Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 297 (2006) 1082–1087
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

On the valid frequency range of Timoshenko beam theory

N.G. Stephen[a,*], S. Puchegger[b]

[a]School of Engineering Sciences, Mechanical Engineering, University of Southampton, Highfield, Southampton SO17 1BJ, UK
[b]Institute of Material Physics, University of Vienna, Boltzmanngasse 5, A-1090, Vienna, Austria

Received 21 February 2006; received in revised form 25 April 2006; accepted 26 April 2006
Available online 27 June 2006

Abstract

The frequency equation of Timoshenko beam theory factorises for hinged–hinged end conditions, leading to a first and second spectrum of natural frequencies; the latter is largely inaccurate and can be isolated and disregarded. For the majority of other end conditions, when the frequency equation does not factorise, one may think in terms of pseudo-second spectrum contributions arising when evanescent waves become propagating above the cut-off frequency $\omega_{co} = \sqrt{\kappa AG/\rho I}$, and it is conjectured that these may have a corrupting effect on the frequency predictions. Comparisons with measured and simulated frequencies lead to the conclusion that Timoshenko predictions above the cut-off frequency should be disregarded for those end conditions for which the frequency equation does not factorise.

1. Introduction

It is well known [1] that the transcendental (or an equivalent algebraic) frequency equation of Timoshenko beam theory (TBT) factorises for hinged–hinged end conditions. This allows one to think in terms of a first and second spectrum of natural frequencies, or first and second branches of a wave propagation dispersion diagram. It has been shown recently [2] that the transcendental frequency equation also factorises for guided–guided and guided–hinged conditions; these new end combinations can be regarded as portions of a multi-span hinged–hinged beam. Below the second spectrum cut-off frequency given by $\omega_{co} = \sqrt{\kappa AG/\rho I}$, TBT predicts both propagating (TBT1) waves, and evanescent (TBT2) waves associated with the hyperbolic spatial functions. Above the cut-off frequency, these hyperbolic functions become trigonometric [3], implying propagation along the structure or, equivalently, the possibility of standing waves at these second spectrum frequencies.

For hinged–hinged end conditions, or equivalently a beam of infinite length, TBT predictions may be compared with those from exact elastodynamic theory, when the accuracy of the first spectrum is beyond reproach. In Ref. [2] comparison was made between "exact" plane stress predictions for a standing wave in a short hinged–hinged beam of thin rectangular cross-section. The agreement between the TBT1 predictions and the exact lower branch asymmetric mode was within the range -0.4% to $+0.54\%$ for the first 20 modes of

*Corresponding author. Tel.: +442380 592 359; fax: +442380 593 230.
E-mail addresses: ngs@soton.ac.uk (N.G. Stephen), stephan.puchegger@univie.ac.at (S. Puchegger).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.04.020

vibration when using the shear coefficient $\kappa = 5(1+v)/(6+5v)$. Experimental support for this value of the coefficient at long wavelength has also been provided by Méndez-Sánchez et al. [4].

As wavelength tends to zero, exact elasticity theory predicts wave propagation at the Rayleigh surface wave velocity (RSWV), c_R, which may be determined from the appropriate root of the equation

$$\left(2-\frac{c_R^2}{c_s^2}\right)^2 = 4\sqrt{\left(1-\frac{c_R^2}{c_s^2}\right)\left(1-\frac{1-2v}{2(1-v)}\frac{c_R^2}{c_s^2}\right)}. \tag{1}$$

TBT1 can be adapted to agree with this, by noting that as wavelength $\lambda \to 0$, the TBT1 phase velocity prediction becomes $c_p = c_s\sqrt{\kappa}$ and requiring that this be equal to the RSWV leads to the requirement [5]

$$(2-\kappa_R)^2 = 4\sqrt{(1-\kappa_R)\left(1-\frac{1-2v}{2(1-v)}\kappa_R\right)},\ 0<\kappa_R<1. \tag{2}$$

If one employed a shear coefficient more appropriate for longer wavelength, such as $\kappa = 5(1+v)/(6+5v)$ for the plane stress rectangle, the error in the limit of zero wavelength can be expressed in terms of these long- and short-wavelength shear coefficients as $\left(\sqrt{\kappa/\kappa_R}-1\right)\times 100\%$. For Poisson's ratio $v = 0.25$, one has $\kappa_R = 0.8453$, and $\kappa = 0.8621$ for the plane stress rectangle, and using the long-wavelength value would lead to an overestimate of the RSWV by about 1%. Employing a shear coefficient more appropriate for plane strain conditions, $\kappa = 5/(6-v)$, would give an overestimate of the RSWV of about 1.5%. Thus one may surmise that a TBT1 prediction can be within 2%, say, of the exact elastodynamic theory across the entire range of wavelengths.

