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Inverse Dynamics of Servo-Constraints Based on the Generalized Inverse

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Abstract. The acceleration form of constraint equations is utilized in this paper to solve for the inverse dynamics of servo-constraints. A condition for the existence of control forces that enforce servo-constraints is derived. For overactuated dynamical systems, the generalized Moore-Penrose inverse of the constraint matrix is used to parameterize the solutions for these control forces in terms of free parameters that can be chosen to satisfy certain requirements or optimize certain criterions. In particular, these free parameters can be chosen to minimize the Gibbsian (i.e., the acceleration energy of the dynamical system), resulting in "ideal" control forces (those satisfying the principle of virtual work when the virtual displacements satisfy the servo-constraint equations). To achieve this, the nonminimal nonholonomic form recently derived by the authors in the context of Kane's method is used to determine the accelerations of the system, and hence to determine the forces to be generated by the redundant manipulators. Finally, an extension to inverse dynamics of servo-constraints involving control variables is made. The procedures are illustrated by two examples.

Key words: generalized inverse, Kane's equations, redundancy resolution, servo-constraints, spacecraft stabilization

1. Introduction

When it is required that a dynamical system performs tasks that can be stated in terms of its configuration and velocities, then these requirements can be modeled as constraints on the dynamics of the system, called servo-constraints (also called active constraints, program constraints, or constraints of the second type). Examples are trajectories that dynamical systems are required to track, and manifolds in the configuration–motion space in which they are required to be.

In viewing servo-constraints from the perspective of analytical dynamics, it is noticed that they differ from passive constraints in several aspects. Mainly, they are enforced by means of control forces, rather than by material objects that exist in the environment of the dynamical system. Therefore, servo-constraints need not be ideal [1]. Also, they are not restricted in terms of the order of the constraint equations [2], they may or may not be satisfied by the dynamical system during the whole course of motion, and they can be nonlinear nonholonomic. Among the earliest works on servo-constraints are [3–5] Later works are cited in [1].

The purpose of this paper is to develop the tools for realizing this type of constraints. In particular, the acceleration form of constraint equations is utilized together with the Moore–Penrose generalized inverse for the purpose of dynamical stabilization of servo-constraints.

The ability to realize servo-constraints is directly related to the controllability of the dynamical system. The issue of whether a point in a servo-constraint manifold belongs to a controllable subspace of the dynamical system is addressed in terms of a simple condition on two related matrices. The required controls are found with the aid of the generalized inverse of one of these matrices.

Another subject is obtaining the ideal form of servo-constraints for the purpose of emulating passive constraints and solving the redundancy resolution of redundant manipulators. In this paper, the constrained full order state-space model derived previously by the authors [6, 7] is utilized for this purpose. An illustrative example is presented.

If the servo-constraint equations involve control variables also, then the acceleration form of these equations can be used together with the Moore–Penrose generalized inverse of the servo-constraint matrix to add a new set of differential equations, equal in number to the number of control variables. This forms an augmented, separated-in-accelerations, state-space model, in which the states become the configuration parameters, the velocity parameters, and the control variables.

2. Servo-Constraint Realization

Servo-constraints realization is the problem of moving the state of the dynamical system in a pre-specified constraint manifold with the aid of the available control forces. In this section, the acceleration form of constraints is used to solve for the forces required to realize servo-constraints, where the generalized inverse of the constraint matrix derived from the servo-constraint equations is the one utilized to express the redundancy. The considered dynamical equations of motion of a controlled dynamical system is of the form

$$\dot{q} = C(q,t)u + D(q,t) \tag{1}$$

$$Q(q,t)\dot{u} = P(q,u,t) + G(q,u,t)\tau, \tag{2}$$

where $q, u \in \mathbb{R}^n$ denote the column matrices containing the configuration parameters and the velocity parameters, $\dot{q}$ and $\dot{u}$ are the derivatives of q and u with respect to t, respectively. The square matrices involved in the two equations above are $C, Q \in \mathbb{R}^{n\times n}$, such that C^{-1}, Q^{-1} exist for all generalized coordinates and for all $t \in \mathbb{R}$. The control matrix $G \in \mathbb{R}^{n\times l}$ is such that $l \leq n$, and the column matrices $D, P \in \mathbb{R}^n$. The column matrix $\tau \in \mathbb{R}^l$ contains the control variables. Equations (1) and (2) form a complete state-space model.

Assume that the dynamical system is required to track the prescribed velocity-dependent trajectory described by the m nonholonomic constraint equations

$$\psi(q,u,t) = 0, \quad \psi \in \mathbb{R}^m, \tag{3}$$

where the servo-constraints ψ may be multi-objective, i.e., represent simultaneous requirements. The purpose is to find the control forces that are necessary to enforce the above equations, and to relate them to the available control authority. The acceleration form of the servo-constraint equations is

$$\dot{\psi}(q,u,\dot{u},t) = \frac{\partial\psi}{\partial u}\dot{u} + X(q,u,t), \tag{4}$$

where X is found from Equation (1) to be

$$X(q,u,t) = \frac{\partial\psi}{\partial q}C(q,t)u + \frac{\partial\psi}{\partial q}D(q,t) + \frac{\partial\psi}{\partial t}. \tag{5}$$

The following modified constraint equations at the acceleration level are considered

$$\dot{\psi}(q, u, \dot{u}, t) - \Theta\psi(q, u, t) = 0, \tag{6}$$

where $\Theta \in \mathbb{R}^{m \times m}$ is a prescribed matrix that has strictly negative-real eigenvalues. Substituting $\dot{u}$ from Equations (2)–(6) yields

$$S(q, u, t)\tau = z(q, u, t), \tag{7}$$

where

$$S = \frac{\partial \psi}{\partial u} Q^{-1} G \tag{8}$$

$$z = -\frac{\partial \psi}{\partial u} Q^{-1} P - X + \Theta\psi. \tag{9}$$

If the above system of equations is consistent at some specific values of configuration parameters and velocity parameters, i.e., z, is in the range space of S, then it is possible to solve for τ,

$$\tau = S^{+} z + (I - S^{+} S) y, \tag{10}$$

where the superscript "+" refers to the Moore–Penrose generalized inverse, and $y \in \mathbb{R}^{l}$ is arbitrary. Therefore, depending on the nature of S and z, the servo-constraints realization problem can be categorized as one of the following:

1. *The problem has a unique solution*: z is in the range space of S, and $l \leq m$.
2. *The problem has no solution*: z is not in the range space of S.
3. *The problem has infinite number of solutions*: z is in the range space of S, and the null space of S^{T} is not trivial. In this case, the flexibility provided by y can be used to achieve further requirements beside realization of servo-constraints.

