\langle V_1'(x_3\kappa_2-x_2\kappa_3)\rangle \\
& + G\left\langle \left[V_{1,2}-\nu x_2x_3\kappa_2' + \frac{\nu\kappa_3'}{2}\left(x_2^2-x_3^2+\frac{I_2-I_3}{S}\right)\right]^2\right\rangle \\
& + G\left\langle \left[V_{1,3}+\nu x_2x_3\kappa_3' + \frac{\nu\kappa_2'}{2}\left(x_2^2-x_3^2+\frac{I_2-I_3}{S}\right)\right]^2\right\rangle . \qquad (46)
\end{aligned}
$$

Then the corresponding Euler-Lagrange equation, Eq. (40), and boundary condition, Eq. (43), will be modified to

$$
V_{1,22}+V_{1,33}=2(x_2\kappa_3'-x_3\kappa_2') \qquad (47)
$$

$$
\begin{aligned}
& n_3\left[V_{1,3}+\nu x_2x_3\kappa_3' + \frac{\nu\kappa_2'}{2}\left(x_2^2-x_3^2+\frac{I_2-I_3}{S}\right)\right] \\
& \quad = -n_2\left[V_{1,2}-\nu x_2x_3\kappa_2' + \frac{\nu\kappa_3'}{2}\left(x_2^2-x_3^2+\frac{I_2-I_3}{S}\right)\right]. \qquad (48)
\end{aligned}
$$

One can introduce a special function as

$$
\begin{aligned}
& V_{1,2}-\nu x_2x_3\kappa_2' + \frac{\nu\kappa_3'}{2}\left(x_2^2-x_3^2+\frac{I_2-I_3}{S}\right) \\
& \quad = \phi_{,3}+(1+\nu)x_2^2\kappa_3'+f(x_3) \\
& V_{1,3}+\nu x_2x_3\kappa_3' + \frac{\nu\kappa_2'}{2}\left(x_2^2-x_3^2+\frac{I_2-I_3}{S}\right) \\
& \quad = -\phi_{,2}-(1+\nu)x_3^2\kappa_2'+g(x_2)
\end{aligned} \qquad (49)
$$

to satisfy Eq. (47) automatically, where $f(x_3)$ and $g(x_2)$ are arbitrary functions. The advantage of introducing ϕ is that Eq. (48) can be made much simpler based on the choice of $f(x_3)$ and $g(x_2)$. Using Eq. (49), the boundary condition, Eq. (48) becomes

$$
\frac{\partial\phi}{\partial s} = -[f(x_3)+(1+\nu)x_2^2\kappa_3']n_2 - [g(x_2)-(1+\nu)x_3^2\kappa_2']n_3 \qquad (50)
$$

where s is the contour coordinate along the cross-sectional boundary. If the arbitrary functions are chosen such that on the boundary

$$
\begin{aligned}
f(x_3) &= \begin{cases} -(1+\nu)x_2^2\kappa_3' & \text{if } n_2\neq 0, \\ \text{arbitrary} & \text{if } n_2=0 \end{cases} \\
g(x_2) &= \begin{cases} (1+\nu)x_3^2\kappa_2' & \text{if } n_3\neq 0, \\ \text{arbitrary} & \text{if } n_3=0 \end{cases}
\end{aligned} \qquad (51)
$$

then the right-hand side of Eq. (50) vanishes and ϕ is constant along the boundary. For simply connected domains, one can choose ϕ to vanish along the boundary. The governing differential equations for ϕ can be deduced from Eq. (49) as

$$
\phi_{,22}+\phi_{,33} = -2\nu x_2\kappa_2' + \frac{dg(x_2)}{dx_2} - 2\nu x_3\kappa_3' - \frac{df(x_3)}{dx_3}. \qquad (52)
$$

This equation is the same as that governing the flexure problem in both directions if one expresses κ_α' in terms of the tip transverse force and multiplies ϕ by the shear modulus G. Therefore, all flexure problems that are solvable by elasticity theory can also be solved analytically by the VAM (the procedure on which VABS is based). After ϕ is obtained, one can find V_1 up to a constant using Eq. (49), where the constant can be determined by the constraint $\langle V_1\rangle = 0$. The portion of asymptotically correct energy Eq. (28) that is needed for constructing the Timoshenko-like model can be found from Eqs. (35) and (46).

