Journal of Sound and Vibration (1991) **145**(2), 169-178

SHAPE FUNCTIONS FOR MIXED p-VERSION FINITE ELEMENTS IN THE TIME DOMAIN

D. H. HODGES AND L. J. HOU

School of Aerospace Engineering, Georgia Institute for Technology, Atlanta, Georgia 30332-0150, *U.S.A.*

(*Received* 23 *October* 1989, *and in final form* 30 *April* 1990)

The advantages of mixed formulations are well known in that they simplify the form of the governing equations and can lead to potentially significant savings in computational effort for given accuracy. Likewise, the p-version of the finite element method is known to offer savings in computational effort for some problems. In this paper, shape functions for a mixed p-version finite element formulation in the time domain are developed. These shape functions produce a high percentage of zeros in certain coefficient matrices. The mixed formulation in time used here is based on a weak form of Hamilton's extended principle. The present formulation allows for accurate and efficient solution of time-domain problems. The accuracy and power of this formulation are shown for initial value problems (such as time marching) and for directly obtaining the periodic response of linear systems. The method is shown to yield a family of accurate, unconditionally stable algorithms for initial-value problems based on a simple linear oscillator.

1. INTRODUCTION

The so-called p-version of the finite element method [1] allows for additional higher order polynomial shape functions within elements, whereas h-version elements have only the minimal set of shape functions. Thus, the p-version can be used to improve the accuracy of the solution by adding more unknowns to each element without refining the mesh.

There are known problems in mechanics for which the p-version finite element method is superior to the h-version. For example, in the usual application of the finite element method as a method of spatial discretization, the p-version allows for achieving greater accuracy for a given number of unknowns (and fewer elements) in the assembled equations, contributing to computational efficiency. Some examples are discussed in depth by Babuska *et al.* [1]. A simple, one-dimensional example is considered in reference [2].

It is convenient for programming purposes if the shape functions are hierarchical (meaning that adding shape functions does not alter the ones already in use) [1]. Often the additional unknowns can be condensed out at the element level further contributing to computational efficiency. Some one-dimensional, hierarchical shape functions for displacement methods have been developed by Hodges [3].

Recently, finite element methods have been developed for the time domain. The earliest works seem to be those of Fried [4] and Argyris and Scharpf [5], although most of the work being done now has its roots in the mid-1970s when the use of Hamilton's principle was introduced. Hamilton's principle has traditionally been used in *analytical* mechanics as a method of obtaining the equations of motion for dynamical systems. Bailey [6, 7], followed by several others, obtained direct solutions to dynamics problems using a form of Hamilton's principle known as the law of varying action thus opening the door for its

0022-460X/91/050169+10 $03.00/0

use in *computational* mechanics; see the papers by Simkins [8], Hitzl and Levinson [9], Baruch and Riff [10], Borri *et al.* [11], and Peters and Izadpanah [12], for example.

More recently, it was shown in references [11, 12] that expression of Hamilton's law as a weak form (commonly referred to as Hamilton's weak principle or HWP) provides a powerful alternative to numerical solution of ordinary differential equations in the time domain. The accuracy of the time-marching procedure derived in these works is competitive with standard ordinary differential equation solvers. However, in order to derive an unconditionally stable algorithm, Borri *et al.* [11] used reduced/selective element quadrature. Later work reported by Borri *et al.* [13] has shown that further computational advantages may be obtained in mixed formulations of HWP in which the generalized co-ordinates and momenta appear as independent unknowns. In this work, an unconditionally stable algorithm emerges for the linear oscillator with exact element quadrature. This result is obtained in spite of the fact that the crudest possible shape functions were used for the equations written in the weakest possible form (i.e., where no field variable is differentiated and all boundary conditions are natural ones).

HWP has also been shown to be an ideal tool for obtaining a periodic solution for dynamical systems, as well as finding the corresponding transition matrix for analysis of the stability of small perturbations about the periodic solution [14]. Also, Hodges and Bless [15] have applied the mixed method to two-point boundary-value problems related to optimal control theory.