The procedure for enforcing servo-constraints, Equation (3), is summarized in the following steps:

1. The expression for $\dot{u}$ obtained from the nonminimal form is substituted in Equation (4).
2. The resulting expression for $\dot{\psi}$ is used to form Equation (6), where Θ is chosen such that the first-order servo-constraint dynamics is stable. Equation (6) is put in the form of Equation (7).
3. Using the generalized Moore–Penrose inverse of S, the expression for τ, Equation (10), is formed, where the column matrix y can be chosen arbitrarily.

A similar treatment for holonomic servo-constraints can be done. In this case, the servo-constraint equations take the form

$$\psi(q, t) = 0. \tag{11}$$

The above equations are twice differentiated, and the desired dynamics takes the form

$$\ddot{\psi}(q, u, \dot{u}, t) - \Theta_1\dot{\psi}(q, u, t) - \Theta_2\psi(q, t) = 0, \tag{12}$$

where Θ_1 and Θ_2 are chosen such that the servo-constraints dynamics is stable. The above equations can be put in the form of Equation (7), from which the procedure follows.

3. Redundancy Resolution

If the number of independent actuators is more than the necessary to enforce servo-constraints, then the set of required control forces is not unique. This redundancy has been studied extensively for over three decades [8] in the area of robotics, at both kinematic and dynamic levels. The Jacobian matrix of the manipulators and its generalized inverse are the main tools in these studies.

The redundancy resolution problem is concerned with finding the control forces that are necessary to enforce predetermined dynamics of the system based on optimizing some criteria, such as the required control effort [9], the kinetic energy of the mechanism [10], or the distance from a desired trajectory [11]. This dynamics may involve holonomic and/or nonholonomic constraints. Also, the desired motion can be a combination of several requirements, e.g., tracking some prescribed trajectory while preserving the total energy of the dynamical system.

The equations of motion derived in the previous section can be used to solve the inverse dynamics for the natural control forces, i.e., those equivalent to passive joint reactions. This is equivalent to minimizing the instantaneous acceleration energy of the dynamical system relative to its unconstrained status, at every configuration and velocity [12]. In doing that, the accelerations of the generated nonminimal constrained model [6] and the controlled equations of motion are matched. Equating the expressions of $\dot{u}$ from the nonminimal form and Equation (2) yields

$$G(q, u, t)\tau = \mathcal{R}(q, u, t), \tag{13}$$

where

$$\mathcal{R}(q, u, t) = QT^{-1}F - P \tag{14}$$

and where T and F are the constrained generalized inertia matrix and load matrix, respectively, satisfying

$$T\dot{u} = F. \tag{15}$$

For some specific values of configuration parameters and velocity parameters, if the matrix $\mathcal{R}$ is in the range space of G, then there exists a solution of τ that is given by

$$\tau = G^{+}\mathcal{R}, \tag{16}$$

where $G^{+}G = I$ holds true because $l \leq n$. If the dynamical system is fully-actuated, i.e., the number of independent control variables is equal to the number of degrees of freedom, we have that $l = n$, and the matrix G is of full-rank. In this case, τ is given by

$$\tau = G^{-1}\mathcal{R}. \tag{17}$$

Remark. If the solution to this inverse dynamics problem exists, then the solution is unique. The control forces $G\tau$ in Equation (13) compensate for reaction forces that correspond to equivalent passive constraints on the dynamical systems. This implies that these control forces satisfy d'Alembert's principle, and the accelerations of the controlled system satisfy Gauss' principle of least constraints.

The inverse dynamics for the ideal control forces can be viewed as a specialization of the servo-constraints realization problem, where the matrices G and $\mathcal{R}$ stand for S and z, respectively, and the second term in the right-hand side of Equation (10) vanishes because $G^+G = I$. Since the nonminimal nonholonomic form that is used to obtain $\mathcal{R}$ results in the accelerations of an equivalent passively constrained system, the interaction between the servo-constraints and the dynamics of the system is ideal, i.e., the reaction forces are normal to the constraint manifold, with the corresponding virtual displacements satisfying the servo-constraint equations. The procedure for solving the inverse dynamics problem by using the nonminimal nonholonomic form is summarized in the following steps:

1. The expressions for $\dot{u}$ obtained from the nonminimal form and Equation (2) are equated, resulting in Equation (13).
2. Equation (13) is solved for τ, resulting in expressions (16) or (17), depending on the degree of actuation of the dynamical system.

4. Example: Double Pendulum

Considering the mechanism shown in Figure 1. It is required to determine the necessary controls to bring the total energy of the mechanism to a prescribed value E_0.

Let the configuration parameters be θ_1 and θ_2, and the velocity parameters be ${}^{\mathcal{N}}\omega^D$ and ${}^{\mathcal{N}}\omega^B$, Equation (1) for the mechanism becomes

$$\dot{\theta}_1 = {}^{\mathcal{N}}\omega^D \tag{18}$$

$$\dot{\theta}_2 = {}^{\mathcal{N}}\omega^B, \tag{19}$$

and the matrices Q, P, and G in Equation (2) are

$$Q = \begin{bmatrix} m_B R^2 + I_D & m_B \frac{L}{2} R \cos(\theta_1 - \theta_2) \\ m_B \frac{L}{2} R \cos(\theta_1 - \theta_2) & m_B \frac{L^2}{4} + I_B \end{bmatrix} \tag{20}$$

$$P = \begin{Bmatrix} -m_B g R \sin\theta_1 - m_B \frac{L}{2} R u_2^2 \sin(\theta_1 - \theta_2) \\ -m_B g \frac{L}{2} \sin\theta_2 + m_B \frac{L}{2} R u_1^2 \sin(\theta_1 - \theta_2) \end{Bmatrix} \tag{21}$$

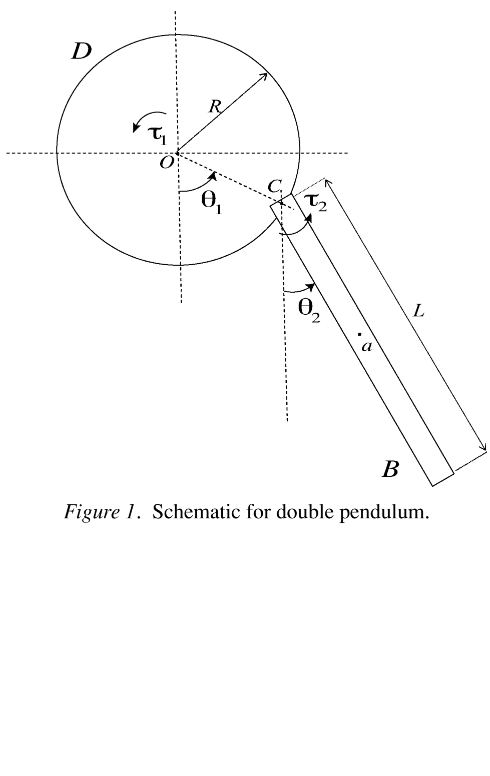

Figure 1. Schematic for double pendulum.

