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Non-classical effects in non-linear analysis of pretwisted anisotropic strips[1]

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Abstract

The literature on classical analysis of anisotropic beams assumes that all 1D "moment strain" measures (i.e. twist and bending curvatures) are of the same order of magnitude, resulting in a linear cross-sectional analysis. The present paper treats the situation in which one or more of the 1D moment strain measures may be larger than the other(s), resulting in a non-linear cross-sectional analysis. This type of non-classical analysis is needed, for example, in problems where the trapeze effect is important, such as in rotor blades. As a precursor to complicated non-linear sectional analysis of arbitrary cross sections, a non-linear sectional analysis is presented for an anisotropic strip with small pretwist, based on the dimensional reduction of laminated shell theory to a non-linear one-dimensional theory using the variational-asymptotic method. Results obtained from this strip-beam analysis are compared with available theoretical and experimental results for a problem in which the trapeze effect is important. In order to demonstrate the usage of the results in the analysis of structures made of an arbitrary geometrical combination of pretwisted generally anisotropic strips, a closed-form expression is derived for the torsional buckling of a column with a cruciform cross section. © 1998 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Modeling of beam-like anisotropic structures having an arbitrary geometry in terms of initial curvature, pretwist and cross-sectional shape has been the focus of several previous papers [1–5]. The primary aim of all of these is to take advantage of the beam-like configuration, which allows consideration of the ratio between a characteristic cross-sectional dimension and the wavelength of the deformation along the beam as a small

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parameter. This greatly reduces the computational effort required relative to that for three-dimensional (3D) modeling of the structure. The simplification of the problem arises from decomposition of the 3D problem into two simpler problems: a two-dimensional (2D) problem, which provides in a compact form the cross-sectional properties using a mathematical technique called the variational-asymptotic method, and a non-linear one-dimensional (1D) problem along the length of the beam.

Almost all of the cross-sectional analyses for composite beams in the literature are linear. Linearity precludes a cross-sectional analysis from capturing, for example, the trapeze effect, which is of well-known importance in rotor blades. Although the trapeze effect is fundamentally a non-linear coupling of extension and twist, in rotor blade analyses it is frequently exhibited as a linear term that increases the effective torsional rigidity as a function of axial force; see, for example, Ref. [6]. This term is not contained in the exact intrinsic equilibrium equations for a beam, for example, those of Reissner [7]. Therefore, it *must* enter the analysis through the constitutive law. Although contained in the analytical treatments of Berdichevsky [8], there are no published treatments of the trapeze effect from an asymptotic point of view for general cross-sectional analysis. The most general cross-sectional analysis in which the non-linear terms needed to model the trapeze effect are included is the work of Borri and Merlini [9], who developed these terms from a "geometric stiffness" point of view.

The objective of this paper is to gain insight into the non-linear cross-sectional modeling of a beam. In order to understand how non-linear phenomena such as the trapeze effect can be captured in a cross-sectional analysis, a simplified configuration is considered for which classical laminated shell theory (CLST) can be used as a starting point, namely a laminated strip-like beam with small pretwist. Even this simplified analysis turns out to be challenging, but it does give a good indication of how a similar analysis can be carried out for more general cross-sectional geometries. Results from this analysis are compared with experiment and another theory. The usage of the theory in analysing beams made of a combination of generally anisotropic pretwisted strips is then indicated through an example — a cruciform cross section. The torsional buckling load is determined for this structure.

2. Analytical development

In order to meet our objective, we attempt to model a thin pretwisted composite strip as a beam, including those non-linear effects that need to be considered, by virtue of the special geometry, to achieve asymptotical correctness. This is achieved by deriving its strain energy function in terms of 1D quantities only. Starting from CLST, the variational-asymptotic method is used as a tool to carry out the dimensional reduction from 2D to 1D.

2.1. Undeformed strip geometry

Asymptotic methods require small parameters. Here the wavelength of deformation along the strip is denoted by ℓ. The width and thickness of the strip are denoted by b and h, respectively. From the geometry of the strip, the natural small parameters are the thickness-to-width ratio $\delta_h = h/b$; the width-to-length ratio $\delta_b = b/\ell$; and the width times pretwist per unit length $\delta_t = bk_1$, where k_1 is the derivative of the pretwist angle with respect to length along the strip. The geometry of the strip is shown in Fig. 1. The Cartesian coordinate measures x_i are directed along the length, width, and thickness of the strip for $i = 1,2$, and 3, respectively, parallel to corresponding unit vectors $\mathbf{b}_i$. (Here and throughout the paper, Latin indices run from 1 to 3 while Greek indices run from 1 to 2; repeated indices are summed over their ranges.) The domain of the strip-beam is such that $0 \leqslant x_1 \leqslant \ell$, $-b/2 \leqslant x_2 \leqslant b/2$ and $-h/2 \leqslant x_3 \leqslant h/2$. The strip has a pretwist rate $k_1(x_1)$, so that the unit vectors associated with the cross-sectional plane, $\mathbf{b}_2$ and $\mathbf{b}_3$, are functions of x_1.

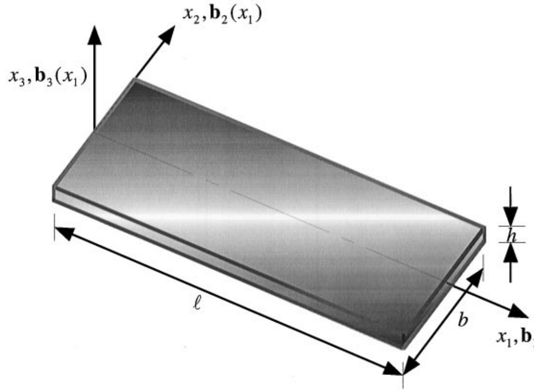

Fig. 1. Pretwisted strip configuration and coordinate system.

The position vector from a point fixed in an inertial reference frame to a generic point on the middle surface of the strip is $\mathbf{r} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2(x_1)$. The position vector of an arbitrary material point in the strip is then

$$\check{\mathbf{r}} = \mathbf{r} + x_3\mathbf{b}_3(x_1) = x_i\mathbf{b}_i. \tag{1}$$

By differentiating $\check{\mathbf{r}}$ with respect to the coordinate measures we find the covariant base vectors for the undeformed geometry to be

$$\mathbf{g}_1 = \mathbf{b}_1 + k_1(x_2\mathbf{b}_3 - x_3\mathbf{b}_2),$$
$$\mathbf{g}_2 = \mathbf{b}_2,$$
$$\mathbf{g}_3 = \mathbf{b}_3, \tag{2}$$

while the contravariant base vectors for the undeformed geometry are

$$\mathbf{g}^1 = \mathbf{b}_1,$$
$$\mathbf{g}^2 = \mathbf{b}_2 + k_1x_3\mathbf{b}_1,$$
$$\mathbf{g}^3 = \mathbf{b}_3 - k_1x_2\mathbf{b}_1. \tag{3}$$

2.2. Kinematical formulation

Before applying the variational-asymptotic method, we will formulate the kinematics of the problem using the procedure outlined by Danielson and Hodges [10].

The position vector $\check{\mathbf{R}}(x_1, x_2, x_3)$ of an arbitrary material point in the deformed configuration is given by

$$\check{\mathbf{R}} = x_1\mathbf{b}_1 + u_i(x_1)\mathbf{b}_i + x_2\mathbf{B}_2(x_1) + x_3\mathbf{B}_3(x_1) + \bar{w}_i(x_1, x_2, x_3)\mathbf{B}_i(x_1), \tag{4}$$

where u_i are rigid-body displacements; $\mathbf{B}_i$ are orthogonal unit vectors introduced by rigid body rotation; and $\bar{w}_i$ are warping displacements of the beam cross section.