In this paper, we explore a relatively narrow portion of the spectrum of problems outlined above. For highly non-linear problems, it has been shown in reference [15] that there are computational advantages of using the weakest possible form of the equations, as in reference [13], so that the crudest shape functions can be used. This h-version formulation allows the element quadrature to be carried out analytically thus avoiding expensive Gaussian quadrature each iteration. However, *for linear problems*, where element quadrature has to be done only once, it is evident from earlier work involving displacement-based [2] methods that the well known computational advantages of the p-version will carry over for mixed methods. Herein, we develop a set of hierarchical polynomials for mixed formulations with the intent to obtain very sparse coefficient matrices. It is also stipulated that the shape functions of the p-version reduce to those of the h-version [13, 15] when the order is at its minimum value. The accuracy and efficiency of this approach for linear problems also will be explored.

2. HAMILTON'S WEAK PRINCIPLE

Let us now consider an arbitrary holonomic mechanical system, the configuration of which is completely defined by a set of generalized co-ordinates q. Furthermore, let us denote by $L(q, \dot{q}, t)$ the Lagrangean of the system, Q the set of non-conservative generalized forces applied to the system, and by $p = \partial L/\partial \dot{q}$ the set of generalized momenta. Then, the variational equation

$$\int_{t_1}^{t_2} \delta L \, \mathrm{d}t + \int_{t_1}^{t_2} \delta q^{\mathrm{T}} Q \, \mathrm{d}t = \delta q^{\mathrm{T}} p \Big|_{t_1}^{t_2} \tag{1}$$

first used by Borri *et al.* [11] and called Hamilton's weak principle (HWP) describes the real motion of the system between the two known times t_1 and t_2. Here p appears only as a discrete quantity at the ends of the time interval, and its relationship to the Lagrangean is enforced as a natural boundary condition. This allows p to be found without differentiation of q. Equation (1) is said to be in displacement form because it only involves the

variation of q. Borri *et al.* [11] applied equation (1) to an initial-value problem where the values of q and p are known at time t_1 and are to be determined at time t_2. This equation, with reduced element quadrature over the element, leads to an explicit step-by-step integration formula which yields a very good approximation. Unfortunately, exact element quadrature results in a conditionally stable time-marching algorithm. To avoid numerical instability, reduced (inaccurate) element quadrature of the stiffness term is required. This is analogous to the locking of elements in finite element structural representations. It is well known that mixed formulations can circumvent these difficulties (e.g., see reference [16]).

To derive the mixed formulation, we begin by defining the Hamiltonian as

$$H(q, p, t) \equiv p^{\mathrm{T}}\dot{q} - L(q, \dot{q}, t). \tag{2}$$

Taking the variation of equation (2) and substituting for δL in equation (1) results in

$$\int_{t_1}^{t_2} (\delta p^{\mathrm{T}}\dot{q} + \delta\dot{q}^{\mathrm{T}}p - \delta H + \delta q^{\mathrm{T}}Q)\,\mathrm{d}t = \delta q^{\mathrm{T}}p\,|_{t_1}^{t_2}. \tag{3}$$

Integrating the first term of the integrand by parts yields

$$\int_{t_1}^{t_2} (\delta\dot{q}^{\mathrm{T}}p - \delta\dot{p}^{\mathrm{T}}q - \delta H + \delta q^{\mathrm{T}}Q)\,\mathrm{d}t = (\delta q^{\mathrm{T}}p - \delta p^{\mathrm{T}}q)|_{t_1}^{t_2}. \tag{4}$$

This form was first used by Borri *et al.* [13] is called a mixed formulation because it contains independent variations of q and p. If the p and q on the right side are regarded as discrete quantities, distinct from their counterparts under the integral on the left side, equation (4) is then in a very *weak* form. All boundary conditions are of the natural type, and all variations are completely arbitrary.

There are two main advantages of the mixed formulation over the displacement formulation. The first advantage is that the mixed formulation generally provides a more accurate solution for a given level of computational effort than does the displacement formulation. The second advantage is that a simpler choice of shape functions is allowed. Note in equation (4) that time derivatives of δq and δp are present. However, no time derivatives of q and p exist. Therefore, it is possible to implement C^0 continuous shape functions for δq and δp and continuous shape functions for q and p within each element. This way, the discrete values of q and p in the trailing terms may take on values distinct from those inside the element integral. In reference [15] it is shown to be advantageous for non-linear problems to use the simplest linear functions for δq and δp and constant shape functions for q and p within each element. This allows one to circumvent numerical quadrature within each element. However, for linear problems one should be able to develop an efficient p-version based on equation (4) since the numerical quadrature need only be undertaken once.

