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# INPLANE BUCKLING OF ANISOTROPIC RINGS

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## ABSTRACT

Inplane buckling of laminated rings is considered based on a non-linear theory for stretching and bending of geometrically and materially symmetric anisotropic beams having constant initial curvature in their plane of symmetry. The ring is formed by initially curving the laminated beam out of the plane of the laminate. For the kinematics, the geometrically exact one-dimensional (1-D) measures of deformation are specialized for small strain, and a 1-D constitutive law is developed via an asymptotically correct dimensional reduction of geometrically non-linear 3-D elasticity. The reduction assumes small strain and comparable magnitudes for the initial radius of curvature $R$ and the wavelength of deformation along the beam reference line. Other small parameters include the ratio of cross-sectional thickness $h$ to initial radius of curvature ($h/R$) and the ratio of cross-sectional thickness to cross-sectional width ($h/b$). In spite of a very simple final expression for the second variation of the total potential, it is shown that the only restriction on the validity of the buckling analysis is that the prebuckling strain remains small. The buckling load obtained exhibits features not found in published formulae.

**Keywords:** buckling, elastic stability, variational-asymptotic method, dimensional reduction, curved beam, ring

## INTRODUCTION

Elegant treatments for planar, large deflections of beams are presented in, for example, (Reissner 1972) and (Epstein and Murray 1976). Relatively few applications of this type of theory for stability analysis of initially curved beams appear in the literature; an analogous treatment for shell stability can be found in (Gellin 1980). The intent is to provide a geometrically-exact theory for this purpose with a minimum of ad hoc approximations. The theory developed herein is a special case of that which appears in (Reissner 1972), but it includes an asymptotic development of the 1-D constitutive law needed to have a complete theory. The asymptotic analysis works on the basis of small parameters related to the strain and the slenderness of the beam and closely follows (Hodges 1999).

We will start with a geometric description of the undeformed and deformed states of the beam. This includes the position vectors to an arbitrary material point and definitions of the reference line and reference cross section in both states. Geometrically-exact force and moment strain measures will be introduced, followed by an asymptotic reduction of the 3-D strain energy to 1-D. Buckling of rings under hydrostatic pressure will be considered as an example,

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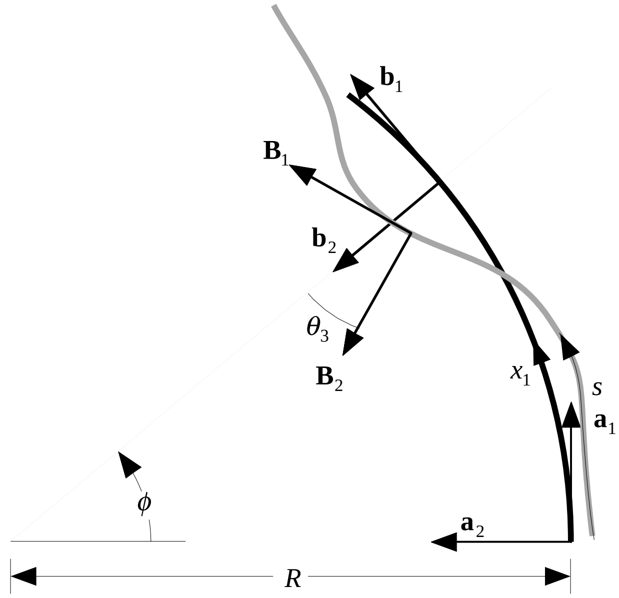


**FIG. 1. Schematic of Undeformed and Deformed Beam.**

and comparisons will be made with (Rasheed and Yousif 2001; Rasheed and Yousif 2002) who used a static condensation to get the effective stiffnesses and a displacement approach for the bifurcation analysis.

## 1-D STRAIN ENERGY

To form the strain energy of a planar, constant-curvature beam, we develop the geometries of both undeformed and deformed states following closely the treatment in (Hodges 1999). The beam is symmetric about the plane in which it is initially curved, and its displacement field is symmetric about that plane. We then make use of the variational-asymptotic method of Berdichevsky (Berdichevsky 1983) to reduce the 3-D strain energy to a 1-D functional for initially curved beams. This functional depends only on the geometrically-exact stretching and bending measures, which we specialize for the case of small strain.