Based on the main exchange rules in [11] and the recovering relations in [12], the warping displacement component that is normal to the local shell surface can be split into two parts: an average across the thickness and an unknown variation due to Poisson-like effects. Thus

$$\bar{w}_3(x_1, x_2, x_3) = w_3(x_1, x_2) + \Delta_3(x_1, x_2, x_3), \tag{5}$$

where

$$\int_{-h/2}^{h/2} \Delta_3(x_1, x_2, x_3)\, \mathrm{d}x_3 = 0. \tag{6}$$

Motivated by the smallness of δ_h and consistent with the above references, the two components of shell warping displacements that are in the local tangential direction are each split into three parts: the average warping, a part linear in the thickness coordinate due to average local rotations, and the rest of the unknown variations, for example, due to the shell curvature:

$$\bar{w}_\alpha(x_1, x_2, x_3) = w_\alpha(x_1, x_2) + x_3\phi_\alpha(x_1, x_2) + \Delta_\alpha(x_1, x_2, x_3), \tag{7}$$

where

$$\int_{-h/2}^{h/2} \Delta_\alpha(x_1, x_2, x_3)\, \mathrm{d}x_3 = 0 \tag{8}$$

and

$$\int_{-h/2}^{h/2} \Delta_{\alpha,3}(x_1, x_2, x_3)\ \mathrm{d}x_3 = 0. \tag{9}$$

Eq. (4) now takes the form

$$\check{\mathbf{R}} = x_1\mathbf{b}_1 + u_i\mathbf{b}_i + x_2\mathbf{B}_2 + w_i\mathbf{B}_i + x_3(\phi_1\mathbf{B}_1 + \phi_2\mathbf{B}_2 + \mathbf{B}_3) + \Delta_i\mathbf{B}_i. \tag{10}$$

Note that $u_i(x_1)$ and $\mathbf{B}_i(x_1)$ are beam quantities, while $\phi_\alpha(x_1, x_2)$ and $w_i(x_1, x_2)$ reflect shell behavior. In particular, $u_i(x_1)$ represent rigid-body translations of the cross section; $\mathbf{B}_i(x_1)$ are dictated by the rigid-body rotations of the cross section; $w_i(x_1, x_2)$ are the warping; and $\phi_\alpha(x_1, x_2)$ are local rotation variables. Thus, the warping displacements are governed by the constraints

$$\langle w_i \rangle = 0, \quad \langle w_{3,2} \rangle = \langle \phi_2 \rangle, \tag{11}$$

where the notation

$$\langle \bullet \rangle \equiv \int_{-b/2}^{b/2} (\bullet)\, \mathrm{d}x_2 \tag{12}$$

is used.

The covariant base vectors for the deformed geometry are determined by

$$\mathbf{G}_i = \frac{\partial \check{\mathbf{R}}}{\partial x_i} \tag{13}$$

in the evaluation of which we eliminate u_i' and $\mathbf{B}_i'$ by introducing the 1D strain measures γ_{11} and κ_i, which satisfy

$$\mathbf{B}_1 = \frac{[(x_1 + u_1)\mathbf{b}_1 + u_2\mathbf{b}_2 + u_3\mathbf{b}_3]'}{s'}, \tag{14}$$

$$\mathbf{B}_i' = [(k_1 + \kappa_1)\mathbf{B}_1 + \kappa_2\mathbf{B}_2 + \kappa_3\mathbf{B}_3] \times \mathbf{B}_i,$$

where

$$\mathbf{b}_i' = k_1 \mathbf{b}_1 \times \mathbf{b}_i,$$

$$s' = \sqrt{(1 + u_1')^2 + (u_2' - k_1 u_3)^2 + (u_3' + k_1 u_2)^2} = 1 + \gamma_{11}, \tag{15}$$

and s is the running arc-length along the beam reference line.

We can now evaluate the deformation gradient tensor $\mathbf{A} = \mathbf{G}_i \mathbf{g}^i$. Following Danielson and Hodges [10] we arrange the components of $\mathbf{A}$ in mixed bases into a matrix A, the elements of which are given by $A_{ij} = \mathbf{B}_i \cdot \mathbf{A} \cdot \mathbf{b}_j$ as follows:

$$\begin{aligned}
A_{11} &= 1 + \gamma_{11} + (x_3 + w_3 + \Delta_3)\kappa_2 - (x_2 + w_2 + x_3\phi_2)\kappa_3 + w_1' + x_3\phi_1' + k_1[x_3(w_{1,2} + x_3\phi_{1,2}) \\
&\qquad - x_2\phi_1 + \Delta_{1,3}x_2 - \Delta_{1,2}x_3] - \Delta_2\kappa_3 + \Delta_1', \\
A_{12} &= w_{1,2} + x_3\phi_{1,2} + \Delta_{1,2}, \\
A_{13} &= \phi_1 + \Delta_{1,3}, \\
A_{21} &= (w_1 + x_3\phi_1)\kappa_3 - (x_3 + w_3 + \Delta_3)\kappa_1 + w_2' + x_3\phi_2' - k_1[w_3 + \Delta_3 - x_3(w_{2,2} + x_3\phi_{2,2}) \\
&\qquad - x_2\phi_2 + \Delta_{2,3}x_2 - \Delta_{2,2}x_3] + \Delta_1\kappa_3 + \Delta_2', \\
A_{22} &= 1 + w_{2,2} + x_3\phi_{2,2} + \Delta_{2,2}, \\
A_{23} &= \phi_2 + \Delta_{2,3}, \\
A_{31} &= (x_2 + w_2 + x_3\phi_2)\kappa_1 - (w_1 + x_3\phi_1)\kappa_2 + w_3' + \Delta_3' + k_1[w_2 + x_3(\phi_2 + w_{3,2} + \Delta_{3,2}) \\
&\qquad + \Delta_2 - x_2\Delta_{3,3}] - \Delta_1\kappa_2 + \Delta_2\kappa_1, \\
A_{32} &= w_{3,2} + \Delta_{3,2}, \\
A_{33} &= 1 + \Delta_{3,3}.
\end{aligned} \tag{16}$$

If small local rotations were to be assumed (not the assumption made in this paper), the 3D strains would be approximated by the difference of the symmetric component of the deformation gradient tensor with the identity matrix I_3, namely the symmetric matrix

$$E = \frac{A + A^{\mathrm{T}}}{2} - I_3 \tag{17}$$

the elements of which are given by

$$\begin{aligned}
E_{11} &= \gamma_{11} + x_3\kappa_2 - x_2\kappa_3 + w_3\kappa_2 + x_3\phi_1' - x_3\phi_2\kappa_3 + w_1' - w_2\kappa_3 + \Delta_3\kappa_2 \\
&\qquad + k_1[x_3(w_{1,2} + x_3\phi_{1,2}) - x_2\phi_1 - x_2\Delta_{1,3} + x_3\Delta_{1,2}] - \Delta_2\kappa_3 + \Delta_1', \\
2E_{12} &= w_{1,2} - x_3\kappa_1 + x_3\phi_{1,2} - w_3\kappa_1 + x_3\phi_2' + x_3\phi_1\kappa_3 + w_1\kappa_3 + w_2' - \Delta_3\kappa_1 \\
&\qquad - k_1[w_3 + \Delta_3 - x_3(x_3\phi_{2,2} + w_{2,2}) + x_2\phi_2 + x_2\Delta_{2,3} - x_3\Delta_{2,2}] + \Delta_1\kappa_3 + \Delta_{1,2} + \Delta_2', \\
2E_{13} &= \phi_1 + x_2\kappa_1 + w_3' + x_3\phi_2\kappa_1 + \Delta_3' - x_3\phi_1\kappa_2 + w_2\kappa_1 - w_1\kappa_2 + k_1[w_2 + x_3(\Delta_{3,2} + w_{3,2} + \phi_2) \\
&\qquad - x_2\Delta_{3,3} + \Delta_2] + \Delta_2\kappa_1 - \Delta_1\kappa_2 + \Delta_{1,3}, \\
E_{22} &= w_{2,2} + x_3\phi_{2,2} + \Delta_{2,2}, \\
2E_{23} &= \phi_2 + w_{3,2} + \Delta_{3,2} + \Delta_{2,3}, \\
E_{33} &= \Delta_{3,3}.
\end{aligned} \tag{18}$$

The variational-asymptotic method can be applied in an iterative manner, where a preliminary order of magnitude analysis is used to develop a somewhat arbitrary estimation scheme. At the end of the first step of applying the method, the results are checked to see if they are actually of the assumed order. If not, an additional step must be taken to obtain asymptotically correct results, as suggested by Berdichevsky [12]. Here, for the preliminary step we note that all the E_{ij} are $O(\varepsilon)$. Also, the geometry implies a relation among the warping variables:

$$(\Delta_{1,3} + \phi_1)^2 + (\Delta_{2,3} + \phi_2)^2 + (1 + \Delta_{3,3})^2 = (1 + e)^2, \tag{19}$$

where $e = O(\varepsilon)$. Based on these observations, one can obtain the following estimation scheme:

$$\begin{aligned} &w_\alpha = O(\varepsilon b), \quad w_3 = O\left(\frac{\varepsilon b}{\delta_h}\right), \\ &\phi_\alpha = O\left(\frac{\varepsilon}{\delta_h}\right), \quad \Delta_i = O(\varepsilon \delta_h b), \\ &\gamma_{11} = O(\varepsilon), \quad \kappa_\alpha = O\left(\frac{\varepsilon}{\delta_h b}\right), \quad \kappa_3 = O\left(\frac{\varepsilon}{b}\right), \end{aligned} \tag{20}$$

where it is noted that $\phi_1 = -x_2\kappa_1 + O(\varepsilon)$ (follows from the smallness of E_{13}) and $\phi_2 = -w_{3,2} + O(\varepsilon)$ (follows from the smallness of E_{23}) are larger in magnitude than the strains. They represent moderate local rotations.