$$G = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \tag{22}$$

The total energy of the system is given by

$$\begin{aligned} E &= K + V \\ &= \frac{1}{2} I_D \, {}^{\mathcal{N}}\omega^D \cdot {}^{\mathcal{N}}\omega^D + \frac{1}{2} I_B \, {}^{\mathcal{N}}\omega^B \cdot {}^{\mathcal{N}}\omega^B + \frac{1}{2} m_B \, {}^{\mathcal{N}}\mathbf{v}^a \cdot {}^{\mathcal{N}}\mathbf{v}^a \\ &\quad + m_B g \left[R(1 - \cos\theta_1) + \frac{L}{2}(1 - \cos\theta_2) \right] \\ &= \frac{1}{2} I_D {u_1}^2 + \frac{1}{2} I_B u_2^2 + \frac{1}{2} m_B \left[R^2 u_1^2 + \frac{L^2}{4} u_2^2 - \frac{L}{2} R u_1 u_2 \cos(\theta_1 - \theta_2) \right] \\ &\quad + m_B g \left[R(1 - \cos\theta_1) + \frac{L}{2}(1 - \cos\theta_2) \right], \end{aligned}$$

where the datum for calculating the potential energy is the vertical position of the center of mass of the bar when $\theta_1 = \theta_2 = 0$. The servo-constraint equation is

$$\psi = E - E_0 = 0. \tag{23}$$

Taking the time derivative of ψ, X in Equation (4) is found to be

$$X = m_B \frac{L}{4} R \left(u_1^2 u_2 - u_1 u_2^2 \right) \sin(\theta_1 - \theta_2) + m_B g \left[R \sin\theta_1 u_1 + \frac{L}{2} \sin\theta_2 u_2 \right],$$

and the desired servo-constraint dynamics, Equation (6), is

$$\dot{E} - \Theta(E - E_0) = 0. \tag{24}$$

The expressions (8) and (9) for S and z are now formed, and Equation (10) is used to solve for τ, where S^+ for the row matrix S is given by [13]

$$S^+ = \frac{S^{\mathrm{T}}}{||S||_2^2}, \tag{25}$$

where $||S||_2$ is the Euclidian norm of the row matrix S. The column matrix y can be chosen arbitrarily. For $\Theta = -1$, $E_0 = 0$, the enforced servo-constraint dynamics is shown in Figure 2. The servo-constraints can be enforced by infinite number of ways, depending on the choice of y. Each choice results in different responses of the configuration parameters and velocity parameters, but all choices yield to the same servo-constraint dynamics, as resembled by Equation (24).

Nevertheless, an interesting choice of the control forces is the "ideal" one. The nonminimal nonholonomic form can be used to solve for this special type of control forces. Let $u_I = u_1$, and $u_D = u_2$. The servo-constraint equation can be put in the form

$$\dot{u}_D = A \dot{u}_I + B, \tag{26}$$

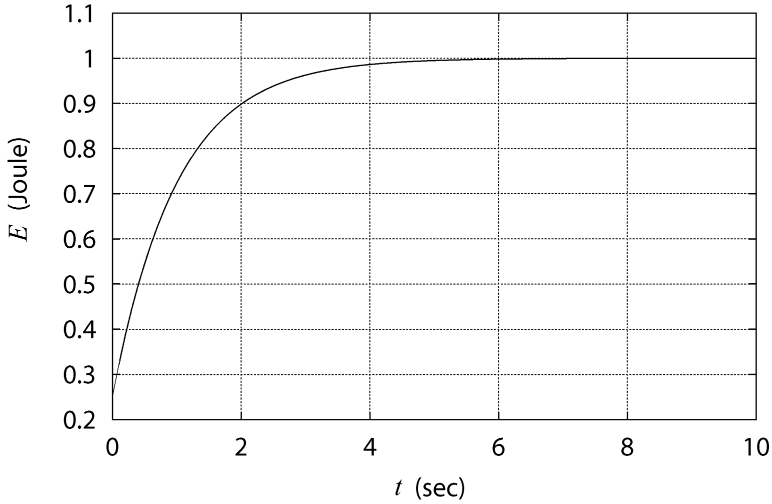

Figure 2. Double Pendulum: Servo-constraint dynamics

where the matrices A and B for this system are

$$A = \frac{-I_D u_1 - m_B R\left[Ru_1 - \frac{L}{4}u_2 \cos(\theta_1 - \theta_2)\right]}{I_B u_2 + m_B \frac{L}{4}[Lu_2 - Ru_1 \cos(\theta_1 - \theta_2)]} \tag{27}$$

$$B = \frac{\mu}{I_B u_2 + m_B \frac{L}{4}[Lu_2 - Ru_1 \cos(\theta_1 - \theta_2)]}, \tag{28}$$

and

$$\begin{aligned}\mu = &-\frac{L}{4} m_B R u_1 u_2 \sin(\theta_1 - \theta_2)(u_1 - u_2) - \frac{1}{2} I_D u_1^2 - \frac{1}{2} I_B u_1^2 - m_B g R \sin\theta_1 u_1 \\ &- m_B g \frac{L}{2} \sin\theta_2 u_2 - m_B g\left[R(1 - \cos\theta_1) + \frac{L}{2}(1 - \cos\theta_2)\right] \\ &- \frac{1}{2} m_B\left[R^2 u_1^2 + \frac{L^2}{4} u_2^2 - \frac{L}{2} R u_1 u_2 \cos(\theta_1 - \theta_2)\right].\end{aligned} \tag{29}$$

The nonminimal form [6], [7] can be constructed for the system.

Next, the corresponding expression, Equation (17), for the control torques matrix τ is formed. Figures 3 and 4 show the responses of the configuration parameters θ_1 and θ_2 and the velocity parameters u_1 and u_2, respectively, and Figure 5 shows the corresponding control torques. It is noticed that, if the servo-constraint dynamics reaches its steady state, then the required control torques reach the zero values. This agrees with the fact that, if the sources of nonconservatism are removed, then the total energy of the system remains unchanged, and confirms that the nonminimal nonholonomic form is the natural choice to enforce the servo-constraint dynamics.