### Undeformed State

Consider an initially curved beam with radius of curvature $R$ in its undeformed state. The undeformed beam reference line (the line of area centroids will suffice in this case) is shown as the dark, heavy line in Fig. 1. The position vector from some fixed point to an arbitrary point $p$ on the beam reference line is denoted by $\mathbf{r}(x_1)$, where $x_1 = R\phi$ is the arc-length coordinate along the undeformed beam reference line. Thus, we can write the position vector to a point in the undeformed beam as

$$\overline{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1) + x_2\mathbf{b}_2(x_1) + x_3\mathbf{b}_3 \tag{1}$$

where the undeformed beam base vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are functions of $x_1$ and where $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 = \mathbf{a}_3$ is not. Spatially-fixed base vectors are denoted by $\mathbf{a}_i$, for $i$ =1, 2, and 3, as shown in Fig. 1; note also that $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2$.

The relationship between these vectors is seen from the geometry to be

$$\begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{Bmatrix} \tag{2}$$

The unit vector tangent to the curve described by $\mathbf{r}(x_1)$ is

$$\frac{d\mathbf{r}}{dx_1} = \mathbf{r}' = \mathbf{b}_1 \tag{3}$$

where $(\ )' = d(\ )/dx_1$. The curvature vector for the undeformed state is $\frac{\mathbf{b}_3}{R}$ so that

$$\mathbf{b}_1' = \frac{\mathbf{b}_2}{R} \qquad \mathbf{b}_2' = -\frac{\mathbf{b}_1}{R} \tag{4}$$

**Deformed State**

The deformed state is a straightforward extension of the above. The position vector for the same material point in the deformed beam to which $\bar{\mathbf{r}}$ points in the undeformed beam is

$$\overline{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{R}(x_1) + x_2\mathbf{B}_2(x_1) + x_3\mathbf{B}_3 + w_i(x_1, x_2, x_3)\mathbf{B}_i(x_1) \tag{5}$$

where $w_i(x_1, x_2, x_3)$ is the displacement of points in the reference cross-sectional plane relative to the rigid-body displacement and rotation reflected by $\mathbf{R}(x_1)$ and $\mathbf{B}_i(x_1)$; for planar deformation $\mathbf{B}_3 = \mathbf{b}_3$, and the curvature vector for the deformed state is $\left(\frac{1}{R} + \kappa\right)\mathbf{b}_3$. In general, $w_i$ describes both in- and out-of-plane warping of the material points which make up the reference cross-sectional plane of the undeformed beam. These functions are not known a priori; they must be calculated subject to constraints which remove redundant degrees of freedom.

Following (Hodges 1999), for the present choice of the reference line, the four constraints on the warping are

$$\langle w_i \rangle = 0 \qquad \langle w_{2,3} - w_{3,2} \rangle = 0 \tag{6}$$

where $\langle\ \rangle$ represents the integral over the cross-sectional plane of the undeformed beam. Notice that our choice of constraints is not unique, but it is necessary that the constraints render the displacement field unique.

The strain is now defined based on the simplest small-local-rotation/small-strain approximation given by Danielson and Hodges (Danielson and Hodges 1987). This choice is appropriate for large deflection analysis of beams with closed cross sections, since there are no restrictions on the magnitudes of reference line displacement or on cross-sectional rotation. The strain components are expressed in a local Cartesian frame parallel to $\mathbf{b}_i$. For cross sections other than those that are open and thin-walled, it can be shown (Cesnik and Hodges 1997) that the first approximation of the warping is of the order $h\varepsilon$, where $h$ is the total thickness of the laminated beam; thus, the order of the maximum strain is $\varepsilon = \max(|\epsilon|, h|\kappa|)$. Now, assuming $(\ )'$ is the order of $(\ )/R$ (which requires that the wavelength of the deformation and $R$ are of the same order) and $h^2/R^2 << 1$, one can write the strain terms as

$$\begin{aligned}
\Gamma_{11} =& \frac{\epsilon - x_2\kappa + w_1' - \frac{w_2}{R}}{\sqrt{g}} \approx \epsilon - x_2\kappa + \underline{w_1' + (\epsilon - x_2\kappa)\frac{x_2}{R} - \frac{w_2}{R}} \\
2\Gamma_{12} =& w_{1,2} + \frac{w_2' + \frac{w_1}{R}}{\sqrt{g}} \approx w_{1,2} + \underline{w_2' + \frac{w_1}{R}} \\
2\Gamma_{13} =& w_{1,3} + \frac{w_3'}{\sqrt{g}} \approx w_{1,3} + \underline{w_3'} \\
\Gamma_{22} =& w_{2,2} \\
2\Gamma_{23} =& w_{2,3} + w_{3,2} \\
\Gamma_{33} =& w_{3,3}
\end{aligned} \tag{7}$$

where $\sqrt{g} = 1 - x_2/R$. Note that the nonunderlined terms in the far right-hand sides are $O(\varepsilon)$, and the underlined ones are $O(h\varepsilon/R)$.