The orders of magnitude of Δ_i are estimated as follows: the stretch through the thickness is approximately $E_{33} = \Delta_{3,3}$, clearly $O(\varepsilon)$. If $\Delta_{2,3}$ were to be larger in magnitude than the strains, then the sum $\phi_2 + w_{3,2}$ would have to be of the same order in order to retain the smallness of E_{23} and furthermore $\Delta_{2,3} = -(\phi_2 + w_{3,2}) + O(\varepsilon)$. This would imply $\Delta_2 = \Delta_{20} - x_3(\phi_2 + w_{3,2}) + O(\varepsilon h)$ where Δ_{20} is independent of x_3. Using Eq. (8), $\Delta_2 = -x_3(\phi_2 + w_{3,2}) + O(\varepsilon h)$. This, however, violates the constraint involving Δ_2 in Eq. (9). Hence, the sum $\phi_2 + w_{3,2}$ and $\Delta_{2,3}$ can only be of the order of the strains. Similarly, if $\Delta_{1,3}$ were to be larger in magnitude than the strains, then $\Delta_{1,3} = -(\phi_1 + x_2\kappa_1) + O(\varepsilon)$ in order to retain the smallness of E_{13}. This would imply $\Delta_1 = \Delta_{10} - x_3(\phi_1 + x_2\kappa_1) + O(\varepsilon h)$ where Δ_{10} is independent of x_3. Using Eq. (8), $\Delta_1 = -x_3(\phi_1 + x_2\kappa_1) + O(\varepsilon h)$. This, however, violates the constraint involving Δ_1 in Eq. (9). Hence $\Delta_{1,3}$ can only be of the order of the strains. Thus Δ_i are all of $O(\varepsilon \delta_h b)$.

The 3D strains Γ are obtained using the moderate local rotation approximation [10] so that

$$\Gamma = E - \frac{\tilde{A}^2}{2} + \frac{E\tilde{A} - \tilde{A}E}{2}, \tag{21}$$

where $\tilde{A}$ is the anti-symmetric component of A. The 3D strain components that are of further interest to us are

$$\begin{aligned} \Gamma_{11} &= \underbrace{\gamma_{11} - x_2\kappa_3 + x_3\kappa_2}_{O(\varepsilon)} + \overbrace{k_1 x_2^2 \kappa_1}^{O(\varepsilon\delta_t/\delta_h)} + \underbrace{\frac{x_2^2\kappa_1^2}{2} + w_3\kappa_2}_{O(\varepsilon^2/\delta_h^2)} + O\left(\varepsilon\delta_b, \varepsilon\delta_t, \frac{\varepsilon^2}{\delta_h}\right), \\ \Gamma_{22} &= \underbrace{w_{2,2} - x_3 w_{3,22}}_{O(\varepsilon)} + \overbrace{\tfrac{1}{2} w_{3,2}{}^2}^{O(\varepsilon^2/\delta_h^2)} + O(\varepsilon\delta_h), \\ 2\Gamma_{12} &= \underbrace{w_{1,2} - 2x_3\kappa_1}_{O(\varepsilon)} + \overbrace{k_1(x_2 w_{3,2} - w_3)}^{O(\varepsilon\delta_t/\delta_h)} + \underbrace{\kappa_1(x_2 w_{3,2} - w_3)}_{O(\varepsilon^2/\delta_h^2)} + O\left(\varepsilon\delta_b, \varepsilon\delta_h, \varepsilon\delta_t, \frac{\varepsilon^2}{\delta_h}\right). \end{aligned} \tag{22}$$

Since the δ's are all small parameters, we ought to retain the terms of $O(\varepsilon\delta_t/\delta_h)$ in comparison to the terms of $O(\varepsilon)$. Furthermore, the terms of $O(\varepsilon^2/\delta_h^2)$ assume significance. Denoting the ratio ε/δ_h^2 by the small parameter Δ, we note that the zeroth-order approximation to the strain energy should contain all the terms up to $O(E\varepsilon^2)$ while the first-order approximation should contain all the terms up to $O(E\varepsilon^2\Delta^2)$, where a typical stiffness coefficient is of $O(E)$. This in turn means that the zeroth-order approximation to the strains should contain all terms up to $O(\varepsilon)$ while the first-order approximation should contain all relevant terms up to $O(\varepsilon\Delta^2)$. On the other hand, terms of $O(\varepsilon\delta_b)$, $O(\varepsilon\delta_h)$, $O(\varepsilon\delta_t)$ and $O(\varepsilon^2/\delta_h)$ can be included in the higher-order approximations to the strains, if necessary.

The 3D strain measures are related to the 2D strain measures by $\Gamma_{\alpha\beta} = \varepsilon_{\alpha\beta} + x_3\rho_{\alpha\beta}$, where $\varepsilon_{\alpha\beta}$ are the membrane strains and $\rho_{\alpha\beta}$ are the middle surface bending curvatures. Hence, by inspection of Eq. (22), we can obtain the relation between 2D (shell) measures and the 1D (beam) measures. The membrane strains are

$$\begin{aligned} \varepsilon_{11} &\approx \gamma_{11} - x_2\kappa_3 + k_1 x_2^2\kappa_1 + \underline{\frac{x_2^2\kappa_1^2}{2}} + \underline{w_3\kappa_2}, \\ \varepsilon_{22} &\approx w_{2,2} + \underline{\frac{1}{2}w_{3,2}^2}, \\ 2\varepsilon_{12} &\approx w_{1,2} + k_1(x_2 w_{3,2} - w_3) + \underline{\kappa_1(x_2 w_{3,2} - w_3)} \end{aligned} \tag{23}$$

while the curvatures are

$$\begin{aligned} \rho_{11} &\approx \kappa_2, \\ \rho_{22} &\approx -w_{3,22}, \\ 2\rho_{12} &\approx -2\kappa_1. \end{aligned} \tag{24}$$

In the above equations, the underlined terms are non-linear and arise due to moderate local rotation. The non-underlined terms in these equations are the dominant ones, the only ones needed for the zeroth-order approximation. The first approximation should include all the above terms.

2.3 Shell theory

The stiffness coefficients for CLST are given by

$$C^{\sigma\nu\omega\theta} = E_{\parallel}^{\alpha\beta\gamma\delta}\bar{\mu}_\alpha^\sigma\bar{\mu}_\beta^\nu\bar{\mu}_\gamma^\omega\bar{\mu}_\delta^\theta, \tag{25}$$

where

$$\begin{aligned} E_{\parallel}^{\alpha\beta\gamma\delta} &= E^{\alpha\beta\gamma\delta} - \frac{E^{\alpha\beta33}E^{\gamma\delta33}}{E^{3333}} - H_{\mu\nu}G^{\alpha\beta\mu}G^{\gamma\delta\nu}, \\ \bar{\mu}_\alpha^\sigma &= \delta_\alpha^\sigma - x_3\frac{\partial^2(\check{\mathbf{r}}\cdot\mathbf{b}_i)}{\partial x_\alpha\partial x_\sigma}n_i, \\ n_i &= \frac{e_{ijk}}{\sqrt{a^0}}\frac{\partial(\check{\mathbf{r}}\cdot\mathbf{b}_j)}{\partial x_1}\frac{\partial(\check{\mathbf{r}}\cdot\mathbf{b}_k)}{\partial x_2}, \\ a^0 &= \det\left[\delta_{ij}\frac{\partial(\check{\mathbf{r}}\cdot\mathbf{b}_i)}{\partial x_\alpha}\frac{\partial(\check{\mathbf{r}}\cdot\mathbf{b}_j)}{\partial x_\beta}\right]. \end{aligned}$$

For our pretwisted rectangular strip,

$$\bar{\mu}_1^1 = \bar{\mu}_2^2 = 1,$$

$$\bar{\mu}_1^2 = \bar{\mu}_2^1 = \frac{-k_1 x_3}{\sqrt{1 + (k_1 x_2)^2}}. \tag{26}$$

Hence, it is seen that the largest correction to any of the stiffness coefficients of $O(E)$ of CLPT, owing to a pretwist per unit length $\delta_t = bk_1$, is $O(\delta_t \delta_h E)$. As the strains to be considered in the first approximation are of $O(\varepsilon, \varepsilon^2/\delta_h^2)$, the largest additional term contributed by a CLST stiffness coefficient to the energy is still smaller than the smallest term contributed by a CLPT stiffness coefficient. Hence, both CLPT and CLST give exactly same zeroth and first-order approximations to the energy.