Remark. Although the redundancy in the control system can be utilized in different manners, the servo-constraints must be kinematically and geometrically possible, for every possible configuration and velocity. For example, if the controls are required in addition to regulating the total energy of the double pendulum to cause it to track the prescribed trajectory

$$R\cos\theta_1 + L\sin\theta_2 = 0, \tag{30}$$

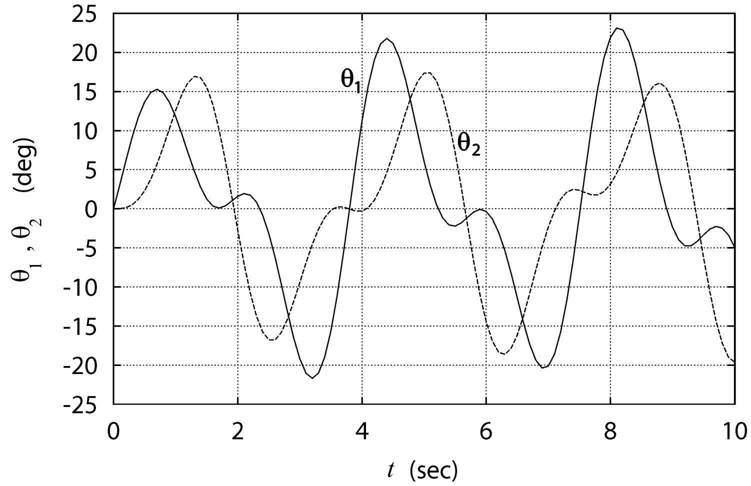

Figure 3. Double pendulum: configuration parameters; ideal controls case

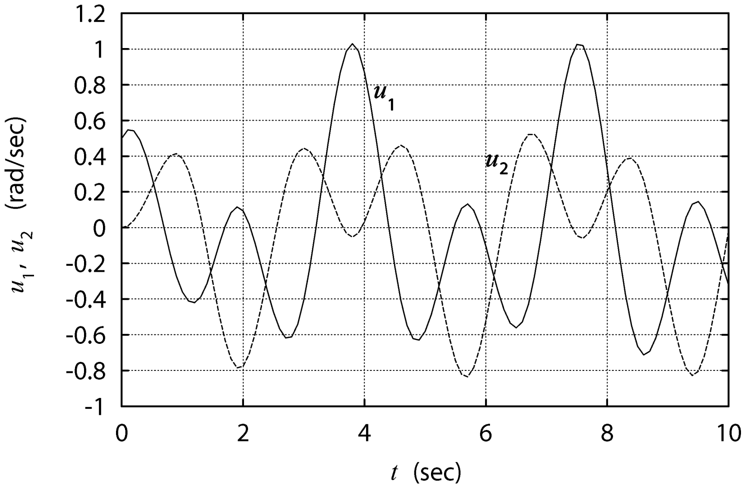

Figure 4. Double pendulum: velocity parameters; ideal controls case.

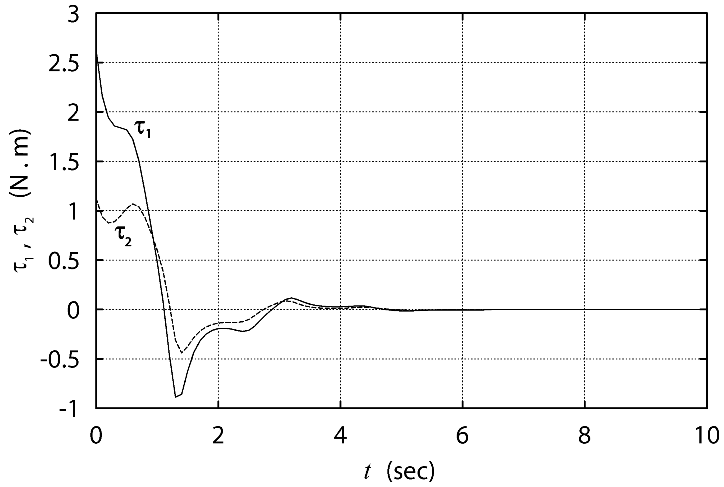

Figure 5. Double pendulum: ideal control torques.

then the requirements can be written as

$$\dot{\psi}_1(q, u, \dot{u}) - \Theta_1 \psi_1(q, u) = 0 \tag{31}$$

$$\ddot{\psi}_2(q, u, \dot{u}) - \Theta_{21} \dot{\psi}_2(q, u) - \Theta_{22} \psi_2(q) = 0, \tag{32}$$

where

$$\psi_1 = E - E_0 \tag{33}$$

$$\psi_2 = R \cos\theta_1 + L \sin\theta_2. \tag{34}$$

The resulting linear in accelerations equations form the matrix system

$$\begin{bmatrix} \dfrac{\partial \psi_1}{\partial u_1} & \dfrac{\partial \psi_1}{\partial u_2} \\ \dfrac{\partial \psi_2}{\partial u_1} & \dfrac{\partial \psi_2}{\partial u_2} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial \psi_1}{\partial q} u + \Theta_1 \psi_1 \\ -u^T \dfrac{\partial^2 \psi_2}{\partial q^2} u + \Theta_{21} \dfrac{\partial \psi_2}{\partial q} u + \Theta_{22} \psi_2 \end{Bmatrix}, \tag{35}$$

for some chosen values of Θ_1, Θ_{21}, and Θ_{22}. It can be verified that the above matrix system has no solution. A sufficient condition for the satisfaction of servo-constraints to render the motion possible is that $m < n$.

5. Controls-Involved Servo-Constraints

It is assumed that the system is required to follow the servo-constraint equations

$$\phi(q, u, \tau, t) = 0, \tag{36}$$

where $\phi \in \mathbb{R}^m$. A solution for τ is obtained by differentiating Equation (36) with respect to time to obtain

$$\frac{\partial \phi}{\partial q} \dot{q} + \frac{\partial \phi}{\partial u} \dot{u} + \frac{\partial \phi}{\partial \tau} \dot{\tau} + \frac{\partial \phi}{\partial t} = 0. \tag{37}$$

Substituting expressions (1) and (2) for $\dot{q}$ and $\dot{u}$ into Equation (37) gives

$$\mathcal{A} \dot{\tau} = \mathcal{B}, \tag{38}$$

where the matrices $\mathcal{A} \in \mathbb{R}^{m \times l}$ and $\mathcal{B} \in \mathbb{R}^m$ are

$$\mathcal{A} = \frac{\partial \phi}{\partial \tau} \tag{39}$$

$$\mathcal{B} = -\frac{\partial \phi}{\partial q} C u - \frac{\partial \phi}{\partial q} D - \frac{\partial \phi}{\partial u} [Q(q, t)^{-1} (P(q, u, t) + G(q, u, t)\tau)] - \frac{\partial \phi}{\partial t}. \tag{40}$$

Solving for $\dot{\tau}$,

$$\dot{\tau} = \mathcal{A}^+ \mathcal{B} + [I - \mathcal{A}^+ \mathcal{A}] y, \tag{41}$$

where $\mathcal{A}^+ \in \mathbb{R}^{l\times m}$ is the Moore-Penrose generalized inverse of $\mathcal{A}$, and $y \in \mathbb{R}^l$ is arbitrary at a specific point, provided that the point is in the controllability subspace of the dynamical system, i.e., the matrix $\mathcal{B}$ is in the range space of the matrix $\mathcal{A}$ at that point. The example in the following section illustrates the method.

