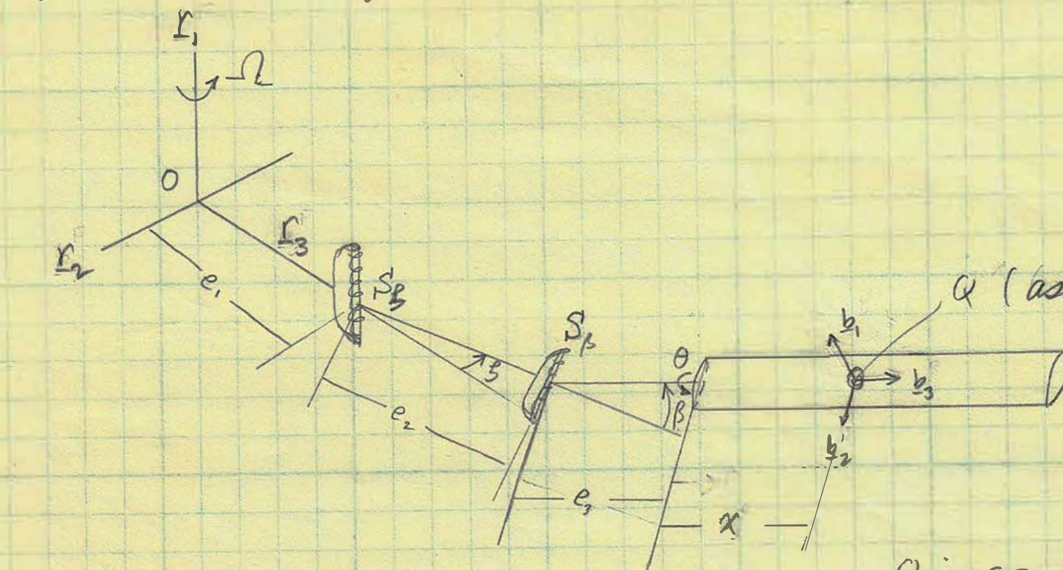
Rigid blade modeling (see chapter 11 of Bramwell)



Q (assume that $S_\beta Q$ passes thru mass center)

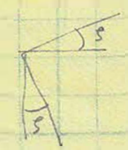
Q is an arbitrary point on the line that passes from S_β to mass center of B

$$\underline{\omega}^{BA} = \underline{\omega}^{BR} + \underline{\omega}^{RA} \qquad \underline{\omega}^{RA} = \Omega \underline{r}_1$$

$$= \Omega \underline{r}_1 + \dot{\xi} \underline{r}_1 + \dot{\beta}\left(c_\xi \underline{r}_2 + s_\xi \underline{r}_3\right) + \dot{\theta}\, \underline{b}_3$$

$$= \Omega \underline{r}_1 + \dot{\xi} \underline{r}_1 + \dot{\beta}\left(s_\theta \underline{b}_1 + c_\theta \underline{b}_2\right) + \dot{\theta}\, \underline{b}_3$$

$$\begin{Bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{Bmatrix} = \begin{bmatrix} c_\beta c_\theta & c_\xi s_\theta + c_\theta s_\beta s_\xi & -c_\theta c_\xi s_\beta + s_\theta s_\xi \\ -c_\beta s_\theta & c_\theta c_\xi - s_\xi s_\beta s_\theta & c_\xi s_\beta s_\theta + c_\theta s_\xi \\ s_\beta & -c_\beta s_\xi & c_\beta c_\xi \end{bmatrix} \begin{Bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \end{Bmatrix}$$



Kinetic energy:

Let $\quad \underline{\underline{I}} = \underline{b}_1 I_1 \underline{b}_1 + \underline{b}_2 I_2 \underline{b}_2 + \underline{b}_3 I_3 \underline{b}_3$

$$\omega_i = \underline{\omega}^{BA} \cdot \underline{b}_i$$

$$\omega_1 = (\Omega + \dot{\xi}) c_\beta c_\theta + s_\theta \dot{\beta}$$

$$\omega_2 = -(\Omega + \dot{\xi}) c_\beta s_\theta + c_\theta \dot{\beta}$$

$$\omega_3 = (\Omega + \dot{\xi}) s_\beta + \dot{\theta}$$

$$\underline{v}^{B^*A} = \cancel{\underline{v}^{OA}}^{\,0} + \underline{\omega}^{RA} \times e_1 \underline{r}_3 + \left(\underline{\omega}^{RA} + \dot{\xi} \underline{r}_1\right) \times e_2 \left(-s_\xi \underline{r}_2 + c_\xi \underline{r}_3\right)$$
$$+ \left[\underline{\omega}^{RA} + \dot{\xi}\, \underline{r}_1 + \dot{\beta}\left(s_\theta \underline{b}_1 + c_\theta \underline{b}_2\right)\right] \times \left(e_3 + \bar{x}\right) \underline{b}_3$$

$\bar{x}$ is the value of x for $Q = B^*$

Let $e_1 = e_2 = e_3 = 0$

$$\underline{v}^{B^*A} = \bar{x}\ \underline{\omega}^{BA} \times \underline{b}_3 = \bar{x}\ \omega_i\ \underline{b}_i \times \underline{b}_3 = \bar{x}\left(\omega_2\ \underline{b}_1 - \omega_1\ \underline{b}_2\right)$$

$$K = \frac{1}{2}\ m\bar{x}^2\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

Let $I_1 = I_2 \qquad I = I_1 + m\bar{x}^2 \qquad I_3 << I$

$$K = \frac{I}{2}\left(\omega_1^2 + \omega_2^2\right)$$

Let the springs be configured so that

$$P = \frac{1}{2}\ K_\beta\left(\beta - \beta_{pc}\right)^2 + \frac{1}{2}\ K_\zeta\ \zeta^2$$

precone angle (usually included in a design to relieve steady-state bending stresses at the root)

Derivation of potential and kinetic energy terms.

$$K = \frac{I}{2}\left(\omega_1^2 + \omega_2^2\right)$$

$$\frac{\partial K}{\partial \dot{\zeta}} = I\left(\omega_1 \frac{\partial \omega_1}{\partial \dot{\zeta}} + \omega_2 \frac{\partial \omega_2}{\partial \dot{\zeta}}\right)$$

$$= I\left\{\left[(\Omega+\dot{\zeta})c_\beta c_\theta + \dot{\beta} s_\theta\right]c_\beta c_\theta - \left[-(\Omega+\dot{\zeta})c_\beta s_\theta + \dot{\beta} c_\theta\right]c_\beta s_\theta\right\}$$

$$= I(\Omega+\dot{\zeta})c_\beta^2$$

$\hat{()}^2 \to 0$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\zeta}}\right) - \cancel{\frac{\partial K}{\partial \zeta}}^{\,0} = I\left[\ddot{\zeta} c_\beta^2 - 2c_\beta s_\beta \dot{\beta}(\Omega + \cancel{\dot{\zeta}})\right] = I\ddot{\hat{\zeta}} c_{\bar\beta}^2 - 2I\Omega c_{\bar\beta} s_{\bar\beta} \dot{\hat{\beta}}$$

$$\frac{\partial K}{\partial \dot{\beta}} = I\left(\omega_1 \frac{\partial \omega_1}{\partial \dot{\beta}} + \omega_2 \frac{\partial \omega_2}{\partial \dot{\beta}}\right)$$

$$= I\left\{\left[(\Omega+\dot{\zeta})c_\beta c_\theta + \dot{\beta} s_\theta\right]s_\theta + \left[-(\Omega+\dot{\zeta})c_\beta s_\theta + \dot{\beta} c_\theta\right]c_\theta\right\}$$