3. DEVELOPMENT OF SHAPE FUNCTIONS

We now turn our attention to obtaining p-version shape functions. Let the time interval be $\Delta t = t_2 - t_1$, where the quantities t_i for $i = 1, 2$ are the nodal values of the time. Now the time can be written as $t = t_1 + \tau\Delta t$, where the non-dimensional time is denoted by

$\tau = (t - t_1)/\Delta t$ and $0 \leq \tau \leq 1$. Rewriting equation (4) in dimensionless form, one then obtains

$$\int_0^1 [\delta q'^{\mathrm{T}} p - \delta p'^{\mathrm{T}} q - \Delta t(\delta H + \delta q^{\mathrm{T}} Q)]\, \mathrm{d}\tau = (\delta q^{\mathrm{T}} p - \delta p^{\mathrm{T}} q)|_0^1. \tag{5}$$

Derivatives with respect to non-dimensional time τ are denoted by $(\ )'$. As noted above, C^0 shape functions may be used to be used to represent δq and δp. For the present development, these functions are chosen to be hierarchical "bubble" functions, similar to those of reference [3]:

$$\delta p = \delta p_1(1-\tau) + \delta p_2 \tau + (1-\tau)\tau \sum_{i=1}^{N-2} \delta p_i^* \beta_i(\tau) f_i, \qquad 0 \leq \tau \leq 1, \tag{6}$$

and

$$\delta q = \delta q_1(1-\tau) + \delta q_2 \tau + (1-\tau)\tau \sum_{i=1}^{N-2} \delta q_i^* \beta_i(\tau) f_i, \qquad 0 \leq \tau \leq 1. \tag{7}$$

Here δq_1 and δp_1 denote values of δq and δp at the left node of the element, δq_2 and δp_2 denote values of δq and δp at the right node of the element, and $N-1$ is the degree of the polynomials describing δp and δq within the element. The summation term is excluded if $N = 2$ (its minimum value). The functions β_i are the Jacobi polynomials so that $\beta_i(\tau) = G_{i-1}(P, Q; \tau)$ and P and Q are parameters to be determined later. The factors f_i are normalizing constants, also to be found below. Detailed mathematical characteristics of Jacobi polynomials may be found in reference [17].

The approximate values of p and q are taken as continuous functions on the element interior while allowing for distinct, discrete values on the element boundaries. Thus,

$$\begin{aligned} p &= \sum_{j=1}^{N-1} \bar{p}_j \alpha_j(\tau), \quad & q &= \sum_{j=1}^{N-1} \bar{q}_j \alpha_j(\tau), \quad & 0 < \tau < 1 \\ p &= \hat{p}_1 & q &= \hat{q}_1 & \tau &= 0 \\ p &= \hat{p}_2 & q &= \hat{q}_2 & \tau &= 1. \end{aligned} \tag{8}$$

Here $\hat{p}_1$ and $\hat{q}_1$ denote discrete values of p and q at the left node of the element, $\hat{p}_2$ and $\hat{q}_2$ denote discrete values of p and q at the right node of the element, and α_i is a yet unspecified polynomial of degree $i-1$ describing p and q within the element. Note that both α_i and β_i are of degree $i-1$ and that these shape functions reduce to those of reference [15] for the simplest case when $N = 2$ if $\alpha_1 = 1$.

In order to show the form of the resulting equations in a simple manner, we specialize the analysis for a one-degree-of-freedom system. The polynomial $\alpha_j(\tau)$ will now be found which will lead to the simplest algebraic equations for each element. These polynomials will be determined based on the first two terms which appear in the integrand of equation (5). These terms always appear in the formulation regardless of the complexity of the system being modeled. These terms are of the form $\int_0^1 p\delta q'\, \mathrm{d}\tau$ and $\int_0^1 q\delta p'\, \mathrm{d}\tau$. Note that δp and δq are assumed as free variations which exist when p and q are prescribed, and the functional forms of δp and δq are quite different from those of p and q. Since p and q are handled identically, as are their variations, it is only necessary to consider one of these integrals.