6. Example: Spacecraft Stabilization

The following Euler equations form a mathematical model for a spacecraft:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1 \tag{42}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2 \tag{43}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3, \tag{44}$$

where ω_1, ω_2, and ω_3 are the angular velocities about the principal axes of the spacecraft. The control variables for the system are the applied torques τ_1, τ_2, τ_3 about the corresponding axes. The principal moments of inertia of the spacecraft are I_1, I_2, and I_3. The servo-constraint equation used to stabilize the spacecraft is the Lyapanov equation

$$\dot{K} + aK = 0, \quad a > 0, \tag{45}$$

where K is the kinetic energy of the spacecraft

$$K = \frac{1}{2}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right]. \tag{46}$$

Therefore, Equation (45) is

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 + \frac{1}{2}a\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right] = 0. \tag{47}$$

Substituting expressions (42)–(44) in the above equation, one obtains

$$\begin{aligned} I_1\omega_1\tau_1 + I_2\omega_2\tau_2 + I_3\omega_3\tau_3 &= -\frac{a}{2}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right] \\ &- \omega_1\omega_2\omega_3\left[\frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3}\right]. \end{aligned} \tag{48}$$

Differentiating the above equation gives

$$\begin{aligned} &I_1\omega_1\dot{\tau}_1 + I_2\omega_2\dot{\tau}_2 + I_3\omega_3\dot{\tau}_3 \\ &\quad = -I_1\dot{\omega}_1\tau_1 - I_2\dot{\omega}_2\tau_2 - I_3\dot{\omega}_3\tau_3 - aI_1\omega_1\dot{\omega}_1 - aI_2\omega_2\dot{\omega}_2 - aI_3\omega_3\dot{\omega}_3 \\ &\qquad - [\dot{\omega}_1\omega_2\omega_3 + \omega_1\dot{\omega}_2\omega_3 + \omega_1\omega_2\dot{\omega}_3]\left[\frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3}\right]. \end{aligned} \tag{49}$$

Substituting the expressions (42)–(44) for angular accelerations in the above equation gives

$$\begin{aligned}
I_1\omega_1\dot{\tau}_1 + I_2\omega_2\dot{\tau}_2 + I_3\omega_3\dot{\tau}_3 &= -I_1\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right]\tau_1 - I_2\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right]\tau_2 \\
&- I_3\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right]\tau_3 - aI_1\omega_1\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right] - aI_2\omega_2\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right] \\
&- aI_3\omega_3\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right] - \left[\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right]\omega_2\omega_3 + \omega_1\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right]\omega_3\right. \\
&\left. + \omega_1\omega_2\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right]\right]\left[\frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3}\right]. \qquad (50)
\end{aligned}$$

Hence, the matrices $\mathcal{A}$ and $\mathcal{B}$ for the system are

$$\mathcal{A} = [I_1\omega_1 \quad I_2\omega_2 \quad I_3\omega_3] \qquad (51)$$

$$\begin{aligned}
\mathcal{B} = &-I_1\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right]\tau_1 - I_2\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right]\tau_2 - I_3\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right]\tau_3 \\
&- aI_1\omega_1\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right] - aI_2\omega_2\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right] - aI_3\omega_3\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right] \\
&- \left[\left[\frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \tau_1\right]\omega_2\omega_3 + \omega_1\left[\frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \tau_2\right]\omega_3\right. \\
&\left. + \omega_1\omega_2\left[\frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \tau_3\right]\right]\left[\frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3}\right]. \qquad (52)
\end{aligned}$$

The Moore–Penrose generalized inverse of $\mathcal{A}$ is

$$\mathcal{A}^{+} = \frac{1}{(I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2}\begin{Bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{Bmatrix}. \qquad (53)$$

The expression (41) for $\dot{\tau}$ is now formed. For some specific choice of y, integrating these equations together with Euler's equations in time gives the trajectories of angular velocities of the dynamical system and the required control torques. All choices of y result in satisfying the servo-constraint Equation (45).

The initial conditions of the control variables should satisfy the servo-constraint equation, and the constant a can be any positive number. The simulations are performed for $I_1 = 10$ kg m^2, $I_2 = 6.3$ kg m^2, $I_3 = 8.5$ kg m^2, and $a = 1$, and the initial conditions on angular velocities $\omega_1(0) = \omega_2(0) = \omega_3(0) = 0.1$ rad/s. The initial conditions on the control variables that satisfy Equation (45) are chosen to be $\tau_1(0) = \tau_2(0) = 0.1$ N m, $\tau_3(0) = -0.3376$ N m. The first-order dynamic of K is shown Figure 6.

Although the servo-constraint dynamics is satisfied irrespective of the choice of y, some choices may result in unsatisfactory performance of the controlled system. For example, choosing $y_1 = y_2 = y_3 = 0$ results in the angular velocities shown in Figures 7–9, and the required control variables shown in Figures 10–12. Clearly, the chattering of the control variables and the angular velocities are undesirable.

A better choice of y is $y_i = -\tau_i,\ i = 1, \ldots, 3$ shown in Figures 13–15. This choice is made based on the structure of the controls dynamics given by Equation (41). These equations can be written as

$$\dot{\tau} = y + \mathcal{A}^{+}[\mathcal{B} - \mathcal{A}y]. \qquad (54)$$

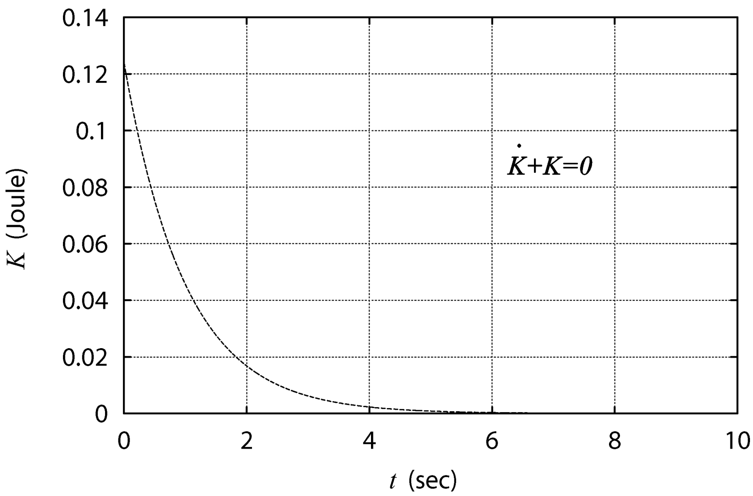

Figure 6. Servo-constraint (desired kinetic energy decay profile).

