Georgia Institute of Technology
School of Aerospace Engineering

Aerospace Engineering 4220

Introduction to Structural Dynamics and Aeroelasticity

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I certify that, in full accord with the Honor Code of the Georgia Institute of Technology, I have neither received assistance from nor given assistance to other students in taking this examination.

Problems: Closed book

1. Consider a uniform beam with circular cross section. The general homogeneous solution for the torsional dynamics of a uniform beam of circular cross section can be shown to be of the form

$$\begin{aligned}\theta(x,t) &= X(x)Y(t)\\ &= [A\sin(\alpha x) + B\cos(\alpha x)]\left[C\sin\left(\alpha t\sqrt{\frac{G}{\rho}}\right) + D\cos\left(\alpha t\sqrt{\frac{G}{\rho}}\right)\right]\end{aligned}$$

where A, B, C, and D are arbitrary constants. In this case the polar moment of inertia $I_p = J$, the St.-Venant torsional constant; and the mass density is denoted by ρ. Consider the case in which the right end is free and the left end has attached to it a rigid flywheel with concentrated mass moment of inertia $I_c = \mu\rho J\ell$ and is spring-restrained by a light, linear, torsional spring of spring constant $k = \kappa GJ/\ell$, as indicated in Fig. 1. Note that κ and μ are dimensionless parameters.

   (a) Write the boundary condition on $\theta(\ell, t)$. (5 points)

   (b) Using free-body diagrams and Euler's law for the dynamics of a rigid body, show that the boundary condition on $\theta(0, t)$ is of the form

$$\beta\theta(0,t) + \beta_x\frac{\partial\theta}{\partial x}(0,t) + \beta_{tt}\frac{\partial^2\theta}{\partial t^2}(0,t) = 0$$

   where β, β_x, and β_{tt} are constants. Find those constants, which may be positive or negative. Hint: This is *not* a case that is worked out in the notes. You'll have to derive it yourself. (15 points)

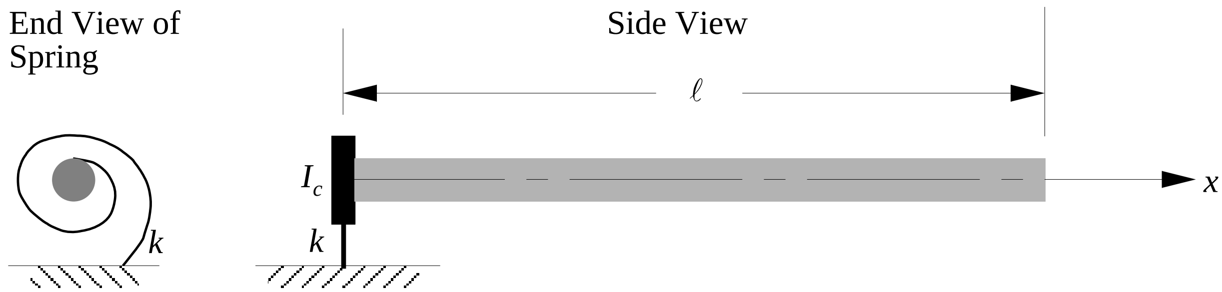

Figure 1: Schematic of uniform beam restrained by a spring and with an attached body, both at the left end

(c) In particular, show that the boundary conditions on X reduce to

$$\ell X'(0) + [\mu(\alpha\ell)^2 - \kappa]X(0) = 0; \qquad X'(\ell) = 0$$

Note: even if you cannot obtain these expressions, please use them in the remainder of the problem. (5 points)

(d) Show that the characteristic equation, i.e., the equation that governs the separation constant, can be put into the form

$$\tan(\alpha\ell) = \frac{\kappa - \mu(\alpha\ell)^2}{\alpha\ell}$$

Hint: Recall that when one has two homogeneous, algebraic equations, you should not just solve one equation and substitute into the other. Instead, one must ascertain that the determinant of the coefficient matrix is zero to have a nontrivial solution. (15 points)

(e) Does a rigid-body mode exist? Why or why not? Supposing κ to be identically zero, would a rigid-body mode exist then? Why or why not? (5 points)

2. A uniform string with mass per unit length m has been stretched taut with tension T to a length ℓ and attached between two rigid, immovable walls. The transverse vibration of the string is excited by two equal and opposite forces of magnitude $\mu 1(t)/\varepsilon$ where ε is the distance each force acts from the mid-point of the string. The force at $x = (\ell - \varepsilon)/2$ acts downward (opposite of positive v), and the force at $x = (\ell + \varepsilon)/2$ acts upward. See Fig. 2. Recall that $1(t)$ has magnitude of zero when $t < 0$ and unity when $t \geq 0$.

(a) Write the generalized equations of motion. In particular, find the generalized force associated with the i^{th} mode for the force system described above and in Fig. 2. Make sure to define ω_i and the generalized mass. (9 points)

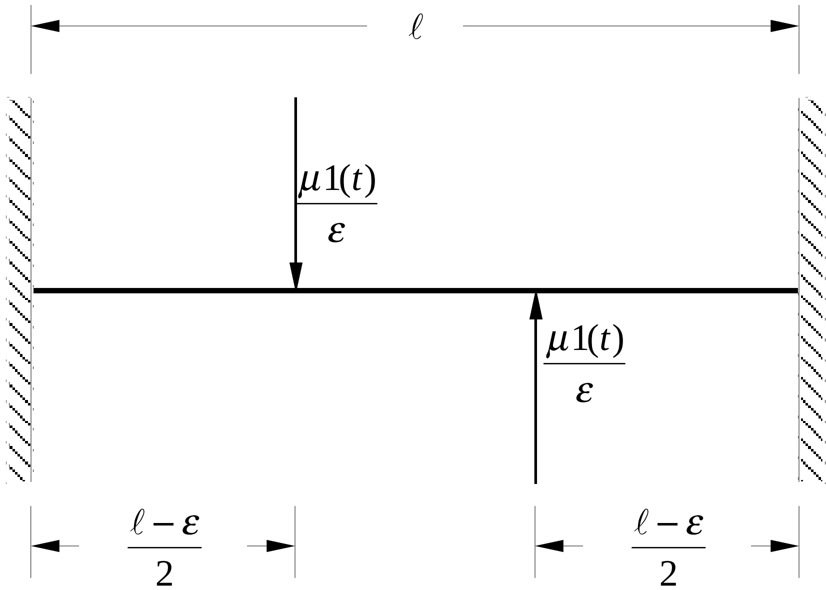

Figure 2: Schematic of string undergoing forced vibration

(b) Using the concept of the Taylor series, show that when ε is taken to be infinitesimal, the generalized force can be reduced to

$$\begin{aligned} \Xi_i &= \frac{\mu 1(t) i\pi (-1)^{\frac{i}{2}}}{\ell} && i \text{ even} \\ &= 0 && i \text{ odd} \end{aligned} \tag{1}$$

(5 bonus points)

(c) Use the result from part 2b for the generalized force to solve the generalized equations of motion for the string subject to initial conditions of zero displacement and velocity at time equal to zero. Indeed, show that

$$v(x,t) = \frac{2\mu}{\pi T} \sum_{i=2,4,\ldots}^{\infty} \frac{(-1)^{\frac{i}{2}}}{i} [1 - \cos(\omega_i t)] \sin\left(\frac{i\pi x}{\ell}\right)$$

for $t \geq 0$. (13 points)

(d) Why are the only modes that are excited those which are antisymmetric about the mid-point? (3 points)