Substitution of equations (7) and (8) into $\int_0^1 \delta q' p\, \mathrm{d}\tau$ yields

$$\begin{aligned} \int_0^1 \delta q' p\, \mathrm{d}\tau = (\delta q_2 - \delta q_1) \sum_{j=1}^{N-1} \bar{p}_j \int_0^1 \alpha_j(\tau)\, \mathrm{d}\tau \\ + \sum_{i=1}^{N-2} \delta q_i^* f_i \sum_{j=1}^{N-1} \bar{p}_j \int_0^1 [\tau(1-\tau)\beta_i(\tau)]' \alpha_j(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{9}$$

For the second term above, with just the coefficient matrix considered, integration by parts yields

$$f_i \int_0^1 [\tau(1-\tau)\beta_i]'\alpha_j \, d\tau = f_i\tau(1-\tau)\beta_i\alpha_j|_0^1 - f_i \int_0^1 \tau(1-\tau)\beta_i\alpha_j' \, d\tau. \tag{10}$$

Obviously, the first term on the right side vanishes. Since β_i is a Jacobi polynomial, the second term will produce a diagonal matrix if α_j' (which is of degree $j-2$) is chosen so that

$$\alpha_i' = \beta_{i-1} f_{i-1} \tag{11}$$

and if the Jacobi polynomial parameters are chosen so that $P=3$ and $Q=2$. Now,

$$f_i \int_0^1 [\tau(1-\tau)\beta_i]'\alpha_j \, d\tau = -f_i f_{j-1} \int_0^1 \tau(1-\tau)\beta_i\beta_{j-1} \, d\tau = -\delta_{i,j-1}. \tag{12}$$

Here $\delta_{i,j-1}$ is the Kronecker symbol which vanishes unless $i=j-1$ where it has the value of unity. Thus, the normalizing factors can be determined as

$$f_i^2 = \left[\int_0^1 \tau(1-\tau)\beta_i^2 \, d\tau \right]^{-1}. \tag{13}$$

For the particular Jacobi polynomials considered here ($P=3$ and $Q=2$), the recursion relation used to compute the polynomials β_i is

$$\beta_1(\tau) = 1, \qquad \beta_2(\tau) = \tau - \tfrac{1}{2}, \qquad \beta_i(\tau) = \beta_2(\tau)\beta_{i-1}(\tau) - \beta_{i-2}(\tau)A_{i-2}, \tag{14}$$

where

$$A_i = i(i+2)/4(2i+1)(2i+3). \tag{15}$$

This way

$$f_1^2 = 6, \quad f_{i+1}^2 = \frac{f_i^2}{A_i}, \qquad i \geq 1. \tag{16}$$

Now one can choose $\alpha_1 = 1$ and solve for α_i for $i \geq 2$ obtaining

$$\alpha_i = f_{i-1} \int \beta_{i-1} \, d\tau + c_i, \qquad i \geq 2. \tag{17}$$

From the first term in equation (9), one can see that the sparsity could be further improved if

$$\int_0^1 \alpha_i(\tau) \, d\tau = 0, \qquad i \geq 2. \tag{18}$$

The integration constants c_i can now be obtained by imposing this condition on α_i in equation (17) so that

$$\alpha_i(\tau) = f_{i-1}\left[\int_0^1 \tau^*\beta_{i-1}(\tau^*) \, d\tau^* - \int_\tau^1 \beta_{i-1}(\tau^*) \, d\tau^* \right], \qquad i \geq 2. \tag{19}$$

For example, with $N=4$ these polynomials can be easily worked out to be

$$f_1\beta_1 = \sqrt{6}, \qquad f_2\beta_2 = \sqrt{120}\,(\tau - \tfrac{1}{2}), \qquad \alpha_1 = 1, \qquad \alpha_2 = \sqrt{6}\,(\tau - \tfrac{1}{2}), \qquad \alpha_3 = \sqrt{30}\,[(\tau - \tfrac{1}{2})^2 - \tfrac{1}{12}], \tag{20}$$

leading to a cubic representation for δq and δp and a quadratic one for p and q.