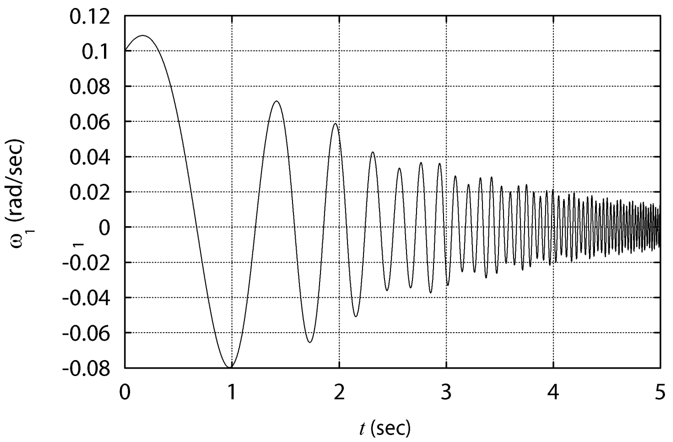

Figure 7. Angular velocity component about spacecraft body axis x; $y_1 = y_2 = y_3 = 0$.

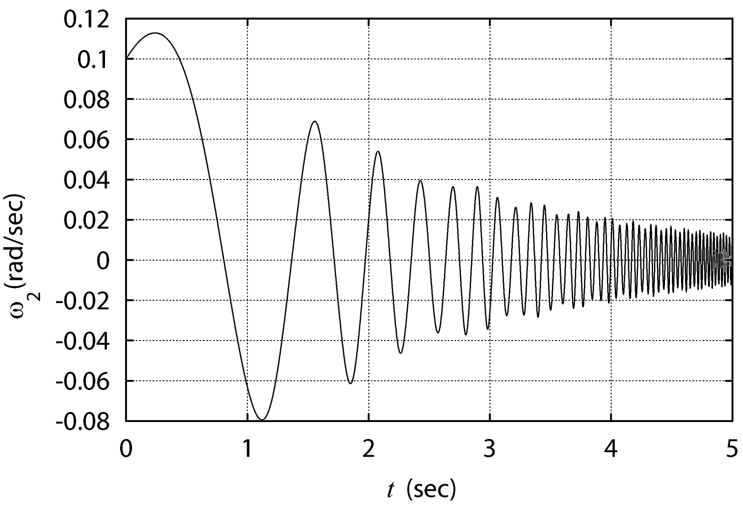

Figure 8. Angular velocity component about spacecraft body axis y; $y_1 = y_2 = y_3 = 0$.

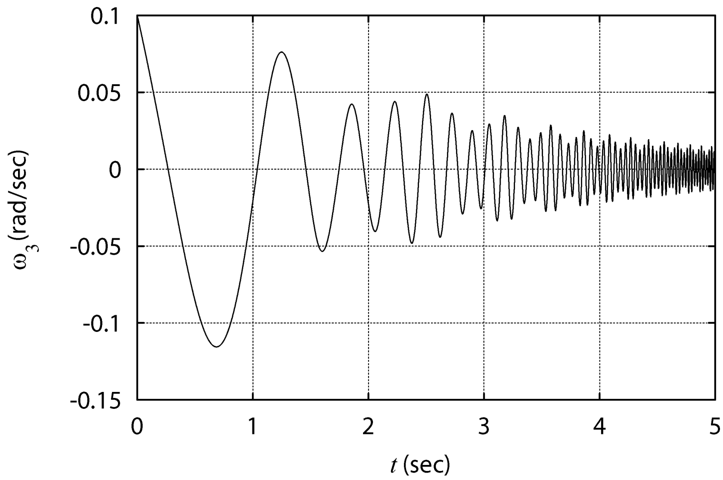

Figure 9. Angular velocity component about spacecraft body axis z; $y_1 = y_2 = y_3 = 0$.

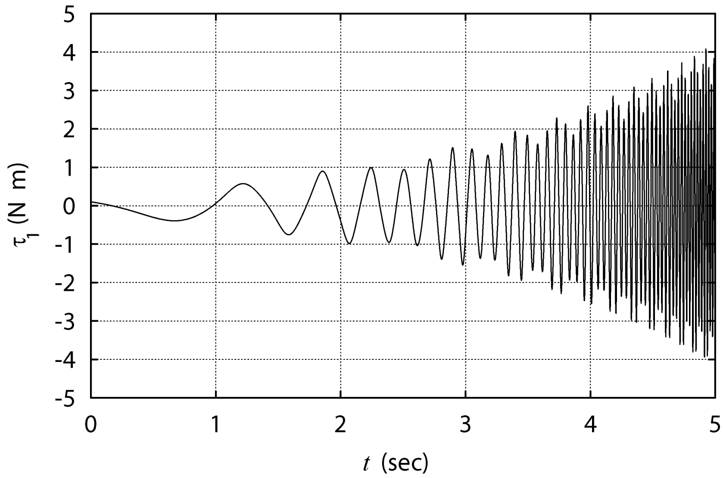

Figure 10. Torque about spacecraft body axis x; $y_1 = y_2 = y_3 = 0$.

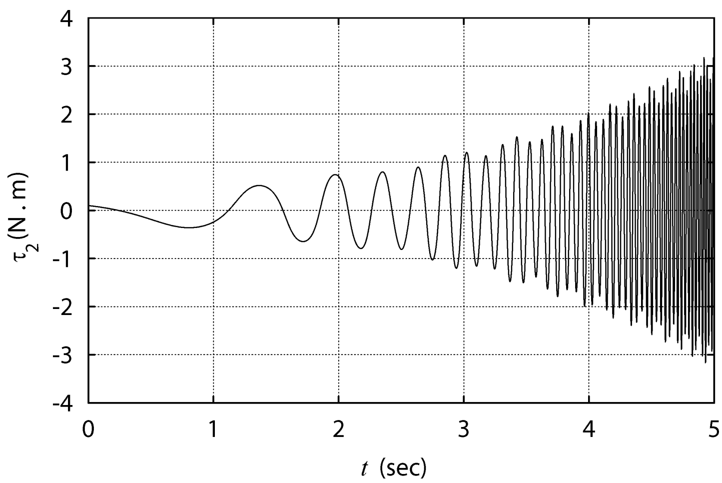

Figure 11. Torque about spacecraft body axis y; $y_1 = y_2 = y_3 = 0$.

