Rigid-Body Contribution to the Generalized Active Force

Georgia Institute of Technology
School of Aerospace Engineering
AE 6210: Advanced Dynamics I

Prof. Dewey H. Hodges

Consider the rigid rod R with cross-sectional area A, partially submerged in a fluid of weight density w, as shown in Fig. 1. The fluid exerts an upward force on R equivalent to a single force

$$\mathbf{F} = -Awy \sec\theta \mathbf{a}_2 \tag{1}$$

applied at R^*, the geometric center of the submerged portion.

The contribution to the generalized active forces can be found as follows: Recognize that $\mathbf{F}$ applied at R^* is equivalent to $\mathbf{F}$ applied at E, the bottom end of R, plus a couple of torque

$$\mathbf{T} = -\frac{Awy^2}{2} \sec\theta \tan\theta \mathbf{a}_3 \tag{2}$$

where $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2$. The partial velocities of E can be found from the velocity of E in A given by

$$^A\mathbf{v}^E = \dot{x}\mathbf{a}_1 + \dot{y}\mathbf{a}_2 \tag{3}$$

and the partial angular velocities from the angular velocity of R in A given by

$$^A\boldsymbol{\omega}^R = \dot{\theta}\mathbf{a}_3 \tag{4}$$

Taking the generalized speeds as

$$u_1 = \dot{x} \qquad u_2 = \dot{y} \qquad u_3 = \dot{\theta} \tag{5}$$

one obtains nonzero values for the partial velocities and partial angular velocities as

$$^A\mathbf{v}_1^E = \mathbf{a}_1 \qquad ^A\mathbf{v}_2^E = \mathbf{a}_2 \qquad ^A\boldsymbol{\omega}_3^R = \mathbf{a}_3 \tag{6}$$

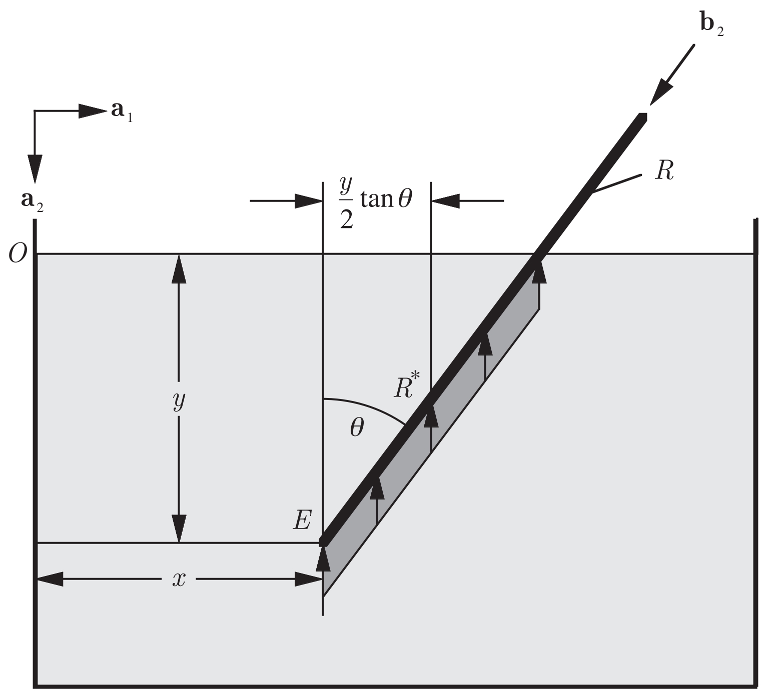

Figure 1: Schematic of a rod R partially submerged in fluid

so that

$$\begin{aligned} F_1 &= 0 \\ F_2 &= -Awy\sec\theta \\ F_3 &= -\frac{Awy^2}{2}\sec\theta\tan\theta \end{aligned} \tag{7}$$

Cannot one simply work with the force at R^*? To do this we note that

$$^A\mathbf{v}^{R^*} = (\ )\mathbf{a}_1 + \frac{\dot{y}}{2}\mathbf{a}_2 \tag{8}$$

Note that the $\mathbf{a}_1$ component is a bit messy, but since the horizontal component of force is zero we do not need to determine it. Thus

$$^A\mathbf{v}_2^{R^*} = (\ )\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2 \tag{9}$$

Since there is zero moment about this point

$$\begin{aligned} F_1 &= 0 \\ F_2 &= -\frac{Awy}{2}\sec\theta \\ F_3 &= 0 \end{aligned} \tag{10}$$

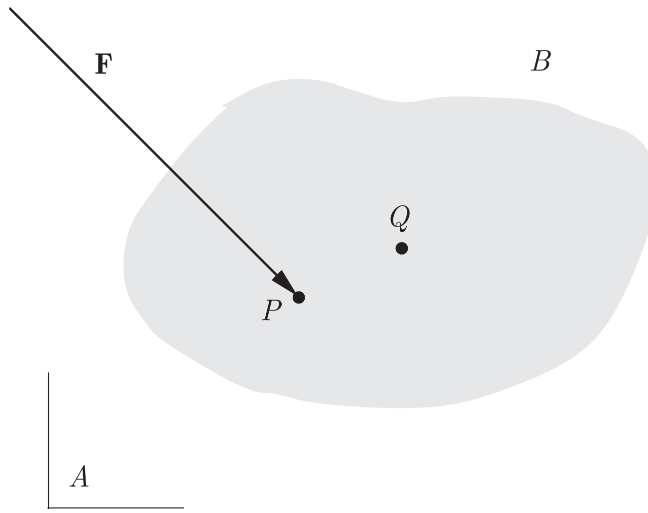

Figure 2: Schematic of a rigid body B moving in a frame A

which are obviously different (and wrong!). Why?