2.4. Strain energy of the strip

As we currently consider the strip to be a 2D elastic body, its strain energy density (i.e. energy per unit middle surface area) is given by

$$U_{2D} = \frac{1}{2} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ \rho_{11} \\ \rho_{22} \\ 2\rho_{12} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ \rho_{11} \\ \rho_{22} \\ 2\rho_{12} \end{Bmatrix}, \tag{27}$$

where $\varepsilon_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are the 2D strain measures defined in Eqs. (23) and (24); and A, D and B are the membrane, bending and coupling (3×3) stiffness matrices respectively.

The beam strain energy density (energy per unit length of the strip) is given by $U_{1D} = \langle U_{2D} \rangle$. In order to carry out this integration we need to obtain the unknown functions of x_2 in Eq. (27) (i.e. w_i; note that ϕ_α have been pre-determined while Δ_i do not appear in the first-order approximation to the strain energy – only in the constraints, so that they are uncoupled from the rest of the problem). This is achieved (at any given order of approximation in the variational–asymptotic method) by minimizing the strain energy functional $U = \int_0^\ell U_{1D} \mathrm{d}x_1$ subject to the constraints in Eqs. (6), (8), (9) and (11).

2.4.1. Zeroth-order approximation

The zeroth-order approximation to the 2D strain energy consists of all terms of $O(E\varepsilon^2)$ thus leading to the asymptotic classical linear theory for generally anisotropic pretwisted rectangular strips. Hence the zeroth-order approximations to the 2D membrane and bending strains are all of the non-underlined terms (of $O(\varepsilon)$) in Eqs. (23) and (24), respectively. All zeroth-order variables will be denoted by a superscript 0. Minimization of the zeroth-order strain energy leads to ε_{12}^0 (the only strain measure which has a w_1^0-dependent term), ε_{22}^0 (the only strain measure which has a w_2^0-dependent term) and ρ_{22}^0 (the only strain measure which has a w_3^0-dependent term) being expressed in terms of the other known 2D strain measures. Closed-form solutions are then obtained for the zeroth-order warping displacements w_i^0 and it is verified that their orders of magnitude agree with our estimations. For example, the shell out-of-plane warping is given by

$$w_3^0 = (12x_2^2 - b^2)\frac{\bar{B}_{12}\gamma_{11} + \bar{D}_{12}\kappa_2 - 2\bar{D}_{26}\kappa_1}{24\bar{D}_{22}} + x_2(b^2 - 4x_2^2)\frac{\bar{B}_{12}\kappa_3}{24\bar{D}_{22}} + (80x_2^4 - b^4)\frac{\bar{B}_{12}k_1\kappa_1}{960\bar{D}_{22}}. \tag{28}$$

The new stiffness variables used above (the quantities with an overbar) are defined in the Appendix. Expressions are also provided in the appendix for the shell in-plane warping field, w^0_α.

2.4.2. First-order approximation

The first-order approximation to the 2D strain energy consists of all terms of $O(E\varepsilon^2)$, $O(E\varepsilon^2\Delta)$ and $O(E\varepsilon^2\Delta^2)$. Hence, the first-order approximations to the 2D strains should contain all terms up to $O(\varepsilon\Delta^2)$. This consists of both the underlined and non-underlined terms in Eqs. (23) and (24). First-order variables will be denoted by a superscript I. The warping displacements are given by $w^{\mathrm{I}}_i = w^0_i + \tilde{w}_i$, where the perturbation quantities $\tilde{w}_i$ are assumed to be of an order higher than the corresponding zeroth-order quantities, w^0_i. Thus, $\tilde{w}_\alpha = O(\varepsilon\Delta b)$ and $\tilde{w}_3 = O(\varepsilon\Delta b/\delta_h)$. The strain measures Eqs. (23) and (24) can be then rewritten as

$$
\begin{aligned}
\varepsilon_{11} &= \underset{\varepsilon}{\gamma_{11}} - \underset{\varepsilon}{x_2\kappa_3} + \underset{\varepsilon}{k_1 x_2^2\kappa_1} + \underset{\varepsilon\Delta}{\frac{x_2^2\kappa_1^2}{2}} + \underset{\varepsilon\Delta}{w^0_3\kappa_2} + \underset{\varepsilon\Delta^2}{\tilde{w}_3\kappa_2},\\
\varepsilon_{12} &= \underset{\varepsilon}{\varepsilon^0_{12}} + \underset{\varepsilon\Delta}{\tilde{\varepsilon}_{12}},\\
\varepsilon_{22} &= \underset{\varepsilon}{\varepsilon^0_{22}} + \underset{\varepsilon\Delta}{\tilde{\varepsilon}_{22}},\\
\rho_{11} &= \underset{\varepsilon}{\kappa_2},\\
\rho_{22} &= \underset{\varepsilon}{w^0_{3,22}} + \underset{\varepsilon\Delta}{\tilde{w}_{3,22}},\\
2\rho_{12} &= -\underset{\varepsilon}{2\kappa_1}.
\end{aligned}
\tag{29}
$$

Minimization with respect to the three unknowns $\tilde{\varepsilon}_{12}$, $\tilde{\varepsilon}_{22}$ and $\tilde{w}_3$ is equivalent to minimization with respect to $\tilde{w}_i$ and is conducted in three steps:

1. Only the terms of order $\varepsilon^2\Delta^2$ are retained upon the substitution of Eq. (29) into the energy.
2. $\varepsilon^{\mathrm{I}}_{12}$ (the only strain measure which has a $\tilde{w}_1$-dependent term) and $\varepsilon^{\mathrm{I}}_{22}$ (the only strain measure which has a $\tilde{w}_2$-dependent term) are expressed in terms of the other 2D strain measures.
3. These expressions are substituted into the first-order strain energy functional, and the result is minimized with respect to $\tilde{w}_3$, subject to the appropriate constraints.

Closed-form solutions are then obtained for the perturbations to the zeroth-order warping displacements w^0_i, and it is verified that their orders of magnitude agree with our estimations. For example, the perturbation to the shell out-of-plane warping is given by

$$
\begin{aligned}
\tilde{w}_3 = &-\frac{(b^4 - 80x_2^4)\bar{B}_{12}\kappa_1^2}{1920\bar{D}_{22}} + \frac{x_2(7b^4 - 40b^2x_2^2 + 48x_2^4)(-2\bar{B}_{12}^2 + \bar{A}_{11}\bar{D}_{22})\kappa_2\kappa_3}{5760\bar{D}_{22}^2}\\
&+\frac{(7b^4 - 120b^2x_2^2 + 240x_2^4)\bar{B}_{12}^2\gamma_{11}\kappa_2}{5760\bar{D}_{22}^2} + \frac{(7b^4 - 120b^2x_2^2 + 240x_2^4)\bar{B}_{12}\bar{D}_{12}\kappa_2^2}{5760\bar{D}_{22}^2}\\
&-\frac{(7b^4 - 120b^2x_2^2 + 240x_2^4)\bar{B}_{12}\bar{D}_{26}\kappa_1\kappa_2}{2880\bar{D}_{22}^2}\\
&+\frac{[(-23b^6 - 336b^4x_2^2 - 560b^2x_2^4 + 896x_2^6)\bar{B}_{12}^2 + (29b^6 - 420b^4x_2^2 + 560b^2x_2^4 - 448x_2^6)\bar{A}_{11}\bar{D}_{22}]k_1\kappa_1\kappa_2}{161280\bar{D}_{22}^2}.
\end{aligned}
\tag{30}
$$

See the appendix for the procedure to obtain closed-form solutions to $\tilde{w}_\alpha$, the first-order perturbations of the shell in-plane warping field.