In contrast, the validity of the second spectrum frequency predictions (TBT2) has been called into question on the basis of two studies. In Ref. [6] comparison was made between the phase velocity predictions from the exact Pochhammer–Chree (PC) theory for wave propagation in an infinite rod of circular cross-section, and the equivalent TBT predictions; excellent agreement was found for the lowest flexural mode and TBT1, using the shear coefficient $\kappa = 6(1+v)^2/(7+12v+4v^2)$. However, no consistent agreement could be found for TBT2; at long wavelength, the TBT2 prediction was close to the second flexural mode of PC theory, as one would wish, but at shorter wavelength TBT2 agreed more closely with the second extensional mode of PC theory. Similarly in Ref. [2], for the thin rectangular cross-section, the TBT2 prediction did not provide consistent agreement with any single mode of vibration; at long wavelength it was very close to the second exact asymmetric mode (as one would wish), but as wavelength shortens it became closer to the second symmetric (extensional) mode, then the third asymmetric mode. In both studies it was concluded that the second spectrum predictions of Timoshenko beam theory should be disregarded.

Beside the three special beam end combinations noted above, in general the transcendental frequency equation does not factorise. This is true for the free–free case, which is the easiest beam end condition to achieve experimentally. Again, above the same cut-off frequency, Chan et al [3] have argued that previously hyperbolic (evanescent) spatial functions become trigonometric, implying disturbance propagation along the structure; here, these are described as pseudo-second spectrum contributions. For the hinged–hinged case, the second spectrum could be readily identified, and disregarded. However, for the free–free case, this is not possible; indeed, the pseudo-second spectrum contributions are necessary to maintain the required conditions of zero bending moment and shearing force at the free ends. Now, if the second spectrum predictions are largely inaccurate for the hinged–hinged case, as seen in Ref. [2], there seems no good reason why the pseudo-second spectrum contributions should be accurate for the free–free; however, if they are necessary, and cannot be disregarded, they may have the effect of corrupting the TBT predictions above the cut-off frequency. This possibility is consistent with Hutchinson's conclusion [7] that "The cut-off frequency … appears to be a reasonable choice for an upper bound … on the Timoshenko solution." This conclusion is clearly not true for the hinged–hinged numerical example considered in Ref. [2]: the cut-off ($n = 0$) frequency for TBT2 was 65 851 rad/s and only the lowest four TBT1 predictions were smaller than this value. The highest ($n = 20$) TBT1 mode considered had a frequency some five and a half times greater than the cut-off frequency, but the difference between the TBT1 and the exact elastodynamic plane stress frequency predictions was only 0.54%.

If the above ideas regarding a corrupting effect of pseudo-second spectrum contributions were to be correct, one would conclude that frequency predictions from TBT above the cut-off frequency should be disregarded as unreliable *for those end conditions for which the transcendental frequency equation does not factorise*. The purpose of the present work was to test this hypothesis.

For a short free–free beam of near-square cross-section, the Timoshenko frequency predictions are compared with experimental observations, commercial finite element (ANSYS) predictions and predictions from a resonant ultrasound spectroscopy (RUS) technique. RUS was proposed first by Demarest [8], and has become a standard method for the determination of elastic constants (see recent review by Leisure and Willis [9]). It is essentially a best-fit between measured and computed Rayleigh–Ritz upper bound natural frequencies of a specimen. For the free-cube (rectangular parallelepiped) the latter employs products of (orthogonal) Legendre polynomials in the three coordinate directions as basis functions. The experimental set-up consists of the beam specimen suspended and excited by two carbon-fibre loops, which are attached to two piezoelectric transducers, and is described fully in Ref. [10]. Resonant frequencies are found by sweeping the frequency range of interest with a network analyser. Even very weak resonances can be detected by this method, since the noise level of the network analyser is about −140 dBm (1 mW reference level).