Although it is necessary to carry out an integration by parts with respect to x_1 for the sets of terms in the first bracket of Eq. (46) to render the present problem as a purely cross-sectional problem, this operation should not be applied at the step where the strain energy is obtained. Previous publications, [7,9], are silent on this seemingly inconsistent practice because a reasonable explanation had not been formulated. However, for the present problem, the transverse shear energy is completely represented by the last two sets of terms and the first set of terms is part of the energy due to extension. If one integrates this first set of terms by parts, this energy represented by it will be transformed into transverse shear energy according to Eq. (32), which is at least physically inappropriate; and, in the worst case, the total transverse shear energy will turn out to be negative. Nevertheless, if one does the integration by parts and also keeps the residual terms at the ends, the fictitious transverse shear energy caused by integration by parts will be canceled by the residual terms at the boundaries, which means the final three-dimensional results will not be affected by this operation. Based on this fact and Eq. (32), the first set of terms in Eq. (46) for the present problem will not affect the final Timoshenko-like model and will be discarded in later calculations.

After constructing the Timoshenko-like model using Eqs. (32) and (30) and using it to solve the one-dimensional beam problem, the three-dimensional strain field can be recovered using Eq. (34) and the displacement field can be recovered similarly as Eq. (27). This procedure will be given in detail for some example cross sections that follow.

Example Cross Sections

In this section two typical examples listed in the elasticity text of Timoshenko and Goodier [15] are studied here using the analytical procedures formulated in previous sections.

Elliptical Section. For an elliptical cross section with semi-axes a and b in the directions of x_2 and x_3, respectively, and $\rho = a/b$ as the aspect ratio, the Saint-Venant warping is found to be

$$\psi = \frac{(b^2 - a^2) x_2 x_3}{a^2 + b^2}. \tag{53}$$

If one chooses the $f(x_3)$ and $g(x_2)$ according to Eq. (51)

$$f(x_3) = -(1+\nu) x_2^2 \kappa_3' = -(1+\nu)\left(1 - \frac{x_3^2}{b^2}\right) a^2 \kappa_3'$$
$$g(x_2) = (1+\nu) x_3^2 \kappa_2' = (1+\nu)\left(1 - \frac{x_2^2}{a^2}\right) b^2 \kappa_2' \tag{54}$$

then both Eqs. (50) and (52) will be satisfied by

$$\phi = m\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) x_3 + n\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) x_2 \tag{55}$$

with

$$m = -\frac{\rho^2[\nu + (1+\nu)\rho^2]}{1 + 3\rho^2} b^2 \kappa_3'$$
$$n = -\frac{[\nu\rho^2 + (1+\nu)]}{3+\rho^2} b^2 \kappa_2'. \tag{56}$$

Then one can obtain V_1 by Eq. (49) as

$$V_1 = p\,\frac{x_3 \kappa_2'}{24(3+\rho^2)} + q\,\frac{x_2 \kappa_3'}{24(1+3\rho^2)} \tag{57}$$

with

$$\begin{aligned} p &= -4x_3^2[4+\nu+(2-\nu)\rho^2] - 12x_2^2(2-\nu+\nu\rho^2) + 3b^2[16+8\rho^2 \\ &\quad + 13\nu + 2\nu\rho^2 + \nu\rho^4)] \\ q &= 4x_2^2[(4+\nu)\rho^2 + 2 - \nu] - 12x_3^2[(2-\nu)\rho^2 + \nu] - 3b^2[16\rho^4 \\ &\quad + 8\rho^2 + 13\nu\rho^4 + 2\nu\rho^2 + \nu)]. \end{aligned} \tag{58}$$