Now we can integrate U_{2D} with respect to x_2 across the width and obtain the 1D strain energy density as

$$U_{1D} = \tfrac{1}{2}\varepsilon_l^T[S_l]\varepsilon_l + \varepsilon_l^T[S_{ln}]\varepsilon_n + \tfrac{1}{2}\varepsilon_n^T[S_n]\varepsilon_n, \tag{31}$$

where the linear and non-linear 1D strain vectors, ε_l and ε_n respectively, are defined as follows:

$$\varepsilon_l = \{\gamma_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3\}^T,$$
$$\varepsilon_n = \{\kappa_1^2 \quad \kappa_2^2 \quad \kappa_2\gamma_{11} \quad \kappa_2\kappa_3 \quad \kappa_2\kappa_1\}^T, \tag{32}$$

and the matrices $[S_l]$, $[S_{ln}]$, and $[S_n]$ can be thought of as partitions of a 9×9 matrix $[S]$.

The linear stiffness matrix $[S_l]$ is given by

$$\begin{bmatrix} b\bar{\bar{A}}_{11} & -2b\bar{\bar{B}}_{16} + \dfrac{b^3\bar{\bar{A}}_{11}k_1}{12} & b\bar{\bar{B}}_{11} & 0 \\ -2b\bar{\bar{B}}_{16} + \dfrac{b^3\bar{\bar{A}}_{11}k_1}{12} & 4b\bar{\bar{D}}_{66} - \dfrac{b^3\bar{\bar{B}}_{16}k_1}{3} + \dfrac{b^5\bar{\bar{A}}_{11}{k_1}^2}{80} & -2b\bar{\bar{D}}_{16} + \dfrac{b^3\bar{\bar{B}}_{11}k_1}{12} & 0 \\ b\bar{\bar{B}}_{11} & -2b\bar{\bar{D}}_{16} + \dfrac{b^3\bar{\bar{B}}_{11}k_1}{12} & b\bar{\bar{D}}_{11} & 0 \\ 0 & 0 & 0 & \dfrac{b^3\bar{\bar{A}}_{11}}{12} \end{bmatrix}. \tag{33}$$

The non-linear stiffness matrix $[S_{ln}]$ is given by

$$\begin{bmatrix} \dfrac{b^3\bar{\bar{A}}_{11}}{24} & 0 & 0 & 0 & 0 \\ -\dfrac{b^3\bar{\bar{B}}_{16}}{12} + \dfrac{b^5\bar{\bar{A}}_{11}}{160}k_1 & \dfrac{b^5\bar{\bar{A}}_{11}\bar{D}_{12}}{360\bar{D}_{22}}k_1 & \dfrac{b^5\bar{\bar{A}}_{11}\bar{B}_{12}}{360\bar{D}_{22}}k_1 & 0 & 0 \\ \dfrac{b^3\bar{\bar{B}}_{11}}{24} - \dfrac{b^5\bar{\bar{A}}_{11}\bar{D}_{26}}{180\bar{D}_{22}}k_1 + \dfrac{b^7\bar{\bar{A}}_{11}\bar{B}_{12}}{10080\bar{D}_{22}}k_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{b^5\bar{\bar{A}}_{11}\bar{B}_{12}}{720\bar{D}_{22}} & 0 \end{bmatrix}. \tag{34}$$

The elements of the non-linear stiffness matrix $[S_n]$ are given by

$$S_{55} = \frac{b^5\bar{\bar{A}}_{11}}{320},$$

$$S_{56} = S_{58} = S_{68} = S_{78} = S_{89} = 0,$$

$$S_{57} = \frac{b^5\bar{\bar{A}}_{11}\bar{B}_{12}}{720\bar{D}_{22}},$$

$$S_{59} = -\frac{b^5 \bar{\bar{A}}_{11} \bar{D}_{26}}{360 \bar{D}_{22}} + \frac{b^7 \bar{\bar{A}}_{11} \bar{B}_{12}}{10080 \bar{D}_{22}} k_1,$$

$$S_{66} = \frac{b^5 \bar{\bar{A}}_{11} \bar{D}_{12}^2}{720 \bar{D}_{22}^2},$$

$$S_{67} = \frac{b^5 \bar{\bar{A}}_{11} \bar{B}_{12} \bar{D}_{12}}{720 \bar{D}_{22}^2},$$

$$S_{69} = -\frac{b^5 \bar{\bar{A}}_{11} \bar{D}_{12} \bar{D}_{26}}{360 \bar{D}_{22}^2} - \frac{b^7 \bar{\bar{A}}_{11} \bar{B}_{12} \bar{D}_{12}}{60480 \bar{D}_{22}^2} k_1,$$

$$S_{77} = \frac{b^5 \bar{\bar{A}}_{11} \bar{B}_{12}^2}{720 \bar{D}_{22}^2},$$

$$S_{79} = -\frac{b^5 \bar{\bar{A}}_{11} \bar{B}_{12} \bar{D}_{26}}{360 \bar{D}_{22}^2} - \frac{b^7 \bar{\bar{A}}_{11} \bar{B}_{12}^2}{60480 \bar{D}_{22}^2} k_1,$$

$$S_{88} = \frac{b^7 \bar{\bar{A}}_{11} \bar{B}_{12}^2}{10080 \bar{D}_{22}^2} - \frac{b^7 \bar{\bar{A}}_{11}^2}{30240 \bar{D}_{22}},$$

$$S_{99} = \frac{b^5 \bar{\bar{A}}_{11} \bar{D}_{26}^2}{180 \bar{D}_{22}^2} + \frac{b^5 \bar{\bar{A}}_{11} \bar{D}_{12}}{360 \bar{D}_{22}} + \frac{b^7 \bar{\bar{A}}_{11} \bar{B}_{12} \bar{D}_{26}}{15120 \bar{D}_{22}^2} k_1 - \left(\frac{b^9 \bar{\bar{A}}_{11}^2}{90720 \bar{D}_{22}} + \frac{b^9 \bar{\bar{A}}_{11} \bar{B}_{12}^2}{403200 \bar{D}_{22}^2} \right) k_1^2. \tag{35}$$

The new stiffness variables used above (the quantities with a double overbar) are defined in the appendix. Linear extension-twist coupling is reflected in S_{12}, while non-linear extension-twist coupling is exhibited in S_{15}. A purely torsion-twist non-linearity is exhibited in S_{25} and S_{55}. Pretwist affects only the *linear* extension-twist coupling stiffness, but it affects both the linear (S_{22}) and non-linear (S_{25}) parts of the purely torsional non-linear terms in the energy.

As expected, the two bending measures differ widely in their behavior. Note that κ_3, which represents bending about the x_3 axis (i.e. the stiff direction), is small compared to the other two 1D curvatures and is coupled only in a very weak non-linear fashion with the other bending, κ_2, through S_{48} and S_{88}. On the other hand, the flatwise bending, κ_2, could be very large and is extensively coupled with extension and twist. Linear bending- twist coupling comes from S_{23}, while non-linear bending-twist coupling stems from a large number of terms like S_{35}, S_{26}, S_{59}, S_{69} and S_{99}. All the above observations are for generally anisotropic strips. Many of the couplings drop out for special layups.

3. Applications

3.1. Extension-twist coupling in pretwisted antisymmetric laminates

In order to evaluate the developed analytical model, we study the special case of cantilevered laminated strips with antisymmetric layups and loaded only by an axial force at the tip. The reason for the selection of this specialized case was two-fold. Antisymmetric layups produce laminates with extension-twist coupling; the non-linear coupling between extension and twist is one of the predominant non-linear effects of the present analysis. Secondly, experimental and theoretical results have been recently made available for this specialized case [14], enabling validation of the present theory.

For antisymmetric laminates, the definitions for the stiffness coefficients with a bar are greatly simplified because

$$A_{16} = A_{26} = 0,$$
$$D_{16} = D_{26} = 0,$$
$$B_{11} = B_{22} = B_{12} = B_{66} = 0. \tag{36}$$

This results in the decoupling of κ_2 from the rest of the problem.