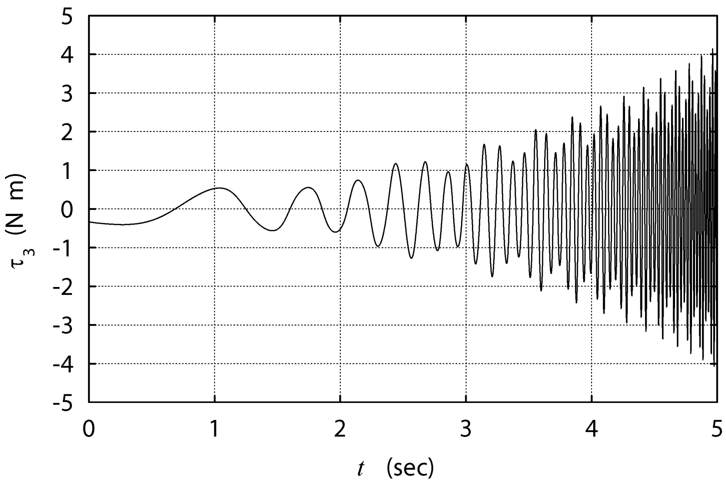

Figure 12. Torque about spacecraft body axis z; $y_1 = y_2 = y_3 = 0$.

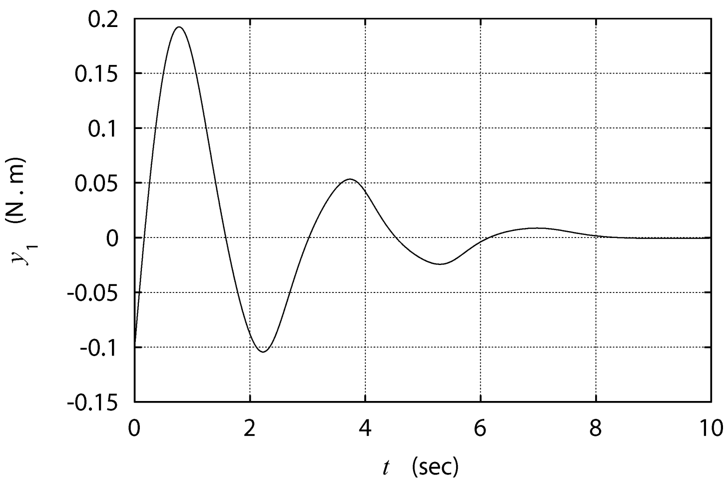

Figure 13. Parameter y_1; $y = -\tau$.

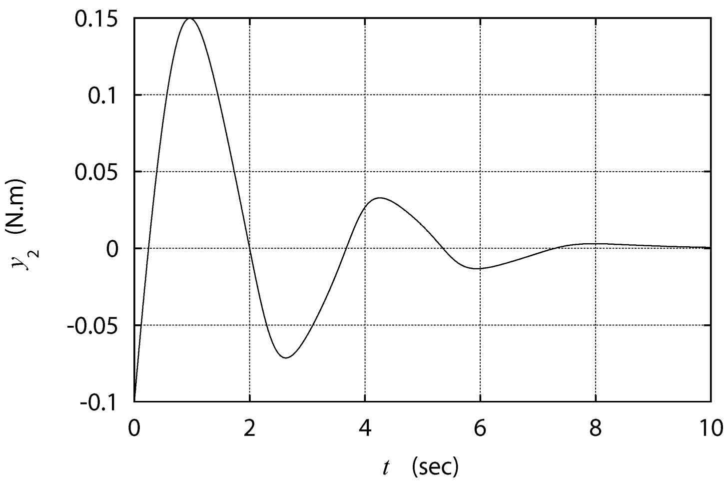

Figure 14. Parameter y_2; $y = -\tau$.

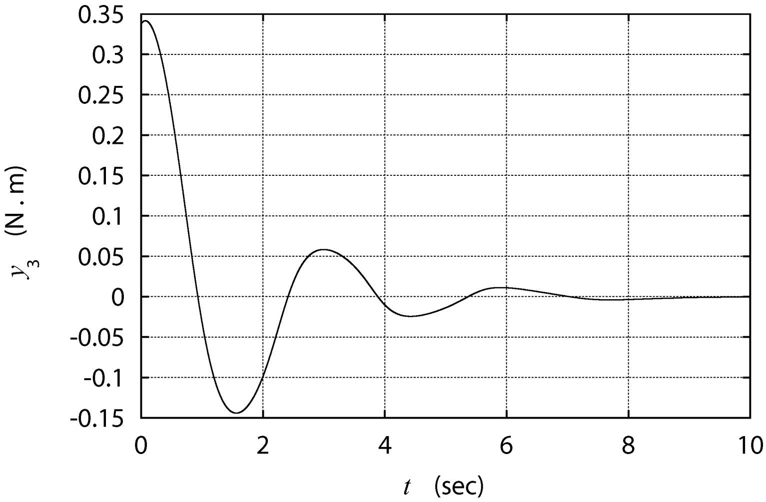

Figure 15. Parameter y_3; $y = -\tau$.

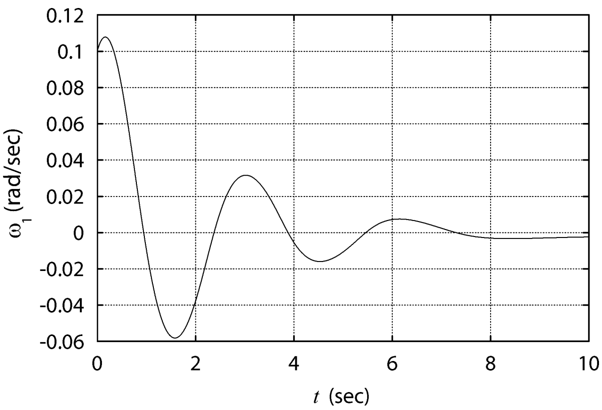

Figure 16. Angular velocity component about spacecraft body axis x; $y = -\tau$.