2. Specimen measurements, finite element analysis, and TBT predictions

A short aluminium alloy beam of near-square cross-section was machined, and found to have dimensions: length $L = 40.04 \pm 0.02$ mm, breadth $b = 10.09 \pm 0.01$ mm, depth $d = 10.04 \pm 0.02$mm; the beam had mass $m = 11.43$ g which implies a density $\rho = 2817.9111\,\mathrm{kg\,m^{-3}}$. The elastic moduli were found by a best-fit between the 16 lowest measured bending frequencies (Table 1)—eight in the more flexible (*f*) and eight in the more stiff (*s*) plane—and the RUS simulations, leading to Young's modulus $E = 72.66611$ GPa, shear modulus $G = 27.17481$ GPa and hence Poisson's ratio $\nu = E/(2G) - 1 = 0.337012$. These values of the elastic constants were used in the subsequent finite element analysis performed using ANSYS. The beam was modelled using 20-node brick elements (SOLID95) with the (approximate) element size defined as one-tenth that of the breadth; this leads to 10×10 elements to define the cross-section, with the length of the beam divided into 40 elements. The TBT predictions were found using the frequency equation for a free–free beam, as given by Levinson and Cooke [1], with shear coefficient $\kappa = 5(1+\nu)/(6+5\nu)$; this was constructed as a

Table 1
Measured and predicted natural frequencies (Hz) for a short free–free aluminium beam

n	Measured	ANSYS	RUS	TBT	Comments
1	27359.6*f*	27417.6*f* (+0.21%)	27417.5*f* (+0.21%)	27407.1*f* (+0.17%)	*s*>*f* for all
	27423.8*s*	27515.6*s* (+0.33%)	27515.5*s* (+0.33%)	27505.3*s* (+0.30%)	
2	60862.0*f*	60882.0*f* (+0.03%)	60881.5*f* (+0.03%)	60851.1*f* (−0.02%)	*s*>*f* for all
	61098.3*s*	61022.0*s* (−0.12%)	61021.5*s* (−0.13%)	60992.1*s* (−0.17%)	
3	97609.5*f*	97734.4*f* (+0.13%)	97732.5*f* (+0.13%)	97796.0*f* (+0.19%)	*s*>*f* for all
	97852.4*s*	97881.5*s* (+0.03%)	97879.7*s* (+0.03%)	97945.4*s* (+0.09%)	
4	131494*f*	131658*f* (+0.12%)	131654*f* (+0.12%)	132277*f* (+0.60%)	*s*>*f* for all
	131732*s*	131675*s* (−0.04%)	131671*s* (−0.05%)	132308*s* (+0.44%)	
5	161352*s*	161390*s* (+0.02%)	161383*s* (+0.02%)	163547*s* (+1.36%)	*f*>*s* for all
	161538*f*	161517*f* (−0.01%)	161511*f* (−0.02%)	163661*f* (+1.31%)	
6	165183*s*	164887*s* (−0.18%)	164884*s* (−0.18%)	169108*s* (+2.38%)	*f*>*s* for all
	165598*f*	165398*f* (−0.12%)	165394*f* (−0.12%)	169634*f* (+2.44%)	
7	194863*f*	194933*f* (+0.04%)	194922*f* (+0.03%)	202352*f* (+3.84%)	*f*>*s* for TBT
	194973*s*	195032*s* (+0.03%)	195022*s* (+0.03%)	202115*s* (+3.66%)	*s*>*f* for ANSYS, RUS, and measured
8	195869*s*	195977*s* (+0.06%)	195966*s* (+0.05%)	203319*s* (+3.80%)	*f*>*s* for all
	195908*f*	196097*f* (+0.10%)	196086*f* (+0.09%)	203518*f* (+3.88%)	
9	213501		213505*f* (+0.00%)	241202*f* (+12.97%)	*f*>*s* for TBT
	213635		213707*s* (+0.03%)	241067*s* (+12.84%)	*s*>*f* for RUS
10/11	220556		220002*f* (−0.25%)	247954*f* (+12.42%)	*f*>*s* for TBT
	220702		220247*f* (−0.21%)	281542*f* (+27.57%)	*s*>*f* for RUS
	221010		220984*s* (−0.01%)	247782*s* (+12.11%)	
	221092		221459*s* (+1.17%)	281408*s* (+27.28%)	

Percentage differences between the predicted and the measured values are shown in parentheses.

frequency function, y, using MATLAB. For frequencies below the cut-off, zero values of y were found using the FZERO command within MATLAB; above the cut-off frequency, the function y is complex, and the FMINS command was used to find a minimum (zero) value of the absolute value of y. The cut-off frequencies are calculated as $f_{co,f} = 159047$ Hz for vibration in the more flexible plane, which is greater than $f_{co,s} = 158259$ Hz for vibration in the stiffer plane.