The equilibrium equations are derived via the principle of virtual work. First the strain energy is given as

$$U = \int_0^\ell U_{1\mathrm{D}}(\gamma_{11}, \kappa_1, \kappa_2, \kappa_3)\,\mathrm{d}x_1, \tag{37}$$

where the geometrically exact strain measures for classical theory can be found, for example, in [15]. The principle of virtual work for an axially loaded strip can be written as

$$\delta U = F_1 \delta u_1(\ell), \tag{38}$$

where F_1 is the axial load applied at the tip $x_1 = \ell$. For the case of an antisymmetric layup under an axial force, both κ_2 and κ_3 are zero. The geometrically exact strain–displacement relations reduce to $\gamma_{11} = u_1'$ and $\kappa_1 = \theta_1'$ where θ_1 is the elastic angle of twist. The two governing equilibrium equations thus reduce to algebraic equations for the coupled extension-twist problem:

$$\frac{\partial U_{1\mathrm{D}}}{\partial \gamma_{11}} = F_1, \quad \frac{\partial U_{1\mathrm{D}}}{\partial \kappa_1} = 0. \tag{39}$$

These equations are solved by using the first equation to eliminate γ_{11} in favor of F_1 and then using the second to express F_1 in terms of κ_1. For constant k_1, the tip pretwist angle $\theta_0 = \ell k_1$ and κ_1 is also a constant so that the elastic tip twist angle $\theta = \ell \kappa_1$. The result is

$$F_1 = \frac{[c_1 + c_2(\theta^2 + 3\theta\theta_0 + 2\theta_0^2)]\theta}{c_3 - (\theta + \theta_0)}, \tag{40}$$

where

$$c_1 = \frac{48}{b}\left(\bar{D}_{66} - \frac{\bar{B}_{16}^2}{A_{11}}\right),$$
$$c_2 = \frac{b^3 \bar{A}_{11}}{30\ell^2},$$
$$c_3 = \frac{24\ell \bar{B}_{16}}{b^2 \bar{A}_{11}}. \tag{41}$$

It is observed that the contribution from the c_2 term is negligible in comparison to that of the c_1 term. As a result the extension-twist relation developed above takes the following simple form:

$$F_1 = \frac{c_1 \theta}{c_3 - (\theta + \theta_0)}. \tag{42}$$

The laminates considered by Armanios *et al.* [14] were fabricated from ICI Fiberite T300/954-3 graphite/cyanate prepreg material using the stacking sequence

$$[\alpha_2/(\alpha - 90^\circ)_4/\alpha_2/-\alpha_2/(90^\circ - \alpha)_4/-\alpha_2]_T. \tag{43}$$

Three specimens each of two laminates are considered with $\alpha = 20^\circ$ and $\alpha = 30^\circ$. They both have a constant pretwist of -0.20°/cm. Results are shown in Fig. 2 for $\alpha = 20^\circ$ laminates and in Fig. 3 for $\alpha = 30^\circ$ laminates. We see that the results of the present theory agree well with the experimental data. It is observed that the inclusion of a small pretwist does not significantly affect the results. Furthermore, the present theory, when specialized for antisymmetric laminates with no bending and torsional loads, reduces to that of Armanios *et al.* [14] for the case of infinite transverse shear stiffness. It was also shown in [14] that the effects of finite transverse shear stiffness are very small and that for the layups considered here for validation, internal stresses due to the curing process are negligibly small.

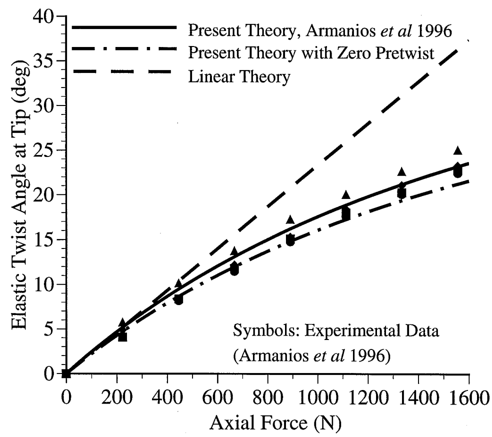

Fig. 2. Extension-twist coupling ($\alpha = 20^\circ$).

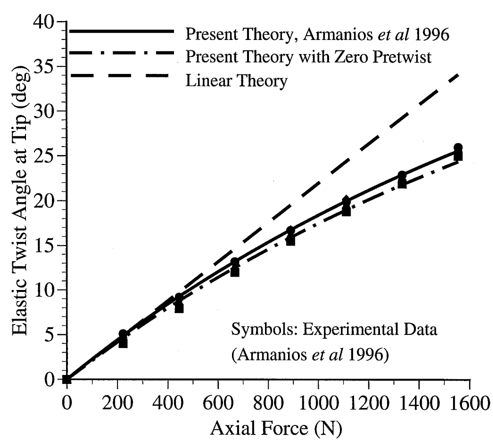

Fig. 3. Extension-twist coupling ($\alpha = 30^\circ$).

3.2. Torsional buckling of a column with cruciform section

Another application of the results obtained in this paper would be in the analysis of torsional buckling of columns made up of an arbitrary geometrical combination of pretwisted generally anisotropic strips. As an example, we consider here a column with a cruciform cross section shown in Fig. 4. It is composed of two untwisted strips, denoted by a and b. The stiffness properties and warping fields of the component strips will be denoted by the use of superscripts a and b, respectively. The strain energy of the section is minimized subject to the following constraints on the warping variables:

$$h_b \int_{-b/2}^{b/2} w_i^b \, dx_3 + h_a \int_{-a/2}^{a/2} w_i^a \, dx_2 = 0,$$

$$h_b \int_{-b/2}^{b/2} (w_{2,3}^b - \phi_3^b) \, dx_3 + h_a \int_{-a/2}^{a/2} (\phi_2^a - w_{3,2}^a) \, dx_2 = 0$$

$$\Delta_i^b(x_1, 0, 0) + w_i^b|_{x_3=0} - \Delta_i^a(x_1, 0, 0) - w_i^a|_{x_2=0} = 0,$$

$$(w_{2,3}^b|_{x_3=0} - \phi_3^b|_{x_3=0}) + (\Delta_{2,3}^b(x_1, 0, 0) - \Delta_{3,2}^b(x_1, 0, 0))$$

$$- (\phi_2^a|_{x_2=0} - w_{3,2}^a|_{x_2=0}) - (\Delta_{2,3}^a(x_1, 0, 0) - \Delta_{3,2}^a(x_1, 0, 0)) = 0,$$

$$\int_{-h_a/2}^{h_a/2} \Delta_i^a \, dx_3 = 0,$$

$$\int_{-h_b/2}^{h_b/2} \Delta_i^b \, dx_2 = 0,$$

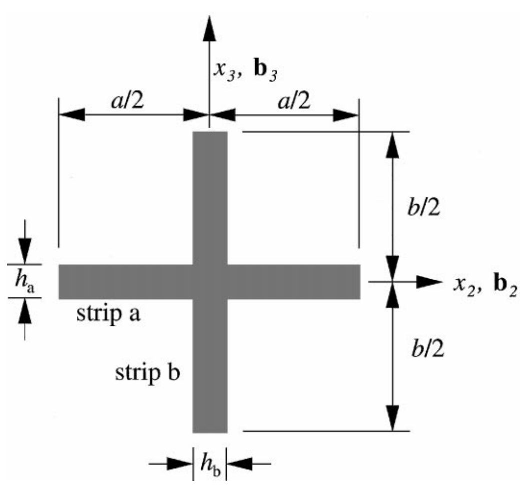

Fig. 4. Cruciform configuration and coordinate system.

$$\int_{-h_a/2}^{h_a/2} \Delta^a_{1,3}\,\mathrm{d}x_3 = 0,$$

$$\int_{-h_b/2}^{h_b/2} \Delta^b_{1,2}\,\mathrm{d}x_2 = 0,$$

$$\int_{-h_a/2}^{h_a/2} \Delta^a_{2,3}\,\mathrm{d}x_3 = 0,$$

$$\int_{-h_b/2}^{h_b/2} \Delta^b_{3,2}\,\mathrm{d}x_2 = 0. \tag{44}$$

We substitute the resulting solution for the warping field into the energy density and integrate over the section to get the 1D energy from which the elements of the 1D stiffness matrix are extracted. We now consider a cantilevered column with a cruciform section loaded only by an axial force F_1 at the tip. As in the case of the strip, the governing equilibrium equations (see Eq. (39)) are developed using the principle of virtual work and solved to obtain F_1 in terms of κ_1. Torsional buckling can now be analysed. The lowest value of the compressive load, $-F_1 = F_{\mathrm{TB}}$, at which the elastic twist approaches infinity is the torsional buckling load. This is given below for the case of a cruciform section made up of strips which have antisymmetric layups and are of equal width (See the appendix for the more general case $a \neq b$.)

$$F_{\mathrm{TB}} = \frac{48}{a}\left[\bar{D}^a_{66} + \bar{D}^b_{66} - \frac{(\bar{B}^a_{16} - \bar{B}^b_{16})^2}{\bar{A}^a_{11} + \bar{A}^b_{11}}\right]. \tag{45}$$

Furthermore if both strips are made of identical material and layup, the torsional buckling load is $96\bar{D}_{66}/a$.