$$= I\dot{\beta}$$

$$\frac{\partial K}{\partial \beta} = I\left(\omega_1 \frac{\partial \omega_1}{\partial \beta} + \omega_2 \frac{\partial \omega_2}{\partial \beta}\right) = I\left\{-\left[(\Omega+\dot{\zeta})c_\beta c_\theta + \dot{\beta} s_\theta\right](\Omega+\dot{\zeta})s_\beta c_\theta + \left[-(\Omega+\dot{\zeta})c_\beta s_\theta + \dot{\beta} c_\theta\right](\Omega+\dot{\zeta})s_\beta s_\theta\right\}$$

$$= -I(\Omega+\dot{\zeta})^2 c_\beta s_\beta$$

$\hat{()}^2 \to 0$

$$= -I\left(\Omega^2 + 2\Omega\dot{\zeta} + \cancel{\dot{\zeta}^2}\right)c_\beta s_\beta$$

$$= -I\Omega^2\left[c_{\bar\beta} s_{\bar\beta} + \hat{\beta}\left(c_{\bar\beta}^2 - s_{\bar\beta}^2\right)\right] - 2\Omega I \dot{\hat{\zeta}} c_{\bar\beta} s_{\bar\beta}$$

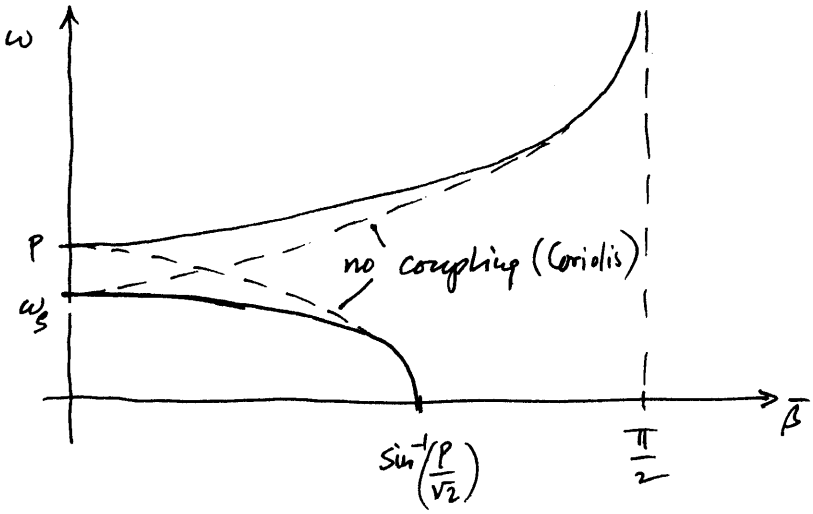
$$\frac{\partial P}{\partial \zeta} = K_\zeta \zeta \qquad \frac{\partial P}{\partial \beta} = K_\beta(\beta - \beta_{pc})$$

$$\boxed{\begin{aligned} \zeta &= \bar{\zeta} + \hat{\zeta}(t) \qquad \hat{()}^2 \Rightarrow 0 \\ \beta &= \bar{\beta} + \hat{\beta}(t) \end{aligned}}$$

Let $\hat{\xi} = \check{\xi} e^{i\omega t}$ $\quad \hat{\beta} = \check{\beta} e^{i\omega t}$

$$\begin{bmatrix} \omega_\xi^2 - \omega^2 c_{\bar\beta}^2 & -2i\omega c_{\bar\beta} s_{\bar\beta} \\ 2i\omega c_{\bar\beta} s_{\bar\beta} & p^2 - \omega^2 - 2 s_{\bar\beta}^2 \end{bmatrix} \begin{Bmatrix} \check{\xi} \\ \check{\beta} \end{Bmatrix} e^{i\omega t} = 0$$

The determinant must vanish. If $\sqrt{2} > p \geq 1 > \omega_\xi$



$\bar\beta > \sin^{-1}\left(\frac{p}{\sqrt{2}}\right)$ unstable!

$p > \sqrt{2}$ Instability not possible

$\hat{\xi}$ and $\hat{\beta}$ are $90°$ out of phase with each other.