Now, the first two terms in the integrand and the trailing terms in equation (5) can be expressed as

$$\int_0^1 p\delta q' \, d\tau - p\delta q \Big|_0^1 = \delta \underline{q}^T \mathbf{A} \underline{p} \tag{21}$$

and

$$\int_0^1 q\delta p' \, d\tau - q\delta p \Big|_0^1 = \delta \underline{p}^T \mathbf{A} \underline{q}, \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}_{N\times(N+1)}, \quad \delta \underline{q} = \begin{Bmatrix} \delta q_1 \\ \delta q_2 \\ \delta q_1^* \\ \delta q_2^* \\ \vdots \\ \delta q_{N-2}^* \end{Bmatrix}, \quad \underline{p} = \begin{Bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \bar{p}_1 \\ \bar{p}_2 \\ \vdots \\ \bar{p}_{N-1} \end{Bmatrix}$$

and

$$\delta \underline{p} = \begin{Bmatrix} \delta p_1 \\ \delta p_2 \\ \delta p_1^* \\ \delta p_2^* \\ \vdots \\ \delta p_{N-2}^* \end{Bmatrix}, \quad \underline{q} = \begin{Bmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \bar{q}_1 \\ \bar{q}_2 \\ \vdots \\ \bar{q}_{N-1} \end{Bmatrix}.$$

For all problems there will be matrices of this type, but there will also be problem-dependent matrices stemming from the δH and virtual work terms. For some problems, such as the linear oscillator example considered in reference [11], there will be terms of the form $\int_0^1 p\,\delta p \, d\tau$ and $\int_0^1 q\,\delta q \, d\tau$. These terms can also be put into a matrix form such that

$$\int_0^1 q\,\delta q \, d\tau = \delta \underline{q}^T \mathbf{B} \underline{q} \quad \text{and} \quad \int_0^1 p\,\delta p \, d\tau = \delta \underline{p}^T \mathbf{B} \underline{p}, \tag{23, 24}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1/2 & -\sqrt{6}/12 & 0 & \cdots & 0 \\ 0 & 0 & 1/2 & \sqrt{6}/12 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{6}/6 & 0 & -\sqrt{5}/30 & \ddots & \vdots \\ 0 & 0 & 0 & \sqrt{5}/10 & \ddots & -\sqrt{70}/140 & 0 \\ \vdots & \vdots & \vdots & \ddots & \sqrt{70}/70 & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & \cdots & \ddots & 0 \end{bmatrix}_{N\times(N+1)}.$$

The first two rows are all zeros except for the third and fourth columns, and the lower $N-2$ rows and last $N-1$ columns have two non-zero diagonals. In view of the sparsity of these matrices, computational efficiency can be improved by using a sparse matrix linear equation solver.

4. RESULTS AND DISCUSSION

For the purpose of evaluating the performance of the method, we consider a single-degree-of-freedom linear oscillator for which the Hamiltonian is given by

$$H = p\dot{q} - \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}kq^2 = p^2/(2m) + kq^2/2. \tag{25}$$

The numerical example in reference [11] was chosen to show both the accuracy and the freedom from numerical instability of the present method. In reference [11] a unit impulse was applied at $t = 0{\cdot}5$ s. Substitution of equation (25) into equation (4) then leads to a matrix form of the equations,

$$\begin{Bmatrix} \delta\underline{q} \\ \delta\underline{p} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} -\mathbf{B}k\Delta t & \mathbf{A} \\ -\mathbf{A} & -\mathbf{B}\dfrac{\Delta t}{m} \end{bmatrix} \begin{Bmatrix} \underline{q} \\ \underline{p} \end{Bmatrix} = \begin{Bmatrix} -Q_1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix},$$

where Q_1 is an impulse applied at the beginning of a time step; Q_1 is set equal to zero for all time steps except $t = 0{\cdot}5$ where $Q_1 = 1$. (Introduction of the unit impulse as a trailing term in equation (5), rather than under the integral, avoids "smearing" the impulse over a time element, as pointed out in reference [11].) For the numerical results, initial conditions were specified as $q(0) = 0$ and $p(0) = 1$ and $k = m = 1$. For one element in time, we can then obtain a time-marching algorithm for generating $\hat{q}_2$ and $\hat{p}_2$ given $\hat{q}_1$ and $\hat{p}_1$. Exact element quadrature is used; the *ad hoc* procedure of reduced element quadrature is not necessary with the present mixed formulation. Since $\hat{q}_1$ and $\hat{p}_1$ are known at any time step, the number of equations and unknowns is $2N$.