**Dimensional Reduction**

To carry out the variational asymptotic dimensional reduction, one minimizes the dominant terms in the strain energy per unit length of the beam subject to the constraints in Eqs. (6). This yields expressions for $w_i$ that can be used to construct an expression for the strain energy per unit length that is valid through $O(Eh\varepsilon^2/R)$, where $E$ is a typical material modulus. This resulting energy expression is given by

$$U = \frac{b}{2}\int_{\mathcal{L}} \left[\left(\overline{A}_{11} - 2k_2\overline{B}_{11}\right)\epsilon^2 + 2\left(\overline{B}_{11} - k_2\overline{D}_{11}\right)\epsilon\kappa + \overline{D}_{11}\kappa^2\right] dx_1 \tag{8}$$

where $\mathcal{L}$ is the domain of $x_1$ and the stiffness variables (the quantities with an overbar) are defined as

$$\begin{aligned} \overline{A}_{11} &= A_{11} + \tfrac{{A_{16}}^2 A_{22} - 2A_{12}A_{16}A_{26} + {A_{12}}^2 A_{66}}{{A_{26}}^2 - A_{22}A_{66}} \\ \overline{B}_{11} &= B_{11} + \tfrac{A_{12}A_{66}B_{12} + A_{16}A_{22}B_{16} - A_{26}(A_{16}B_{12} + A_{12}B_{16})}{{A_{26}}^2 - A_{22}A_{66}} \\ \overline{D}_{11} &= D_{11} + \tfrac{A_{66}{B_{12}}^2 - 2A_{26}B_{12}B_{16} + A_{22}{B_{16}}^2}{{A_{26}}^2 - A_{22}A_{66}} \end{aligned} \tag{9}$$

The laminate thickness direction is along $x_2$ locally, and to maintain the type of symmetry described above, it is necessary to require that

$$\begin{aligned} B_{16} + \tfrac{A_{12}A_{66}B_{26} + A_{16}A_{22}B_{66} - A_{26}(A_{16}B_{26} + A_{12}B_{66})}{{A_{26}}^2 - A_{22}A_{66}} &= 0 \\ D_{16} + \tfrac{A_{66}B_{12}B_{26} + A_{22}B_{16}B_{66} - A_{26}(B_{16}B_{26} + B_{12}B_{66})}{{A_{26}}^2 - A_{22}A_{66}} &= 0 \end{aligned} \tag{10}$$

The details of these operations follow closely with (Hodges 1999); they and the resulting warping will be included in a later paper.

It is important to note that the stiffness measures have contributions from the laminate properties and from the initial curvature of the beam. The latter corrections are $O(h/R)$ relative to the leading terms. The only approximations in the dimensional reduction are thus $\varepsilon << 1$ and $h^2/R^2 << 1$. Later it will be shown that these conditions dovetail into one condition for ring- and high-arch-buckling problems. The next approximation would produce terms in the 1-D energy that are $O(h^2/R^2)$ relative to the leading terms. These are associated with large initial curvature and transverse shear effects, not considered in the present treatment. Finally, although one can formally write the asymptotically correct 3-D strain field to $O(h\varepsilon/R)$, without calculation of $O(h/R)$ perturbations to the warping the present analysis only allows recovery of stresses to $O(E\varepsilon)$.

**1-D Strain-Displacement Relations**

Following the treatment of (Hodges 1999), the 1-D strain-displacement relations are

$$\begin{aligned} \epsilon &= \sqrt{\left(1 + u_1' - \frac{u_2}{R}\right)^2 + \left(u_2' + \frac{u_1}{R}\right)^2} - 1 \\ \kappa &= \left(1 + u_1' - \frac{u_2}{R}\right)\left(u_2'' + \frac{u_1'}{R}\right) - \left(u_2' + \frac{u_1}{R}\right)\left(u_1'' - \frac{u_2'}{R}\right) \end{aligned} \tag{11}$$

Other than small stretching strain $\epsilon << 1$, there are no approximations in the 1-D variables.