4. Conclusions

Few approaches are available in the literature to analyze the non-linear "trapeze effect" for anisotropic beams of arbitrary geometry. Since they are not based on asymptotic approaches, the asymptotical correctness of these analyses is difficult to assess. Motivated by the need for a rigorous approach to this problem, we have undertaken a preliminary study of the trapeze phenomenon in anisotropic beams by developing a geometrically non-linear cross-sectional analysis for pretwisted anisotropic strips. The resulting non-linear terms appearing in the 1D constitutive relations are essential to satisfy asymptotical correctness of the theory. They occur naturally owing to the existence of small geometric parameters in the structure. The dominant additional term in this analysis, relative to those in a strictly classical approach, is due to elastic twist and flatwise bending.

Our results confirm the known importance in isotropic strips of non-linear extension-twist coupling for anisotropic ones, in agreement with recently obtained experimental data. Our study also shows that cross-sectional stiffness constants that are derived from a linear cross-sectional analysis are insufficient for modeling thin anisotropic strips in the presence of large axial loads.

The extension of the theory to beams made of a combination of pretwisted anisotropic strips is indicated through an example. The torsional buckling load of a column with a cruciform cross section made up of untwisted antisymmetric laminates is derived. As expected, D_{66} is the main determining factor while the length and the small pretwist do not affect the buckling load in the first approximation. It should be observed that without non-linear effects of the type derived herein, it is not possible to predict the torsional buckling load.

The theory presented in this paper is part of an overall framework of tools under development for cross-sectional analysis of rotor blades and other slender, anisotropic structural members. The goals include development of finite element approaches for arbitrary, built-up cross-sectional geometries, and closed-form expressions for thin-walled cross sections with arbitrary geometry. To meet these goals, the cross-sectional analysis for classical theory, already developed [5], is being extended to include the dominant non-classical effects in the 1D theory [16], such as numerical treatment of non-linear effects [17], Timoshenko- and Vlasov-like end effects [18], higher-frequency dynamic effects [19], other non-linear phenomena such as the Brazier effect, etc. Work that is underway will be reported in later papers.

Acknowledgements

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Appendix

The stiffness variables used in the paper are defined as follows:

$$\bar{A}_{11} = A_{11} + \frac{A_{16}^2 A_{22} - 2A_{12}A_{16}A_{26} + A_{12}^2 A_{66}}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{B}_{11} = B_{11} + \frac{A_{12}A_{66}B_{12} + A_{16}A_{22}B_{16} - A_{26}(A_{16}B_{12} + A_{12}B_{16})}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{B}_{12} = B_{12} + \frac{A_{12}A_{66}B_{22} + A_{16}A_{22}B_{26} - A_{26}(A_{16}B_{22} + A_{12}B_{26})}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{B}_{16} = B_{16} + \frac{A_{12}A_{66}B_{26} + A_{16}A_{22}B_{66} - A_{26}(A_{16}B_{26} + A_{12}B_{66})}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{D}_{11} = D_{11} + \frac{A_{66}B_{12}^2 - 2A_{26}B_{12}B_{16} + A_{22}B_{16}^2}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{D}_{12} = D_{12} + \frac{A_{66}B_{12}B_{22} + A_{22}B_{16}B_{26} - A_{26}(B_{16}B_{22} + B_{12}B_{26})}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{D}_{22} = D_{22} + \frac{A_{66}B_{22}^2 - 2A_{26}B_{22}B_{26} + A_{22}B_{26}^2}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{D}_{16} = D_{16} + \frac{A_{66}B_{12}B_{26} + A_{22}B_{16}B_{66} - A_{26}(B_{16}B_{26} + B_{12}B_{66})}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{D}_{26} = D_{26} + \frac{A_{66}B_{22}B_{26} + A_{22}B_{26}B_{66} - A_{26}({B_{26}}^2 + B_{22}B_{66})}{A_{26}^2 - A_{22}A_{66}},$$

$$\bar{D}_{66} = D_{66} + \frac{A_{66}B_{26}^2 - 2A_{26}B_{26}B_{66} + A_{22}B_{66}^2}{A_{26}^2 - A_{22}A_{66}}. \tag{A1}$$

$$\bar{\bar{A}}_{11} = \bar{A}_{11} - \frac{\bar{B}_{12}^2}{\bar{D}_{22}},$$

$$\bar{\bar{B}}_{11} = \bar{B}_{11} - \frac{\bar{B}_{12}\bar{D}_{12}}{\bar{D}_{22}},$$

$$\bar{\bar{B}}_{16} = \bar{B}_{16} - \frac{\bar{B}_{12}\bar{D}_{26}}{\bar{D}_{22}},$$

$$\bar{\bar{D}}_{11} = \bar{D}_{11} - \frac{\bar{D}_{12}^2}{\bar{D}_{22}},$$

$$\bar{\bar{D}}_{16} = \bar{D}_{16} - \frac{\bar{D}_{12}\bar{D}_{26}}{\bar{D}_{22}},$$

$$\bar{\bar{D}}_{66} = \bar{D}_{66} - \frac{\bar{D}_{26}^2}{D_{22}}. \tag{A2}$$

The zeroth-order approximation to the shell in-plane warping field of a generally anisotropic pretwisted strip is

$$\begin{aligned}
w_1^0 = {} & \frac{x_2\gamma_{11}[(-b^2 - 4x_2^2)A_{26}^2 k_1 \bar{B}_{12} + 24A_{26}(B_{22}\bar{B}_{12} - A_{12}\bar{D}_{22})]}{24(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{x_2\gamma_{11}A_{22}(-24B_{26}\bar{B}_{12} + b^2 A_{66}k_1\bar{B}_{12} + 4x_2^2 A_{66}k_1\bar{B}_{12} + 24A_{16}\bar{D}_{22})}{24(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{\kappa_3[(-b^4 + 80x_2^4)A_{26}^2 k_1 B_{12} + 40(b^2 - 12x_2^2)A_{26}(B_{22}\bar{B}_{12} - A_{12}\bar{D}_{22})]}{960(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{\kappa_3 A_{22}[-40(b^2 - 12x_2^2)B_{26}\bar{B}_{12} + b^4 A_{66}k_1\bar{B}_{12} - 80x_2^4 A_{66}k_1\bar{B}_{12} + 40b^2 A_{16}\bar{D}_{22} - 480x_2^2 A_{16}\bar{D}_{22}]}{960(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{x_2\kappa_2[(-b^2 - 4x_2^2)A_{26}^2 k_1\bar{D}_{12} + 24A_{26}(B_{22}\bar{D}_{12} - B_{12}\bar{D}_{22})]}{24(A_{26}^2{}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{x_2\kappa_2 A_{22}(-24B_{26}\bar{D}_{12} + b^2 A_{66}k_1\bar{D}_{12} + 4x_2^2 A_{66}k_1\bar{D}_{12} + 24B_{16}\bar{D}_{22})}{24(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{320x_2\kappa_1 A_{26}[(6B_{26} - x_2^2 A_{12}k_1)\bar{D}_{22} + B_{22}(x_2^2 k_1\bar{B}_{12} - 6\bar{D}_{26})]}{960(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{x_2\kappa_1 A_{26}^2 k_1[(-b^4 - 48x_2^4)k_1\bar{B}_{12} + 80(b^2 + 4x_2^2)\bar{D}_{26}]}{960(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{x_2\kappa_1 A_{22}(b^4 A_{66}k_1^2\bar{B}_{12} + 48x_2^4 A_{66}k_1^2\bar{B}_{12} - 1920B_{66}\bar{D}_{22} + 320x_2^2 A_{16}k_1\bar{D}_{22})}{960(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}} \\
& + \frac{x_2\kappa_1 A_{22}[-320B_{26}(x_2^2 k_1\bar{B}_{12} - 6\bar{D}_{26}) - 80b^2 A_{66}k_1\bar{D}_{26} - 320x_2^2 A_{66}k_1\bar{D}_{26}]}{960(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}},
\end{aligned} \tag{A3}$$