Coriolis/gyroscopic terms do not change stability boundary, but they do change eigenvalues!

Perturbation analysis with $\bar\beta$ small yields

$$\omega^2 = \begin{cases} \omega_\xi^2 - \bar\beta^2 \dfrac{\omega_\xi^2}{p^2 - \omega_\xi^2}\left(4 + \omega_\xi^2 - p^2\right) + O(\bar\beta^4) \\ p^2 + \dfrac{2\bar\beta^2 (p^2 + \omega_\xi^2)}{p^2 - \omega_\xi^2} + O(\bar\beta^4) \end{cases}$$

Talk about HW1

for $\omega \cong \omega_\xi$ $\quad \hat{\beta} \cong -\dfrac{2i\omega_\xi \bar\beta}{p^2 - \omega_\xi^2}$

$\omega \cong p$ $\quad \hat{\xi} \cong -\dfrac{2ip\bar\beta}{p^2 - \omega_\xi^2}$

Ignoring aerodynamics:

$$M = \begin{bmatrix} c_{\bar\beta}^2 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = 2 c_{\bar\beta} s_{\bar\beta} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} \omega_\zeta^2 & 0 \\ 0 & p^2 - 2 s_{\bar\beta}^2 \end{bmatrix}$$

$$M \hat{q}'' + C \hat{q}' + K \hat{q} = 0$$

$$\hat{q} = \begin{Bmatrix} \hat\zeta \\ \hat\beta \end{Bmatrix} \qquad (\ )' = \frac{1}{\Omega} \frac{d}{dt} (\ )$$

$$\omega_\zeta^2 = \frac{K_\zeta}{I\Omega^2} \qquad p^2 = \frac{K_\beta + I\Omega^2}{I\Omega^2} \qquad \therefore p^2 \geq 1 \qquad p^2 = 1 + \omega_\beta^2$$

$$\omega_\zeta^2 \geq 0$$

$\bar\zeta = 0$ (never appears in perturbation equations) – why

for $R = 0$

It does for $R \neq$ [illegible]

It will when there is a ~~to~~ hub of

$$(p^2 - 1)(\bar\beta - \beta_{pc}) + c_{\bar\beta} s_{\bar\beta} = 0$$

For $\beta_{pc} = 0 \Rightarrow \bar\beta = 0$

Two uncoupled modes: pure lag with frequency ω_ζ
pure flap with frequency p

For $\bar\beta \neq 0$, $\ll 1$ flapping dominates one mode and lead-lag the other. Coupling increases with $\bar\beta$. Coupling is so large when $\bar\beta = O(1)$ that dominant type of motion is difficult to determine.

For $\bar\beta = \frac{\pi}{2}$ the system looks like a rotating shaft with stiffness in the spin direction $= K_\zeta$. One frequency goes to infinity. The other is $\omega = \sqrt{p^2 - 2}$ or

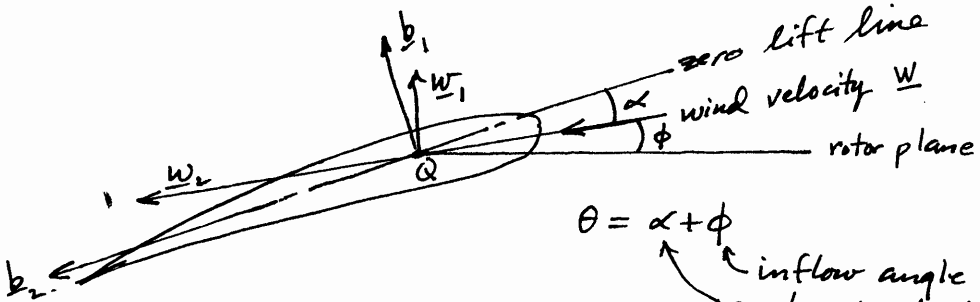
$$\omega^2 = \frac{K_\beta + I\Omega^2}{I\Omega^2} - 2 \geq 0 \Rightarrow K_\beta \geq I\Omega^2$$