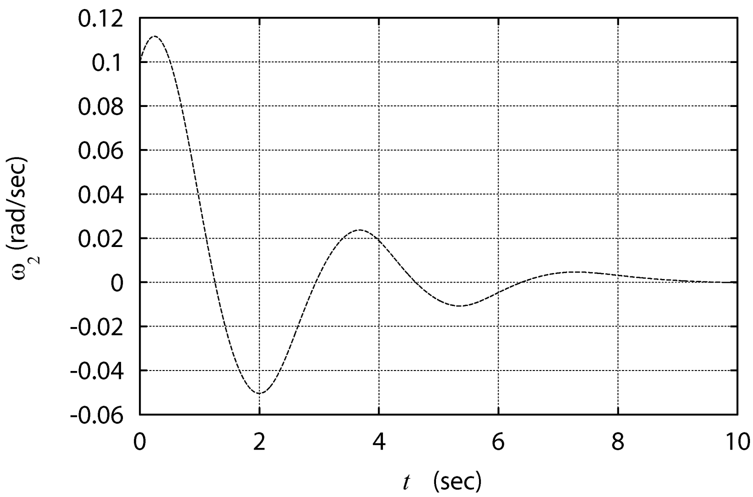

Figure 17. Angular velocity component about spacecraft body axis y; $y = -\tau$

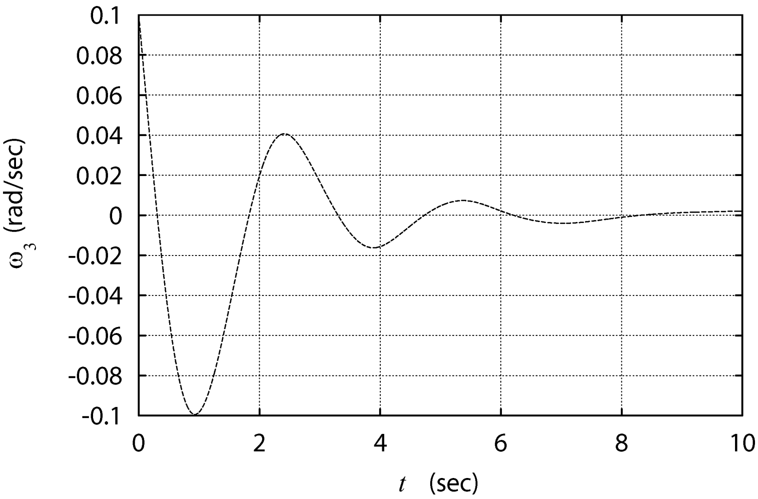

Figure 18. Angular velocity component about spacecraft body axis z; $y = -\tau$.

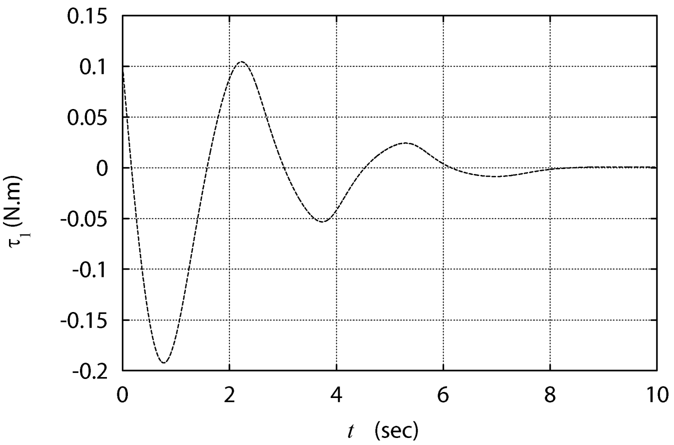

Figure 19. Torque about spacecraft body axis x; $y = -\tau$.

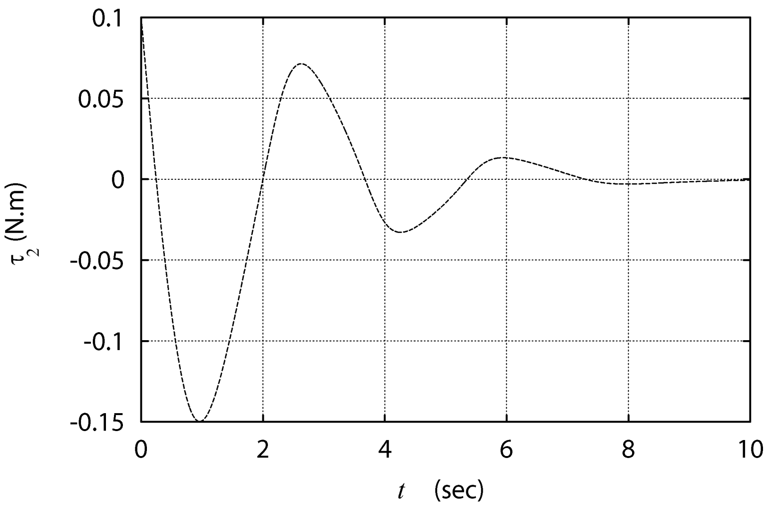

Figure 20. Torque about spacecraft body axis y; $y = -\tau$.

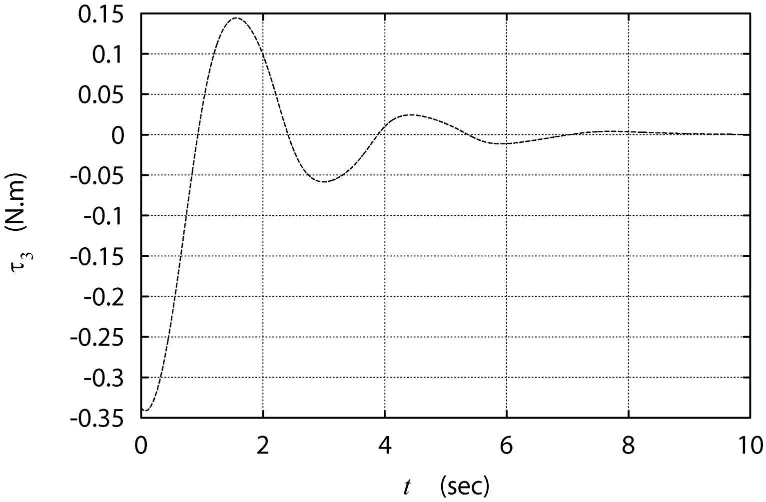

Figure 21. Torque about spacecraft body axis z; $y = -\tau$.

Hence, the first term will have a stabilizing effect if $y_i = -\tau_i,\ i = 1, \ldots, 3$ are chosen. This stabilizing effect dominates the dynamics of the system, as noticed from the corresponding smooth behaviors of the control variables and the angular velocities shown Figures 16–21.

7. Conclusion

The acceleration form of constraint equations is utilized in this paper to solve for the inverse dynamics of servo-constraints. A condition for the existence of controls that enforce servo-constraints is derived, together with a parametrization of the solution for these controls in terms of the generalized Moore–Penrose inverse.

In the case of redundant manipulators, the separation in the generalized accelerations of the non-minimal nonholonomic form provides a convenient way to obtain the ideal control forces, that is, those satisfying the principle of virtual work for virtual displacements satisfying the servo-constraint equations, and minimizing the Gibbsian (i.e., the acceleration energy of the dynamical system). The corresponding time marching of the configuration parameters and the velocity parameters shows the way that the dynamical system will evolve in time if the constraints were passive ideal.

Another procedure is introduced for converting algebraic servo-constraint equations involving control variables into dynamic constraint equations that complement the state-space model of the dynamical system, provided that the variables are in the controllable subspace of the dynamical system.

The introduction of the free parameters y as fictitious control variables is beneficial in affining the control problem, i.e., making the state-space model linear in the control variables, and hence allowing using the wealth of related methodologies for control systems analysis and design.

The present approach complements and generalizes the computed torque methods, in a unified treatment of holonomic and nonholonomic servo-constraints, where the constraints-free nature of the equations alleviates the need to measure, or to incorporate the effects of the constraint forces.

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