The results for displacement and momentum are shown in Tables 1 and 2 for a unit impulse applied at $t = 0{\cdot}5$ s. First, note that the present formulation with $N = 2$ gives the same result as those obtained by the displacement formulation with reduced element quadrature of reference [11]. The present $N = 3$ approximation gives extremely good results—accurate to six digits. The present $N = 4$ results reproduce the exact solution to as many places as are displayed in the tables. Also, it should be noted that by using these shape functions, the formulation yields an unconditionally stable algorithm for all $N \geq 2$.

In order to verify that the time-marching algorithm is conditionally stable, one may eliminate the $\bar{q}$s and $\bar{p}$s and write the finite element approximation for the system transition matrix. For $k = m = 1$, this approximation has the form

$$\begin{Bmatrix} \hat{q}_2 \\ \hat{p}_2 \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} \hat{q}_1 \\ \hat{p}_1 \end{Bmatrix}. \tag{26}$$

TABLE 1

Displacement response of linear oscillator due to unit impulse function at $t = 0{\cdot}5$ s

t	Displacement method [11] (exact element quadrature; conditionally stable)	Mixed method $N = 2$ and displacement method [11] (reduced element quadrature)	Mixed method $N = 3$	Mixed method $N = 4$ and exact
0·0	0·00000000	0·00000000	0·00000000	0·00000000
0·1	0·09983361	0·09975062	0·09983340	0·09983342
0·2	0·19867054	0·19850623	0·19866930	0·19866933
0·3	0·29552408	0·29528172	0·29552017	0·29552021
0·4	0·38942729	0·38911176	0·38941829	0·38941834
0·5	0·47944271	0·47906039	0·47942548	0·47942554
0·6	0·66450529	0·66398098	0·66447581	0·66447589
0·7	0·84293388	0·84227832	0·84288692	0·84288702
0·8	1·01294715	1·01217388	1·01217618	1·01287630
0·9	1·17284780	1·17197294	1·17274512	1·17274525
1·0	1·32103949	1·32008150	1·32089639	1·32089652

TABLE 2

Momentum response of linear oscillator due to unit impulse function at $t = 0{\cdot}5$ s

t	Displacement method [11] (exact element quadrature; conditionally stable)	Mixed method $N = 2$ and displacement method [11] (reduced element quadrature)	Mixed method $N = 3$	Mixed method $N = 4$ and exact
0·0	1·00000000	1·00000000	1·00000000	1·00000000
0·1	0·99500832	0·99501247	0·99500417	0·99500416
0·2	0·98008311	0·98009963	0·98006658	0·98006658
0·3	0·95537338	0·95541023	0·95533650	0·95533649
0·4	0·92112581	0·92119056	0·92106102	0·92106099
0·5	0·87768231	0·87778195	0·87758259	0·87758256
0·6	1·82048491	1·82062988	1·82033983	1·82033978
0·7	1·74511295	1·74531692	1·74490883	1·74490877
0·8	1·65231890	1·65259431	1·65204329	1·65204320
0·9	1·54302915	1·54338696	1·54267108	1·54267096
1·0	1·41833479	1·41878424	1·41788502	1·41788487

The elements of the approximate transition matrix, c and s, are rational functions of Δt which are distinctly different for each value of N. They satisfy the constraint $c^2 + s^2 = 1$ for all values of Δt. For the exact transition matrix, c and s are $\cos \Delta t$ and $\sin \Delta t$, respectively. For the approximate transition matrix, $c = \cos \Delta t + O(e)\Delta t$ and $s = \sin \Delta t - O(e)$, where the error function $e = e(N)$ is also tabulated in Table 3. The error after M equal-length time steps can be shown to be approximately Me. The accuracy of the momentum is comparable to that of the displacement. (Recall that the momentum value is obtained directly from the solution of the algebraic equations—not by differentiation of the displacement.) As shown in reference [11], if the eigenvalues of the transition