Then the energy of the order $(h/l)^2$ excluding the terms that do not affect the final Timoshenko-like model can be computed as

$$\frac{2\Pi_2}{EAb^4} = \frac{[\rho^4\nu^2 + 2\rho^2(1+\nu)^2 + 5(1+\nu)^2]\kappa_2'^2}{12(3+\rho^2)(1+\nu)} + \frac{\rho^2[\nu^2 + 2\rho^2(1+\nu)^2 + 5\rho^4(1+\nu)^2]\kappa_3'^2}{12(1+3\rho^2)(1+\nu)}. \tag{59}$$

The final Timoshenko-like model can be expressed as

$$2U = \begin{Bmatrix} \gamma_{11} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}^T \begin{bmatrix} ES & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & EI_3 \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} + \begin{Bmatrix} 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix}^T \begin{bmatrix} S_2 & 0 \\ 0 & S_3 \end{bmatrix} \begin{Bmatrix} 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix} \tag{60}$$

where

$$S = \pi ab \quad J = \frac{\pi a^3 b^3}{a^2 + b^2} \quad I_2 = \frac{\pi}{4} ab^3 \quad I_3 = \frac{\pi}{4} a^3 b \tag{61}$$

and

$$S_2 = \frac{3a^2(3a^2+b^2)(1+\nu)^2 GS}{2[b^4\nu^2 + 5a^4(1+\nu)^2 + 2a^2b^2(1+\nu)^2]}$$
$$S_3 = \frac{3b^2(a^2+3b^2)(1+\nu)^2 GS}{2[a^4\nu^2 + 5b^4(1+\nu)^2 + 2a^2b^2(1+\nu)^2]}. \tag{62}$$

The results are the same as those in [10], but in that work the results are obtained by using the Ritz method and assuming a third-order polynomial which is of the exact form as shown here. This result is the same as what is in [16] which has been obtained through elasticity theory. However, the result provided in [17] is an approximation of the exact solution.

Rectangular Section. For a rectangular section of width $2a$ in x_2-direction and height $2b$ in x_3-direction (see Fig. 3), the Saint-Venant warping can be expressed in a form of infinite series such as

$$\psi = -x_2 x_3 + \frac{32b^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\sinh\left(\frac{2n+1}{2}\,\frac{\pi x_2}{b}\right)}{\cosh\left(\frac{2n+1}{2}\,\frac{\pi a}{b}\right)} \sin\left(\frac{2n+1}{2}\,\frac{\pi x_3}{b}\right). \tag{63}$$

To solve for ϕ, we should choose the arbitrary functions $f(x_3)$ and $g(x_2)$ first. Along $x_2 = \pm a$, $n_2 \neq 0$, so we can choose $f(x_3) = -(1+\nu)a^2\kappa_3'$ and $g(x_2)$ can be arbitrary. Along $x_3 = \pm b$, $n_3 \neq 0$, we can choose $g(x_2) = (1+\nu)b^2\kappa_2'$ and $f(x_3)$ can be arbitrary. Solving Eq. (52), one finds

$$\begin{aligned} \phi &= -\frac{\nu}{3}(x_2^2 - a^2) x_2 \kappa_2' + \frac{4\nu a^3 \kappa_2'}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n \cosh\left(\frac{n\pi x_3}{a}\right) \sin\left(\frac{n\pi x_2}{a}\right)}{n^3 \cosh\left(\frac{n\pi b}{a}\right)} - \frac{\nu}{3}(x_3^2 \\ &\quad - b^2) x_3 \kappa_3' + \frac{4\nu b^3 \kappa_3'}{\pi^3} \sum_{m=1}^{\infty} \frac{(-1)^m \cosh\left(\frac{m\pi x_2}{b}\right) \sin\left(\frac{m\pi x_3}{b}\right)}{m^3 \cosh\left(\frac{m\pi a}{b}\right)}. \end{aligned} \tag{64}$$