3. Results and discussion

The beam cross-section is almost square, so one expects natural frequencies to occur as near equal pairs; these are denoted in Table 1 by the same mode number n, but qualified as s and f, according to whether vibration takes place within the stiff or flexible plane, respectively.

First note that the ANSYS prediction is always slightly greater than the RUS, but agreement is near perfect; typically, the difference (not shown in Table 1) is less than 0.006%. Thus one may regard the ANSYS and RUS predictions as a unity, and henceforth these are referred to as the simulated frequencies. These simulated frequencies are at times greater, and at times smaller than the measured frequencies, as one would expect from a best-fit analysis for the elastic constants. Best-fit will minimise the absolute differences between the measured and simulated frequencies, so it is no surprise that the largest percentage difference should occur for the fundamental ($n = 1$) modes. Overall, agreement between the measured and simulated frequencies is excellent for all of the modes shown in Table 1. It is clear that the simulated values are quite capable of providing excellent agreement above the cut-off frequencies; indeed, short beams or cubes are typical RUS samples and the simulated frequencies can be as accurate $\pm 0.2\%$ for the first fifty modes of vibration (see Fig. 3 of Ref. [9]). In contrast, agreement between the TBT predictions, and the measured and simulated frequencies, is excellent for frequencies below cut-off, that is, $n \leq 4$ in Table 1. Above the cut-off frequencies, TBT predictions are increasingly greater than both the measured and simulated values. Thus for $n = 8$, the TBT prediction is nearly 4% greater than the measured value, which is approaching the limit of what might be considered acceptable as an engineering approximation (typically 5%, say), while the simulated values are just some 0.1% greater. For higher modes, $n > 8$, the TBT predictions greatly exceed the measured and simulated frequencies.

The question of when to disregard the TBT predictions is now answered according to the accuracy one requires. The results presented in Table 1 are quite consistent with Hutchinson's conclusion that the cut-off frequency is a reasonable choice. Thus for modes $n = 1$ to 5 (the latter slightly exceed the cut-off frequency) the TBT predictions are within about -0.2% and $+1.4\%$ of the measured and simulated frequencies, while for $n \geq 6$ the simulated frequencies are still in very good agreement with the measured, but the TBT predictions are increasingly inaccurate.

The following points are also noted:

1. For the first four mode pairs, $n = 1$ to 4 in Table 1, it is seen that the natural frequency in the stiff plane is slightly greater than in the flexible plane, as one would expect; this is so for the measured, simulated and TBT frequencies. However, this is not always the case for frequencies above the cut-off, that is for $n > 5$: all of the TBT, and some of the measured, RUS and ANSYS frequencies are now greater in the flexible plane. This counter-intuitive feature appears not to have been reported previously. Of course, our definition of *flexible* and *stiff* is based upon the magnitude of the bending stiffness EI and in turn the second moment of area, I, in the two planes. This is consistent with the Euler–Bernoulli frequency predictions $f_{EB} = k^2\sqrt{EI/\rho A}\Big/(2\pi)$, with $kL = 4.7300$, 7.8532, etc. for free–free end conditions, and TBT regarded as an improvement for these predominantly bending frequencies. On the other hand, the second moment of area also appears in the denominator of the expression for the (pure shear) cut-off frequency $f_{co} = \sqrt{\kappa AG/\rho I}\Big/(2\pi)$; in turn, the cut-off frequency is greater in the more flexible plane, and *vice versa*. The dichotomy arises because the second moment of area is a measure of both bending stiffness and rotatory inertia: a large value increases predominantly bending frequencies but decreases predominantly shear frequencies. T his suggests that all TBT modes of vibration above the cut-off frequency are dominated by shear, while the measured and simulated frequencies are consistent with some predominantly shear and some predominantly bending modes.