TABLE 3

Elements of the transition matrix and error versus N

N	c	s	e
2	$\dfrac{1-\dfrac{\Delta t^2}{4}}{1+\dfrac{\Delta t^2}{4}}$	$\dfrac{\Delta t}{1+\dfrac{\Delta t^2}{4}}$	$\dfrac{\Delta t^3}{12}$
3	$\dfrac{1-\dfrac{5\Delta t^2}{12}+\dfrac{\Delta t^4}{144}}{1+\dfrac{\Delta t^2}{12}+\dfrac{\Delta t^4}{144}}$	$\dfrac{\Delta t\left(1-\dfrac{\Delta t^2}{12}\right)}{1+\dfrac{\Delta t^2}{12}+\dfrac{\Delta t^4}{144}}$	$\dfrac{\Delta t^5}{720}$
4	$\dfrac{1-\dfrac{9\Delta t^2}{20}+\dfrac{11\Delta t^4}{600}-\dfrac{\Delta t^6}{14\,400}}{1+\dfrac{\Delta t^2}{20}+\dfrac{\Delta t^4}{600}+\dfrac{\Delta t^6}{14\,400}}$	$\dfrac{\Delta t\left(1-\dfrac{7\Delta t^2}{60}+\dfrac{\Delta r^4}{600}\right)}{1+\dfrac{\Delta t^2}{20}+\dfrac{\Delta t^4}{600}+\dfrac{\Delta t^6}{14\,400}}$	$\dfrac{\Delta t^7}{100\,800}$
Exact	$\cos \Delta t$	$\sin \Delta t$	—

matrix lie on the unit circle in the complex plane for all values of Δt, the algorithm is unconditionally stable. For the present formulation, the moduli of the eigenvalues are exactly 1 for all N regardless of the value of Δt. Thus, it is concluded that the present formulation is unconditionally stable for the linear oscillator.

Another important measure of the accuracy of a time-marching method is its periodicity error. We obtained it by using the finite element method to take M equal-length time steps, the length of which is adjusted so that the last step yields values of p and q that are the same as their starting values. Then, the approximate period $M\Delta t$ is compared with the exact period. The computational effort is related to the size of the element matrix $2N$ times the number of steps.

Figure 1 shows the periodicity error versus the total numbers of unknowns in an assembled set of finite elements over one period with the degree of shape function as the parameter. It can be seen that the higher order elements give very high accuracy, as has come to be expected from p-version formulations. Also, it is clear that for a given size of the element coefficient matrix times the number of elements, given by $2MN$, the higher the order N the lower the error. This means that higher order elements will be more efficient for a given error. In other words, with the higher order element one can take much larger time steps in time marching without compromising accuracy. The sparsity of the present formulation is not taken into account in generation of these results. However, it is believed that some savings in computational effort can be demonstrated by taking advantage of the high level of sparsity. Figure 1 also exhibits the accuracy one could expect from direct calculation of the transition matrix by the mixed method in linear systems with periodic coefficients. This aspect of the work is presently under investigation and will be treated in a later paper.

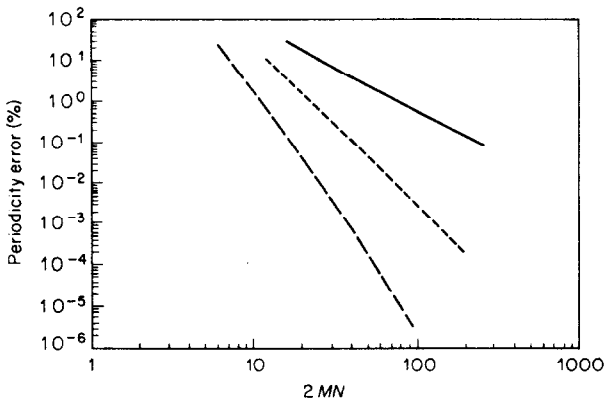

Figure 1. Periodicity error versus matrix size for solving periodic response problems. ——, $N = 2$; · · · · ·, $N = 3$; - - -, $N = 4$.