The contribution of a system of forces acting on a rigid body to the generalized active forces is given by Eq. 5.5.2 on page 113 of DTAoKM as

$$\left(\widetilde{F}_r\right)_B = {}^A\widetilde{\boldsymbol{\omega}}_r^B \cdot \mathbf{T} + {}^A\widetilde{\mathbf{v}}_r^Q \cdot \mathbf{R} \qquad (r = 1, 2, \ldots, p) \tag{11}$$

The following remarks concern the application of this formula:

1. This represents the contribution for only one rigid body. If there are additional rigid bodies that make up the system, this equation must be applied to each one in turn and the results summed over all bodies that make up the system under consideration. This is the case with problem 8.4, for example.

2. Expressions for the resultant of the set of applied forces and the moment of the set of applied forces about the point Q must be valid for the entire time interval during which the resulting expressions for the generalized active force are to be applicable.

3. Often the most convenient point to choose for Q is one about which the moment of the set of forces can be obtained simply. Although the equation is valid for a rigid body, this is not the only equation that can be used to obtain the desired results. For example, there are cases in which it is preferable to work with individual applied forces and the partial velocities of points along the lines of action of the individual forces.

4. The point Q must be fixed in the body.

The fourth item is not satisfied in the second approach to the present example problem, and the violation of it is the reason for the conflict above. Let us now derive a formula for the case of a moving load. Consider a rigid body B moving in a frame A with a force $\mathbf{F}$ acting on it at a point P that is moving in B as shown in Fig. 2. A point Q fixed in B is also shown. The velocity of P in A can be determined to be

$$ {}^{A}\mathbf{v}^{P} = {}^{A}\mathbf{v}^{Q} + {}^{A}\boldsymbol{\omega}^{B} \times \mathbf{p}^{QP} + {}^{B}\mathbf{v}^{P} \tag{12} $$

Now, according to Eq. 4.6.1 we can write

$$ \begin{aligned} \widetilde{F}_r =& \mathbf{F} \cdot \left({}^{A}\widetilde{\mathbf{v}}_r^{P} - {}^{A}\widetilde{\boldsymbol{\omega}}_r^{B} \times \mathbf{p}^{QP} - {}^{B}\widetilde{\mathbf{v}}_r^{P} \right) + \left(\mathbf{p}^{QP} \times \mathbf{F} \right) \cdot {}^{A}\widetilde{\boldsymbol{\omega}}_r^{B} \\ =& \mathbf{F} \cdot \left({}^{A}\widetilde{\mathbf{v}}_r^{P} - {}^{B}\widetilde{\mathbf{v}}_r^{P} \right) \end{aligned} \tag{13} $$

where the two triple-scalar-product terms cancel out.

Now, let's return to the example problem. In addition to the above we need ${}^{R}\mathbf{v}_r^{R^*}$. To get it, we note that

$$ {}^{R}\mathbf{v}^{R^*} = -\dot{\delta}\mathbf{b}_2 \tag{14} $$

where

$$ \delta = \frac{y}{2} \sec\theta \tag{15} $$

and

$$ \mathbf{b}_2 = -\mathbf{a}_1 \sin\theta + \mathbf{a}_2 \cos\theta \tag{16} $$

Thus,

$$ {}^{R}\mathbf{v}^{R^*} = (\ )\mathbf{a}_1 - \frac{1}{2}\left(\dot{y} + y\dot{\theta}\tan\theta \right) \mathbf{a}_2 \tag{17} $$

This gives

$$ \begin{aligned} {}^{A}\mathbf{v}_1^{R^*} - {}^{R}\mathbf{v}_1^{R^*} =& (\ )\mathbf{a}_1 \\ {}^{A}\mathbf{v}_2^{R^*} - {}^{R}\mathbf{v}_2^{R^*} =& (\ )\mathbf{a}_1 + \mathbf{a}_2 \\ {}^{A}\mathbf{v}_3^{R^*} - {}^{R}\mathbf{v}_3^{R^*} =& (\ )\mathbf{a}_1 + \frac{y\mathbf{a}_2}{2}\tan\theta \end{aligned} \tag{18} $$

so that the correct values, i.e.,

$$ \begin{aligned} F_1 &= 0 \\ F_2 &= -Awy\sec\theta \\ F_3 &= -\frac{Awy^2}{2}\sec\theta\tan\theta \end{aligned} \tag{19} $$

are obtained. (It turns out that the $\mathbf{a}_1$ component of the second of Eqs. (18) is zero.)