Then one can derive V_1 to be

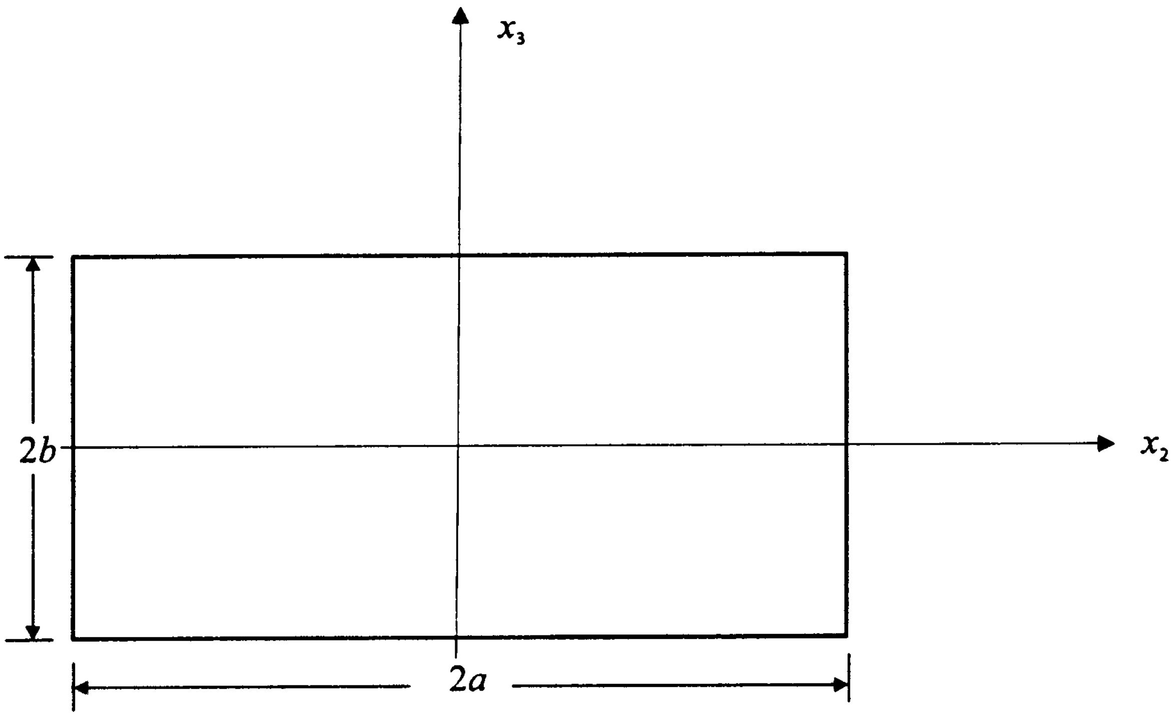

Fig. 3 Sketch of a rectangular cross section

$$V_1 = -\frac{4a^3\nu\kappa_2'}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n\sinh\left(\frac{n\pi x_3}{a}\right)\cos\left(\frac{n\pi x_2}{a}\right)}{n^3\cosh\left(\frac{n\pi b}{a}\right)} + \frac{4\nu b^3\kappa_3'}{\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^m\sinh\left(\frac{m\pi x_2}{b}\right)\cos\left(\frac{m\pi x_3}{b}\right)}{m^3\cosh\left(\frac{m\pi a}{b}\right)} + \left[\left(\frac{\nu}{6}+\frac{1}{3}\right)x_2^3 - \left(a^2+\frac{5a^2\nu}{6}-\frac{b^2\nu}{6}\right)x_2 - \frac{\nu x_2 x_3^2}{2}\right]\kappa_3' - \left[\left(\frac{\nu}{6}+\frac{1}{3}\right)x_3^3 - \left(b^2+\frac{5b^2\nu}{6}-\frac{a^2\nu}{6}\right)x_3 - \frac{\nu x_2^2 x_3}{2}\right]\kappa_2'. \tag{65}$$