$$\Omega \leq \sqrt{\frac{K_\beta}{I}} \quad \text{critical speed}$$

Incorporation of aerodynamics

$$\overline{\delta W} = \underbrace{\int_0^l \underline{F} \cdot \underline{\delta r}^{QA} \, dx}_{\text{lift and drag}} + \underbrace{\int_0^l \underline{M} \cdot \underline{\delta \psi}^{BA} \, dx}_{\substack{\text{pitching moment} \\ \text{(ignore)}}}$$

$$\underline{F} = L \underline{w}_1 + D \underline{w}_2$$



$\theta = \alpha + \phi$

inflow angle

angle of attack

Q aerodynamic center (~ quarter chord)

$$\begin{Bmatrix} \underline{w}_1 \\ \underline{w}_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{Bmatrix}$$

Take a simple strip-theory approach

$$L = \rho \frac{W^2 c \, c_l(\alpha)}{2} \qquad W = |\underline{W}|$$

$\underline{W}$ = relative wind velocity $= W \underline{w}_2$

$$D = \rho \frac{W^2 c \, c_d(\alpha)}{2}$$

To further simplify, take $c_d = c_{d_0}$ ("c_{d_0}" constant)

$c_l = a \sin\alpha$ ("a" constant)

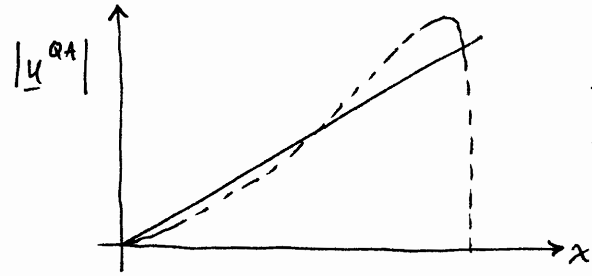
$$\underline{F} = \frac{1}{2} \rho c W^2 \left(a \sin\alpha \, \underline{w}_1 + c_{d_0} \underline{w}_2 \right)$$

$$\underline{W} = \underline{u}^{QA} - \underline{v}^{QA}$$

air velocity

$$\underline{v}^{QA} = x\,(\omega_2\,\underline{b}_1 - \omega_1\,\underline{b}_2)$$

$$\underline{u}^{QA} = -\underbrace{x C_\beta \nu}_{\text{thrust induced inflow velocity}}\,\underline{r}_1 \qquad \nu = \text{const.}$$



$$\underline{W} = W\,\underline{w}_2 = W_{B1}\,\underline{b}_1 + W_{B2}\,\underline{b}_2 = W(\sin\alpha\,\underline{b}_1 + \cos\alpha\,\underline{b}_2)$$

$$\therefore \frac{W_{B1}}{W_{B2}} = \tan\alpha \qquad W_{B1} = x(\Omega+\dot{\zeta})c_\beta s_\theta - x\dot{\beta}c_\theta - x c_\beta^2 c_\theta \nu$$

$$W^2 = W_{B1}^2 + W_{B2}^2 \qquad W_{B2} = x(\Omega+\dot{\zeta})c_\beta c_\theta + x\dot{\beta}s_\theta + x c_\beta^2 s_\theta \nu$$

$$\tan\alpha = \frac{(\Omega+\dot{\zeta})c_\beta s_\theta - \dot{\beta}c_\theta - \nu c_\beta^2 c_\theta}{(\Omega+\dot{\zeta})c_\beta c_\theta + \dot{\beta}s_\theta + \nu c_\beta^2 s_\theta}$$

$$= \tan(\theta-\phi)$$

$$\therefore \tan\phi = \frac{\frac{\nu c_\beta}{\Omega} + \frac{\dot{\beta}}{\Omega c_\beta}}{1+\frac{\dot{\zeta}}{\Omega}} \cong \frac{\nu c_{\bar\beta}}{\Omega}\left(1-\frac{\dot{\hat\zeta}}{\Omega}\right) + \frac{\dot{\hat\beta}}{\Omega c_{\bar\beta}} - \frac{\nu\hat\beta s_{\bar\beta}}{\Omega}$$

anticipating $\zeta = \bar\zeta + \hat\zeta(t)$; $\beta = \bar\beta + \hat\beta(t)$