**Final 1-D Strain Energy**

The resulting strain energy per unit length can be written as

$$\begin{aligned}\overline{U}_1^* =& \frac{1}{2}\left(S_{11}\epsilon^2 + 2S_{12}\epsilon\kappa + S_{22}\kappa^2\right)\\ =& \frac{D}{2}\left(A\epsilon^2 + 2hB\epsilon\kappa + h^2\kappa^2\right)\end{aligned} \tag{12}$$

where

$$A = \frac{\overline{A}_{11} - 2b\overline{B}_{11}k_2}{D}; \qquad B = \frac{b\overline{B}_{11} - bk_2\overline{D}_{11}}{hD}; \qquad D = \frac{b\overline{D}_{11}}{h^2} \tag{13}$$

As one can see, the strain energy density becomes quite complicated when Eqs. (11) are substituted into Eq. (12). There are many problems for which the result does become tractable, however, and for this reason this approach is to be preferred over ad hoc approaches in which one cannot easily assess the error associated with particular approximations.

## POTENTIAL ENERGY OF APPLIED PRESSURE LOADING

In anticipation of applying the above theory to inplane buckling, here we make use of the potential energy developed in (Hodges 1999) which was proved to be valid for cases in which the ends of the beam are not allowed to displace, or if the beam is a closed ring, for which the ends are joined so that $u_1(\ell)\delta u_2(\ell) = u_1(0)\delta u_2(0)$. This functional is

$$V = -Rf_2\int_{-\alpha}^{\alpha}\left(u_2 - \frac{u_1^2}{2R} - \frac{u_2^2}{2R} - u_1u_2'\right)d\phi \tag{14}$$

where $\alpha = \pi$ for rings and where $f_2$ is the force per unit length of the deformed beam directed along $\mathbf{B}_2$.

## APPLICATIONS

The buckling of circular rings is considered as an application and will be developed from the total potential energy. To facilitate this analysis, it is now helpful to nondimensionalize the equations. This we do by dividing through the total potential $U + V$ by $DR$ while simultaneously changing the meaning of certain symbols. We replace $u_1$ and $u_2$ by $Ru_1$ and $Ru_2$, respectively; we replace $\kappa$ by $\kappa/R$; and we finally let $(\ )'$ denote $d(\ )/d\phi$. We also introduce the new symbols $\rho^2 = h^2/R^2$ and $\lambda = f_2R^3/D$. All these operations yield, for the nondimensional total potential $\Phi = (U+V)/(DR)$

$$\Phi = \int_{-\alpha}^{\alpha}\left[\frac{A\epsilon^2}{2} + B\rho\epsilon\kappa + \frac{\rho^2\kappa^2}{2} - \lambda\rho^2\left(u_2 - \frac{u_1^2}{2} - \frac{u_2^2}{2} - u_1u_2'\right)\right]d\phi \tag{15}$$

where

$$\epsilon = \sqrt{\left(1 + u_1' - u_2\right)^2 + \left(u_2' + u_1\right)^2} - 1 \tag{16}$$

and

$$\kappa = \left(1 + u_1' - u_2\right)\left(u_2'' + u_1'\right) - \left(u_2' + u_1\right)\left(u_1'' - u_2'\right) \tag{17}$$

Note that $\rho^2 << 1$. It is helpful, before proceeding further, to rewrite $\kappa^2$ in a more compact way. To do so, we make use of the result in (Hodges 1999) that

$$\kappa^2 = \left(u_2'' + u_1'\right)^2 + \left(u_1'' - u_2'\right)^2 \tag{18}$$

**Buckling of Rings**

For the first application we consider the buckling of rings.

*Prebuckled state*

In the prebuckled state, we note that the ring remains circular so that all derivatives with respect to $\phi$ vanish. Denoting the prebuckled state variables with overbars and noting that $\overline{u}_2$ is the only nonzero variable, we find that $\overline{\epsilon} = -\overline{u}_2$, $\overline{\kappa} = 0$, and the functional reduces to

$$\overline{\Phi} = \int_{-\alpha}^{\alpha} \left[ \frac{A\overline{u}_2^2}{2} - \lambda\rho^2 \left( \overline{u}_2 - \frac{\overline{u}_2^2}{2} \right) \right] d\phi \tag{19}$$

from which we find, upon equating the variation to zero

$$\overline{u}_2 = \frac{\lambda\rho^2}{A + \lambda\rho^2} \tag{20}$$

Here let us make an important observation: the strain in the prebuckled state $\overline{\epsilon} = -\overline{u}_2$ is of the order of $\rho^2$. So, for a consistent small-strain analysis we may ignore $\rho^2$ with respect to unity, so that

$$\overline{\epsilon} = -\overline{u}_2 = -\frac{\lambda\rho^2}{A + \lambda\rho^2} = -\frac{\lambda\rho^2}{A} \tag{21}$$