Then the energy of the order $(h/l)^2$ excluding the terms that do not affect the final Timoshenko-like model can be computed as

$$\frac{2\Pi_2}{G} = \left[\frac{16\nu^2 ba^5 + 96b^5 a(1+\nu)^2}{45} - \frac{32\nu^2 a^6}{\pi^5}\times\sum_{n=1}^{\infty}\frac{\tanh\left(\frac{bn\pi}{a}\right)}{n^5}\right]\kappa_2'^2 + \left[\frac{16\nu^2 b^5 a + 96a^5 b(1+\nu)^2}{45} - \frac{32\nu^2 b^6}{\pi^5}\sum_{m=1}^{\infty}\frac{\tanh\left(\frac{am\pi}{b}\right)}{m^5}\right]\kappa_3'^2. \tag{66}$$

The final Timoshenko-like model can be expressed as Eq. (60) with

$$S = 4ab \quad J = \beta ab^3 \quad I_2 = \frac{4}{3}ab^3 \quad I_3 = \frac{4}{3}a^3 b \quad S_\alpha = \frac{GS}{k_\alpha} \tag{67}$$

where β can be found in elasticity textbooks such as [15] and k_α are the so-called shear correction factors, given by

$$k_2 = \frac{6}{5} + \left(\frac{\nu}{1+\nu}\right)^2 \rho^{-4}\left[\frac{1}{5} - \frac{18}{\rho\pi^5}\sum_{m=1}^{\infty}\frac{\tanh(m\pi\rho)}{m^5}\right]$$
$$k_3 = \frac{6}{5} + \left(\frac{\nu}{1+\nu}\right)^2 \rho^{4}\left[\frac{1}{5} - \frac{18}{\pi^5}\sum_{n=1}^{\infty}\frac{\tanh(n\pi\rho^{-1})}{n^5}\right]. \tag{68}$$

Although the form of the shear correction factors are different from those of Renton [16], the numerical values for different aspect ratios are the same. The reason the two results are of different form is because in [16] the flexure problem is solved by using a double trigonometric series while here hyperbolic series are used along with the trigonometric series which converge to a fixed value more rapidly. Please note that although [17] is also based on the VAM, the shear correction factors presented therein for the rectangular section are approximations of the elasticity solution.

Conclusions

The variational asymptotic method, on which the finite-element-based cross-sectional analysis VABS (variational asymptotic beam sectional analysis) is based, has been used to analytically solve the isotropic prismatic beam problem. The same governing equations for Saint-Venant warping and the general flexure problem have been shown to correspond with those of the theory of elasticity. Identical results have been found between elasticity and VAM solutions for beams with elliptical and rectangular cross sections. It has been proven mathematically that for an isotropic prismatic bar with an arbitrary cross section the classical model of VABS is the same as the superposition of elasticity solutions for extension, pure bending in two directions and torsion. Moreover, the Timoshenko-like model of VABS consists of these plus the solution for the general flexure problem in both directions.

The fact that the numerical procedure in VABS reproduces the results of elasticity theory clearly demonstrates that the VAM, the mathematical foundation of VABS, is a valid methodology that can be used to avoid the difficulties of dealing with three-dimensional elasticity while obtaining results that are coincident with the exact solutions. Although it may not be possible to validate the general theory of VABS for anisotropic beams in this same way, it is a natural deduction from the above demonstrations to conclude that the results for generally anisotropic beams should be the same as those calculated by methods based on three-dimensional elasticity theory, such as three-dimensional finite elements. Indeed, as three-dimensional finite elements allow one to go beyond the limitations of three-dimensional elasticity, VABS may also be considered as a means for going beyond those limits when considering the cross-sectional analysis of beams.