$(\hat{\ })^2 = 0$

$$\tan(\bar\phi+\hat\phi) \cong \tan\bar\phi + \hat\phi\sec^2\bar\phi \cong \frac{\nu c_{\bar\beta}}{\Omega}\left(1-\frac{\dot{\hat\zeta}}{\Omega}\right) + \frac{\dot{\hat\beta}}{\Omega c_{\bar\beta}} - \frac{\nu\hat\beta s_{\bar\beta}}{\Omega}$$

$$\tan\bar\phi = \frac{\nu c_{\bar\beta}}{\Omega}$$

$$\hat\phi\sec^2\bar\phi = \frac{\dot{\hat\beta}}{\Omega c_{\bar\beta}} - \frac{\nu c_{\bar\beta}\dot{\hat\zeta}}{\Omega^2} - \frac{\nu\hat\beta s_{\bar\beta}}{\Omega} = \frac{\dot{\hat\beta}}{\Omega c_{\bar\beta}} - \tan\bar\phi\,\frac{\dot{\hat\zeta}}{\Omega} - \frac{\hat\beta s_{\bar\beta}\tan\bar\phi}{c_{\bar\beta}}$$

$$\therefore \hat{\phi} = \frac{\dot{\hat{\beta}} c_{\bar\phi}^2}{\Omega c_{\bar\beta}} - \frac{\hat{\beta} s_{\bar\beta} c_{\bar\phi} s_{\bar\phi}}{c_{\bar\beta}} - \frac{\dot{\hat{\xi}}}{\Omega} c_{\bar\phi} s_{\bar\phi} = -\hat{\alpha}$$

$$W^2 = \left[(\Omega+\dot\xi)^2 c_\beta^2 + \dot\beta^2 + 2\nu\dot\beta c_\beta^2 + \nu^2 c_\beta^4\right] x^2$$

$$= \left[(\Omega^2 + 2\Omega\dot\xi) c_\beta^2 + 2 c_\beta \dot\beta \Omega \tan\bar\phi + \frac{\Omega^2 \tan^2\bar\phi}{c_{\bar\beta}^2} c_\beta^4\right] x^2$$

$$\therefore \frac{\bar{W}^2}{x^2} = \Omega^2 c_{\bar\beta}^2 + \Omega^2 \tan^2\bar\phi\, c_{\bar\beta}^2 = \frac{\Omega^2 c_{\bar\beta}^2}{c_{\bar\phi}^2}$$

$$\frac{\widehat{W^2}}{x^2} = 2\Omega\dot{\hat\xi} c_{\bar\beta}^2 + 2\Omega\dot{\hat\beta} c_{\bar\beta}\tan\bar\phi - 4\Omega^2\tan^2\bar\phi\, c_{\bar\beta} s_{\bar\beta}\hat\beta - 2\Omega^2 c_{\bar\beta} s_{\bar\beta}\hat\beta$$

$$= 2\Omega\dot{\hat\xi} c_{\bar\beta}^2 + 2\Omega\dot{\hat\beta}\frac{c_{\bar\beta} s_{\bar\phi}}{c_{\bar\phi}} - 2\Omega^2 c_{\bar\beta} s_{\bar\beta}\hat\beta\,(1 + 2\tan^2\bar\phi)$$

$$1 + 2\tan^2\bar\phi = \sec^2\bar\phi + \tan^2\bar\phi = \frac{1}{c_{\bar\phi}^2} + \frac{s_{\bar\phi}^2}{c_{\bar\phi}^2} = \frac{1}{c_{\bar\phi}^2}(1 + s_{\bar\phi}^2)$$