To improve on this analysis we would need to keep $\rho^2$ compared to unity everywhere, which is much more complicated. If we restrict the discussion to slender rings with prebuckling strain that is small compared to unity, this leads to great simplifications in analysis.

*Buckling analysis*

To further simplify the total potential, we consider that the perturbations of the prebuckled state at the onset of buckling can be regarded as arbitrarily small. We need to keep all terms of power 1 and 2 in perturbations of $\Phi$. Using the concept of the Taylor series to make certain all such terms are retained, we note that

$$\begin{aligned} \epsilon &= \overline{\epsilon} + \hat{\epsilon}_1 + \hat{\epsilon}_2 \\ \kappa &= \hat{\kappa}_1 + \hat{\kappa}_2 \end{aligned} \tag{22}$$

The subscripts indicate the power of the perturbation displacements. Because of the nonzero value of $\overline{\epsilon}$, we need both first and second order terms. For small strain, we find

$$\begin{aligned} \hat{\epsilon}_1 &= \hat{u}_1' - \hat{u}_2 \\ \hat{\epsilon}_2 &= \frac{1}{2(1+\overline{\epsilon})} \left( \hat{u}_2' + \hat{u}_1 \right)^2 = \frac{1}{2} \left( \hat{u}_2' + \hat{u}_1 \right)^2 \\ \hat{\kappa}_1 &= \hat{u}_2'' + \hat{u}_1' \\ \hat{\kappa}_2 &= \left( \hat{u}_1' - \hat{u}_2 \right) \left( \hat{u}_2'' + \hat{u}_1' \right) + \left( \hat{u}_2' + \hat{u}_1 \right) \left( \hat{u}_1'' - \hat{u}_2' \right) \end{aligned} \tag{23}$$

Now we can write the perturbations of the energy. First, keeping only the terms including first powers of the $\hat{(\ )}$ quantities, we obtain

$$\hat{\Phi}_1 = \int_{-\alpha}^{\alpha} \left[ A\overline{\epsilon}\hat{\epsilon}_1 + B\rho\overline{\epsilon}\hat{\kappa}_1 - \lambda\rho^2(1+\overline{\epsilon})\hat{u}_2 \right] d\phi \tag{24}$$

the variation of which is identically zero for rings, as expected.

Now, let us consider the second-order terms (which amounts to a second variation):

$$\begin{aligned}\hat{\Phi}_2 =& \frac{1}{2}\int_{-\alpha}^{\alpha}\Big[2A\bar{\epsilon}\hat{\epsilon}_2 + A\hat{\epsilon}_1^2 + 2B\rho\bar{\epsilon}\hat{\kappa}_2 + 2B\rho\hat{\epsilon}_1\hat{\kappa}_1 + \rho^2\left(\hat{u}_2'' + \hat{u}_1'\right)^2 \\ &+ \rho^2\left(\hat{u}_1'' - \hat{u}_2'\right)^2 + \lambda\rho^2\left(\hat{u}_1^2 + \hat{u}_2^2 + 2\hat{u}_1\hat{u}_2'\right)\Big]d\phi\end{aligned} \tag{25}$$

The $\hat{\epsilon}_1^2$ term does not contain $\rho$. When $\bar{\epsilon} = -\lambda\rho^2/A$ is substituted into Eq. (25), the third term becomes $O(\rho^3)$. The remaining terms in $\hat{\Phi}_2$ are proportional to $\rho$ and powers thereof. The $O(\rho^0)$ term must be killed. Minimization of $\hat{\Phi}_2$ with respect to $\hat{u}_1$ shows that

$$\hat{u}_2 = \hat{u}_1' + \rho v \tag{26}$$

where $v$ is an unknown function that is $O(\rho^0)$. Substitution of Eq. (26) into $\hat{\Phi}_2$ yields a functional that is $O(\rho^2)$. Variation of that resulting functional with respect to $v$ yields