$$w_2^0 = \frac{x_2\gamma_{11}[A_{66}(A_{12}\bar{D}_{22} - B_{22}\bar{B}_{12}) + A_{26}(B_{26}\bar{B}_{12} - A_{16}\bar{D}_{22})]}{(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}}$$
$$- \frac{(b^2 - 12x_2^2)\kappa_3[A_{66}(B_{22}\bar{B}_{12} - A_{12}\bar{D}_{22}) + A_{26}(-B_{26}\bar{B}_{12} + A_{16}\bar{D}_{22})]}{24(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}}$$
$$+ \frac{x_2\kappa_2[A_{66}(-B_{22}\bar{D}_{12} + B_{12}\bar{D}_{22}) + A_{26}(B_{26}\bar{D}_{12} - B_{16}\bar{D}_{22})]}{(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}}$$
$$+ \frac{x_2\kappa_1 A_{26}[(6B_{66} - x_2^2A_{16}k_1)\bar{D}_{22} + B_{26}(x_2^2k_1\bar{B}_{12} - 6\bar{D}_{26})]}{3(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}}$$
$$+ \frac{x_2\kappa_1 A_{66}[(-6B_{26} + x_2^2A_{12}k_1)\bar{D}_{22} + B_{22}(-x_2^2k_1\bar{B}_{12} + 6\bar{D}_{26})]}{3(A_{26}^2 - A_{22}A_{66})\bar{D}_{22}}. \tag{A4}$$

Closed-form solutions to the first-order approximation, w_α^{I}, to the shell in-plane warping field of a generally anisotropic pretwisted strip were obtained by the following procedure:

$$w_1^{\mathrm{I}} = C_1 - (k_1 + \kappa_1)\int(x_2w_{3,2}^0 - w_3^0 + x_2\tilde{w}_{3,2} - \tilde{w}_3)\,\mathrm{d}x_2$$
$$- \int\left[\frac{-A_{26}(A_{12}\varepsilon_{11}^{\mathrm{I}} + B_{12}\rho_{11}^{\mathrm{I}} + 2B_{26}\rho_{12}^{\mathrm{I}} + B_{22}\rho_{22}^{\mathrm{I}}) + A_{22}(A_{16}\varepsilon_{11}^{\mathrm{I}} + B_{16}\rho_{11}^{\mathrm{I}} + 2B_{66}\rho_{12}^{\mathrm{I}} + B_{26}\rho_{22}^{\mathrm{I}})}{-A_{26}^2 + A_{22}A_{66}}\right]\mathrm{d}x_2,$$

$$w_2^{\mathrm{I}} = C_2 - \int\left(\frac{w_{3,2}^{0\,2}}{2} + w_{3,2}^0\tilde{w}_{3,2}\right)\mathrm{d}x_2$$
$$- \int\left(\frac{A_{66}B_{26}2\rho_{12}^{\mathrm{I}} - A_{26}B_{66}2\rho_{12}^{\mathrm{I}} - A_{16}A_{26}\varepsilon_{11}^{\mathrm{I}} + A_{12}A_{66}\varepsilon_{11}^{\mathrm{I}}}{-A_{26}^2 + A_{22}A_{66}}\right)$$
$$+ \left(\frac{A_{66}B_{12}\rho_{11}^{\mathrm{I}} - A_{26}B_{16}\rho_{11}^{\mathrm{I}} + A_{66}B_{22}\rho_{22}^{\mathrm{I}} - A_{26}B_{26}\rho_{22}^{\mathrm{I}}}{-A_{26}^2 + A_{22}A_{66}}\right)\mathrm{d}x_2, \tag{A5}$$

where the integration constants, C_α, are found using the constraints $\langle w_\alpha\rangle = 0$.

For antisymmetric layups, the torsional buckling load, F_{TB}, of a column with cruciform section in the general case $a \neq b$, is given by

$$\frac{48b(a\bar{A}_{11}^a + b\bar{A}_{11}^b)\bar{D}_{66}^b}{a^3\bar{A}_{11}^a + b^3\bar{A}_{11}^b}$$
$$+ \frac{24}{(a^3\bar{A}_{11}^a + b^3\bar{A}_{11}^b)(4a^6\bar{A}_{11}^{a\;2} + 9a^5b\bar{A}_{11}^a\bar{A}_{11}^b - 10a^3b^3\bar{A}_{11}^a\bar{A}_{11}^b + 9ab^5\bar{A}_{11}^a\bar{A}_{11}^b + 4b^6\bar{A}_{11}^{b\;2})}$$
$$\times(-8a^8\bar{A}_{11}^{a2}\bar{B}_{16}^{a2} - 18a^7b\bar{A}_{11}^a\bar{A}_{11}^b\bar{B}_{16}^{a2} + 20a^5b^3\bar{A}_{11}^a\bar{A}_{11}^b\bar{B}_{16}^{a2}$$
$$- 18a^3b^5\bar{A}_{11}^a\bar{A}_{11}^b\bar{B}_{16}^{a2} - 15a^6b^2\bar{A}_{11}^{b2}\bar{B}_{16}^{a2} + 30a^4b^4\bar{A}_{11}^{b2}\bar{B}_{16}^{a2}$$
$$- 23a^2b^6\bar{A}_{11}^{b2}\bar{B}_{16}^{a2} + 16a^7b\bar{A}_{11}^{a2}\bar{B}_{16}^a\bar{B}_{16}^b + 6a^6b^2\bar{A}_{11}^a\bar{A}_{11}^b\bar{B}_{16}^a\bar{B}_{16}^b$$
$$+ 20a^4b^4\bar{A}_{11}^a\bar{A}_{11}^b\bar{B}_{16}^a\bar{B}_{16}^b + 6a^2b^6\bar{A}_{11}^a\bar{A}_{11}^b\bar{B}_{16}^a\bar{B}_{16}^b + 16ab^7\bar{A}_{11}^{b2}\bar{B}_{16}^a\bar{B}_{16}^b$$
$$- 23a^6b^2\bar{A}_{11}^{a2}\bar{B}_{16}^{b2} + 30a^4b^4\bar{A}_{11}^{a2}\bar{B}_{16}^{b2} - 15a^2b^6\bar{A}_{11}^{a2}\bar{B}_{16}^{b2}$$

$$
\begin{aligned}
& - 18a^5b^3\bar{A}^a_{11}\bar{A}^b_{11}\bar{B}^{b2}_{16} + 20a^3b^5\bar{A}^a_{11}\bar{A}^b_{11}\bar{B}^{b2}_{16} - 18ab^7\bar{A}^a_{11}\bar{A}^b_{11}\bar{B}^{b2}_{16} \\
& - 8b^8\bar{A}^{b2}_{11}\bar{B}^{b2}_{16} + 8a^8\bar{A}^{a3}_{11}\bar{D}^a_{66} + 26a^7b\bar{A}^{a2}_{11}\bar{A}^b_{11}\bar{D}^a_{66} \\
& - 20a^5b^3\bar{A}^{a2}_{11}\bar{A}^b_{11}\bar{D}^a_{66} + 18a^3b^5\bar{A}^{a2}_{11}\bar{A}^b_{11}\bar{D}^a_{66} + 18a^6b^2\bar{A}^a_{11}\bar{A}^{b2}_{11}\bar{D}^a_{66} \\
& - 20a^4b^4\bar{A}^a_{11}\bar{A}^{b2}_{11}\bar{D}^a_{66} + 26a^2b^6\bar{A}^a_{11}\bar{A}^{b2}_{11}\bar{D}^a_{66} + 8ab^7\bar{A}^{b3}_{11}\bar{D}^a_{66}). \qquad \text{(A6)}
\end{aligned}
$$

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