$$= 2\Omega\dot{\hat\xi} c_{\bar\beta}^2 + 2\Omega\dot{\hat\beta}\frac{c_{\bar\beta} s_{\bar\phi}}{c_{\bar\phi}} - 2\Omega^2\frac{c_{\bar\beta} s_{\bar\beta}\hat\beta}{c_{\bar\phi}^2}(1 + s_{\bar\phi}^2)$$

$$Q_\xi = \int_0^l \tfrac{1}{2}\rho c W^2 (a\sin\alpha\, \underline{w}_1 + c_{d_0}\underline{w}_2)\cdot\frac{\partial \underline{v}^{QA}}{\partial\dot\xi}\,dx$$

Recall $\underline{v}^{QA} = x(\omega_2 \underline{b}_1 - \omega_1 \underline{b}_2)$

$$\frac{\partial \underline{v}^{QA}}{\partial\dot\xi} = -x\,\underline{b}_1 c_\beta s_\theta - x\,\underline{b}_2 c_\beta c_\theta$$
$$= -x c_\beta (\underline{b}_1 s_\theta + \underline{b}_2 c_\theta)$$

$$\therefore Q_\xi = -\tfrac{1}{2}\rho c \left(\frac{W}{x}\right)^2 \frac{l^4}{4} c_\beta \Big[(\cos\alpha\sin\theta - \sin\alpha\cos\theta)\, a\sin\alpha$$
$$+ (\sin\alpha\sin\theta + \cos\alpha\cos\theta)\, c_{d_0}\Big]$$

$$Q_\xi = -\frac{\rho a c l^4}{8} c_\beta \left(\frac{W}{x}\right)^2 \left(\sin\phi \sin\alpha + d\cos\phi\right)$$

$$d = \frac{c_{d_0}}{a}$$

$$= -\frac{\rho a c l^4}{8} c_\beta^3 \left(\Omega^2 + 2\Omega\dot{\xi} + 2\nu\dot{\beta} + \nu^2 c_\beta^2\right)\left(s_\phi s_\alpha + d c_\phi\right)$$

$$Q_\xi = \bar{Q}_\xi + \hat{Q}_\xi(t)$$

$$\bar{Q}_\xi = -\frac{\rho a c l^4}{8} c_{\bar\beta}^3 \left(\Omega^2 + \nu^2 c_{\bar\beta}^2\right)\left(s_{\bar\phi} s_{\bar\alpha} + d c_{\bar\phi}\right)$$

$$= -\frac{\rho a c l^4}{8} c_{\bar\beta}^3 \Omega^2 \left(1 + \tan^2\bar\phi\right)\left(s_{\bar\phi} s_{\bar\alpha} + d c_{\bar\phi}\right)$$

$$= -\frac{\rho a c l^4}{8} \Omega^2 c_{\bar\beta}^3 \left(\frac{s_{\bar\alpha} s_{\bar\phi}}{c_{\bar\phi}^2} + \frac{d}{c_{\bar\phi}}\right)$$

$$= -\frac{\rho a c l^4}{8} \Omega^2 \frac{c_{\bar\beta}^3}{c_{\bar\phi}} \left(s_{\bar\alpha} \tan\bar\phi + d\right)$$

$$\hat{Q}_\xi = -\frac{\rho a c l^4}{8}\left(-3 c_{\bar\beta}^2 \hat\beta s_{\bar\beta}\right)\left(s_{\bar\phi} s_{\bar\alpha} + d c_{\bar\phi}\right)\left(\Omega^2 + \nu^2 c_{\bar\beta}^2\right)$$

$$-\frac{\rho a c l^4}{8} c_{\bar\beta}^3 \left(2\Omega\dot{\hat\xi} + 2\nu\dot{\hat\beta} - 2\nu^2 c_{\bar\beta} s_{\bar\beta}\hat\beta\right)\left(s_{\bar\phi} s_{\bar\alpha} + d c_{\bar\phi}\right)$$