$$v = \frac{B}{A}\left(\hat{u}_1''' + \hat{u}_1'\right) + \rho w \tag{27}$$

where $w$ is another unknown function that is $O(\rho^0)$. When this expression for $v$ is substituted into the energy functional, then all terms $O(\rho^2)$ and $O(\rho^3)$ are functions only of $\hat{u}_1$ and its derivatives. Writing $\lambda = \lambda_0 + \rho\lambda_1$, one can find a Rayleigh quotient for $\lambda_0$ from the $O(\rho^2)$ terms given by

$$\lambda_0 = \frac{(A - B^2)\int_{-\alpha}^{\alpha}(\hat{u}_1''' + \hat{u}_1')^2 d\phi}{A\int_{-\alpha}^{\alpha}\left(\hat{u}_1''^2 - \hat{u}_1'^2\right)d\phi} \tag{28}$$

The $O(\rho^3)$ terms can then, in turn, be used to find an expression for $\lambda_1$ given by

$$\lambda_1 = \frac{-2B(A - B^2)\int_{-\alpha}^{\alpha}\left[(\hat{u}_1' + \hat{u}_1''')(3\hat{u}_1' + \hat{u}_1''' + \hat{u}_1^{v}) - 3\hat{u}_1''\left(\hat{u}_1'' + \hat{u}_1^{iv}\right)\right]d\phi}{A^2\int_{-\alpha}^{\alpha}\left(\hat{u}_1''^2 - \hat{u}_1'^2\right)d\phi} \tag{29}$$

In spite of the simplicity of these results, the only approximation employed was that $\bar{\epsilon} << 1$, which, because of the prebuckling state, is equivalent to $\rho^2 << 1$.

Using the expressions for the Rayleigh quotient and assuming that $\hat{u}_1 = \sin m\phi$, one finds that the minimum of $\lambda_0$ is at $m = 2$ and

$$\lambda_0 = \frac{3(A - B^2)}{A} \tag{30}$$

which is in agreement with published results (Simitses 1976) for the isotropic case, where $B = 0$. The correction term is

$$\lambda_1 = -\frac{6B(A - B^2)}{A^2} \tag{31}$$

The result in (Rasheed and Yousif 2001; Rasheed and Yousif 2002) was not obtained via an asymptotic method. Nevertheless, to ascertain to what extent our result is in agreement with theirs, one can expand their expression in terms of the small parameter $\rho$. Doing so, one finds that the $O(\rho^0)$ terms are in agreement, but the $O(\rho^1)$ terms are different because our variables $A$ and $B$ have corrections that are $O(\rho^1)$. Numerical differences because of these corrections are not known at this stage but will be explored in a later paper.

For the buckling analysis of high arches, one may follow the usual approach of assuming that the boundary conditions are such that the displacements in the prebuckled state are the same as those for a ring with the same values of $\lambda$, $A$, $B$, and $\rho$. This has the effect of simplifying the analysis of the prebuckled state, but it does not affect the resulting bifurcation load. For those cases described in (Simitses 1976), one can verify that Eq. (28) provides an upper bound for the published symmetric or antisymmetric buckling load when either symmetric or antisymmetric admissible or comparison functions are substituted therein. Eq. (29) provides an approximation for the $O(\rho)$ correction term.

## CONCLUDING REMARKS

An inplane buckling analysis for laminated composite rings is considered based on a non-linear theory for stretching and bending of geometrically and materially symmetric anisotropic beams having constant initial curvature in their plane of symmetry. The ring is formed by initially curving the laminate out of its plane. The theory for the buckling analysis is derived from geometrically non-linear 3-D elasticity for deformation of such beams in the plane of their symmetry. The dimensional reduction is performed via the variational-asymptotic method. The resulting theory is subject only to the restrictions that the strain and the ratio $\rho^2 = h^2/R^2$ are small compared to unity. The theory contains a term in the 1-D strain energy which couples stretching and bending and both this term and the stretching energy depend on $\rho$.

When applied to the buckling of rings, the theory shows that the prebuckling strain is of the order of $\rho^2$. This means that there is really only one restriction on the theory for this application, that of prebuckling strain being small compared to unity. The buckling analysis which follows is quite simple, boiling down to the minimization of a single functional. This buckling analysis follows from a theory that has fewer restrictions, and exhibits a considerably simpler final form, than those typically found in textbooks. Finally, it is shown that the buckling load depends on the $O(\rho)$ corrections to the stretching energy and the stretch-bending coupling not present in published treatments of this problem.

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