5. CONCLUSION

A set of higher order shape functions for a p-version finite element formulation in time based on the mixed form of Hamilton's weak principle has been derived. This method is suitable for determination of the dynamic response of linear systems. These shape functions lead to very sparse coefficient matrices and are suitable for obtaining extremely high accuracy solutions for both time marching and the determination of the transition matrix in periodic coefficient problems. As pointed out in reference [12], p-version finite

elements in time can be made competitive with conventional time-marching algorithms, particularly if high accuracy is needed. With the present approach, it is believed that by taking appropriate advantage of sparsity, this method can be made computationally more efficient than conventional linear system simulation techniques. For the linear oscillator, the present technique yields unconditionally stable time marching algorithms. As pointed out in reference [15], for non-linear systems, one is more likely to develop computationally efficient algorithms based on an h-version algorithm because of the need in non-linear problems to continually recalculate the element integrals. For the linear case, however, these integrals need to be calculated only once, and the p-version is believed to be superior in this case.

ACKNOWLEDGMENTS

This work was supported, in part, by the U.S. Army Research Office under contract DAAL03-88-K-0164, the technical monitor of which is Dr Gary L. Anderson. Technical discussions with Professor Marco Borri of Politecnico di Milano are gratefully acknowledged.

REFERENCES

1. I. BABUSHKA, B. A. SZABO and I. N. KATZ 1981 *SIAM Journal of Numerical Analysis* **18**, 515-545. The p-version of the finite element method.
2. D. H. HODGES and M. J. RUTKOWSKI 1981 *American Institute of Aeronautics and Astronautics Journal* **19**, 1459-1466. Free-vibration analysis of rotating beams by a variable-order finite-element method.
3. D. H. HODGES 1983 *American Institute of Aeronautics and Astronautics Journal* **21**, 796-797. Orthogonal polynomials as variable-order finite element shape functions.
4. I. FRIED 1969 *American Institute of Aeronautics and Astronautics Journal* 7, 1170-1173. Finite element analysis of time dependent phenomena.
5. J. H. ARGYRIS and D. W. SCHARPF 1969 *Journal of the Royal Aeronautical Society* **73**, 1041-1044. Finite elements in time and space.
6. C. D. BAILEY 1975 *American Institute of Aeronautics and Astronautics Journal* **13**, 1154-1157. Application of Hamilton's law of varying action.
7. C. D. BAILEY 1975 *Foundations of Physics* **5**, 433-451. A new look at Hamilton's principle.
8. T. E. SIMKINS 1978 *American Institute of Aeronautics and Astronautics Journal* **16**, 559-563. Unconstrained variational statements for initial and boundary value problems.
9. D. L. HITZL and D. A. LEVINSON 1980 *Celestial Mechanics* **22**, 255-266. Application of Hamilton's law of varying action to the restricted three-body problem.
10. M. BARUCH and R. RIFF 1982 *American Institute of Aeronautics and Astronautics Journal* **20**, 687-692. Hamilton's principle, Hamilton's law, 6″ correct formulations.
11. M. BORRI, G. L. GHIRINGHELLI, M. LANZ, P. MANTEGAZZA and T. MERLINI 1985 *Computers and Structures* **20**, 495-508. Dynamic response of mechanical systems by a weak Hamiltonian formulation.
12. D. A. PETERS and A. IZADPANAH 1988 *Computational Mechanics* **3**, 73-88. *hp*-version finite elements for the space-time domain.
13. M. BORRI, F. MELLO, M. IURA and S. N. ATLURI 1988 *ARO/AFOSR Workshop on Nonlinear Dynamics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, June.* Primal and mixed forms of Hamilton's principle for constrained and flexible dynamical systems: numerical studies.
14. M. BORRI 1986 *Computers and Mathematics with Applications* **12A**, 149-160. Helicopter rotor dynamics by finite element time approximation.
15. D. H. HODGES and R. R. BLESS 1991 *Journal of Guidance, Control, and Dynamics*, **14**, 148-156. A weak Hamiltonian finite element method for optimal control problems.
16. A. K. NOOR and J. M. PETERS 1983 in *Hybrid and Mixed Finite Element Methods* (S. N. Atluri, R. H. Gallagher and O. C. Zienkiewicz, editors) 537-564. New York: John Wiley. Mixed models and reduced selective integration displacement models for vibration analysis of shells.
17. M. ABRAMOWITZ and I. A. STEGUN (editors) 1970 *Handbook of Mathematical Functions*, Washington, D.C.: National Bureau of Standards. See Chapter 22.