2. The 16 natural frequencies listed in Table 1 are those for which the ANSYS animations confirm the flexural nature of the mode; not shown are a further 11 natural frequencies within the frequency range covered in Table 1. Five of these are torsional, three extensional (or longitudinal), and two are for warping modes. One of the warping modes is evanescent, and is characterised by large axial displacements (reminiscent of that produced by a bi-moment) at the ends of the beam, which decay rapidly in magnitude as one moves toward the centre; its frequency is 165 450 Hz, which is greater than the two cut-off frequencies of TBT. One mode has displacements that are largely independent of the axial coordinate, with a shearing (lozenging) of the near-square cross-section into a diamond. None of the mode shapes bears any resemblance to the thickness-shear (independent of axial coordinate) mode, which is the TBT cut-off mode. Indeed it may be shown that free–free end conditions for TBT (zero shearing force and moment) are not consistent with the thickness-shear mode, which has a finite shearing force that does not vary along the length of the beam. Moreover, the exact linear elastodynamic theory requires zero traction on both the surface generators and the ends of the beam, and the thickness-shear mode [11] satisfies only the former; thus one should not expect the ANSYS simulations to show such a mode for free–free end conditions.
3. ANSYS predictions are not shown for frequencies greater than 200 kHz, as it becomes increasingly difficult to justify describing any of the abundance of vibration modes as bending or flexural, at least from the ANSYS animations. In turn, it is reasonable to question whether the measured frequencies are indeed truly flexural in their character. The procedure adopted to discriminate flexural modes involved the axial rotation of the specimen beam within the carbon-fibre loops; flexural modes were identified as those for which the intensity of the resonance changed from the stiff to the more flexible plane, and vice versa. However, this becomes increasingly difficult for the higher modes, and the measured frequencies are labelled only for $n \leq 8$.

Last, the authors accept that the present investigation is limited in its scope, and would encourage other researchers to conduct similar investigations so that a definitive view on the valid range of Timoshenko beam theory can be agreed; to this end, the ANSYS and MATLAB files are available from the corresponding author for scrutiny, and possible development.

4. Conclusions

A comparison has been made of the natural frequencies of bending vibration of a short free–free aluminium alloy beam, in order to test the valid frequency range of Timoshenko beam theory (TBT). Below the cut-off frequency, there is excellent agreement between the measured frequencies, and those predicted by TBT, finite element analysis (ANSYS) and a resonant ultrasound spectroscopy (RUS) technique. Above the cut-off frequency, there continues to be good agreement between the measured and simulated (ANSYS and RUS) frequencies, but the TBT predictions becomes increasing inaccurate. This is consistent with Hutchinson's conclusion [7] that "The cut-off frequency … appears to be a reasonable choice for an upper bound … on the Timoshenko solution." On the other hand, Stephen [2] has shown that first spectrum TBT predictions for a hinged–hinged beam are very accurate at frequencies well in excess of the cut-off frequency, while second spectrum predictions are accurate only at long wavelengths and, in general, should be disregarded. From the present work, it is concluded that TBT frequency predictions above the cut-off frequency should be disregarded as unreliable for all those end conditions for which the transcendental frequency equation does not factorise, as second spectrum contributions cannot be isolated and then disregarded.

References

[1] M. Levinson, D.W. Cooke, On the two frequency spectra of Timoshenko beams, *Journal of Sound and Vibration* 84 (1982) 319–326.
[2] N.G. Stephen, The second spectrum of Timoshenko beam theory—Further assessment, *Journal of Sound and Vibration* 292 (2006) 372–389.
[3] K.T. Chan, X.Q. Wang, R.M.C. So, S.R. Reid, Superposed standing waves in a Timoshenko beam, *Proceedings of the Royal Society A* 458 (2002) 83–108.

[4] R.A. Méndez-Sánchez, A. Morales, J. Flores, Experimental check on the accuracy of Timoshenko's beam theory, *Journal of Sound and Vibration* 279 (2005) 508–512.

[5] R.D. Mindlin, Influence of rotatory inertia and shear on the flexural motions of isotropic elastic plates, *Journal of Applied Mechanics.* 18 (1951) 31–38 (*ASME Transactions* vol. 73)

[6] N.G. Stephen, The second frequency spectrum of Timoshenko beams, *Journal of Sound and Vibration* 80 (1982) 578–582.

[7] J.R. Hutchinson, Transverse vibrations of beams, exact versus approximate solutions, *Journal of Applied Mechanics* 48 (1981) 923–928.

[8] H.H. Demarest, Cube-resonance method to determine the elastic constants of solids, *Journal of the Acoustical Society of America* 49 (1971) 768–775.

[9] R.G. Leisure, F.A. Willis, Resonant ultrasound spectroscopy, *Journal of Physics: Condensed Matter* 9 (1997) 6001–6029.

[10] W. Lins, G. Kaindl, H. Peterlik, K. Kromp, A novel resonant beam technique to determine the elastic moduli in dependence on orientation and temperature up to 2000 °C, *Review of Scientific Instruments* 70 (1999) 3052–3058.

[11] R.D. Mindlin, H. Deresiewicz, Timoshenko's shear coefficient for flexural vibration of beams, *Proceedings of the Second U.S. National Congress of Applied Mechanics*, Vol. 4, 1954, pp. 175–178.