References

[1] Le, K. C., 1999, *Vibrations of Shells and Rods*, 1st Ed., Springer, Berlin.

[2] Ciarlet, P. G., and Destuynder, P., 1979, ''A Justification of a Nonlinear Model in Plate Theory,'' Comput. Methods Appl. Mech. Eng., **17/18**, pp. 227–258.

[3] Berdichevsky, V. L., 1979, ''Variational-Asymptotic Method of Constructing a Theory of Shells,'' Prikl. Mat. Mekh., **43**(4), pp. 664–687.

[4] Hodges, D. H., Atilgan, A. R., Cesnik, C. E. S., and Fulton, M. V., 1992, ''On a Simplified Strain Energy Function for Geometrically Nonlinear Behavior of Anisotropic Beams,'' Composites Eng., **2**(5–7), pp. 513–526.

[5] Cesnik, C. E. S., and Hodges, D. H., 1993, ''Stiffness Constants for Initially Twisted and Curved Composite Beams,'' Appl. Mech. Rev., **46**(11, Part 2), pp. S211–S220.

[6] Cesnik, C. E. S., and Hodges, D. H., 1997, ''VABS: A New Concept for Composite Rotor Blade Cross-Sectional Modeling,'' J. Am. Helicopter Soc., **42**(1), pp. 27–38.

[7] Popescu, B., and Hodges, D. H., 1999, ''On Asymptotically Correct Timoshenko-Like Anisotropic Beam Theory,'' Int. J. Solids Struct., **37**(3), pp. 535–558.

[8] Popescu, B., Hodges, D. H., and Cesnik, C. E. S., 2000, ''Obliqueness Effects in Asymptotic Cross-Sectional Analysis of Composite Beams,'' Comput. Struct., **76**(4), pp. 533–543.

[9] Yu, W., Hodges, D. H., Volovoi, V. V., and Cesnik, C. E. S., 2002, ''On Timoshenko-Like Modeling of Initially Curved and Twisted Composite Beams,'' Int. J. Solids Struct., **39**(19), pp. 5101–5121.

[10] Yu, W., Volovoi, V. V., Hodges, D. H., and Hong, X., 2002, ''Validation of the Variational Asymptotic Beam Sectional Analysis,'' AIAA J., **40**(10), pp. 2105–2112.

[11] Trabucho, L., and Viano, J., 1996, ''Mathematical Modeling of Rods,'' *Handbook of Numerical Analysis*, *Vol. IV*, P. Ciarlet and J. Lions, eds., Elsevier, New York, pp. 487–974.

[12] Volovoi, V. V., Hodges, D. H., Cesnik, C. E. S., and Popescu, B., 2001, ''Assessment of Beam Modeling Methods for Rotor Blade Applications,'' Math. Comput. Modell., **33**(10–11), pp. 1099–1112.

[13] Danielson, D. A., and Hodges, D. H., 1987, ''Nonlinear Beam Kinematics by Decomposition of the Rotation Tensor,'' ASME J. Appl. Mech., **54**(2), pp. 258–262.

[14] Hodges, D. H., 1999, ''Non-linear Inplane Deformation and Buckling of Rings and High Arches,'' Int. J. Non-Linear Mech., **34**(4), pp. 723–737.

[15] Timoshenko, S. P., and Goodier, J. N., 1970, *Theory of Elasticity*, McGraw-Hill, Maidenhead, UK.

[16] Renton, J. D., 1991, ''Generalized Beam Theory Applied to Shear Stiffness,'' Int. J. Solids Struct., **27**(15), pp. 1955–1967.

[17] Berdichevsky, V. L., and Kvashnina, S. S., 1976, ''On Equations Describing the Transverse Vibrations of Elastic Bars,'' Prikl. Mat. Mekh., **40**, pp. 120–135.