$$-\frac{\rho a c l^4}{8} c_{\bar\beta}^3 \left(\Omega^2 + \nu^2 c_{\bar\beta}^2\right)\left(\hat\phi c_{\bar\phi} s_{\bar\alpha} - \hat\phi s_{\bar\phi} c_{\bar\alpha} - d\hat\phi s_{\bar\phi}\right)$$

$$= -\frac{\rho a c l^4}{8}\Big[2 c_{\bar\beta}^3 \left(s_{\bar\phi} s_{\bar\alpha} + d c_{\bar\phi}\right)\left(\Omega\dot{\hat\xi} + \nu\dot{\hat\beta} - \nu^2 c_{\bar\beta} s_{\bar\beta}\hat\beta\right)$$

$$- 3\left(\Omega^2 + \nu^2 c_{\bar\beta}^2\right) c_{\bar\beta}^2 s_{\bar\beta}\hat\beta \left(s_{\bar\phi} s_{\bar\alpha} + d c_{\bar\phi}\right)$$

$$+ c_{\bar\beta}^3\left(\Omega^2 + \nu^2 c_{\bar\beta}^2\right)\left(c_{\bar\phi} s_{\bar\alpha} - s_{\bar\phi} c_{\bar\alpha} - d s_{\bar\phi}\right)\hat\phi\Big]$$

Collecting terms, we get

$$\hat{Q}_\zeta = -\frac{\rho a c \ell^4}{8}\left\{ \frac{\Omega c_{\bar\beta}^{3}}{c_{\bar\phi}}\left(s_\theta s_{\bar\phi} + 2d - d s_{\bar\phi}^{2}\right)\dot{\hat{\zeta}} + \Omega c_{\bar\beta}^{2}\left[\left(s_\theta c_{\bar\phi} - 2 s_{\bar\phi} c_\theta\right)\frac{1}{c_{\bar\phi}} + d s_{\bar\phi}\right]\dot{\hat{\beta}} + \Omega^2 \frac{c_{\bar\beta}^{2} s_{\bar\beta}}{c_{\bar\phi}^{2}}\left[s_{\bar\phi}\left(5 s_{\bar\phi} c_\theta - 4 c_{\bar\phi} s_\theta\right) - d c_{\bar\phi}\left(3 + s_{\bar\phi}^{2}\right)\right]\hat{\beta}\right\}$$

Similarly,

$$Q_\beta = \int_0^\ell \frac{1}{2}\rho c W^2 \left(a \sin\alpha \, \underline{w}_1 + c_{d_0} \underline{w}_2\right)\cdot \frac{\partial \underline{v}^{QA}}{\partial \dot\beta}\, dx$$

$$\frac{\partial \underline{v}^{QA}}{\partial \dot\beta} = x\left(c_\theta \underline{b}_1 - s_\theta \underline{b}_2\right)$$

$$\therefore Q_\beta = \frac{1}{2}\rho c \left(\frac{W}{X}\right)^2 \frac{\ell^4}{4}\left(s_\alpha c_\phi - d s_\phi\right)$$

$$= \frac{\rho a c \ell^4}{8}\left(\frac{W}{X}\right)^2\left(c_\phi s_\alpha - d s_\phi\right)$$

$$Q_\beta = \bar{Q}_\beta + \hat{Q}_\beta(t)$$

$$\bar{Q}_\beta = \frac{\rho a c \ell^4}{8}\frac{c_{\bar\beta}^{2}}{c_{\bar\phi}^{2}}\Omega^2\left(s_{\bar\alpha} c_{\bar\phi} - d s_{\bar\phi}\right)$$

$$\hat{Q}_\beta = -\frac{\rho a c \ell^4}{8}\left\{\frac{\Omega c_{\bar\beta}^{2}}{c_{\bar\phi}}\left(s_{\bar\phi} c_\theta - 2 c_{\bar\phi} s_\theta + d s_{\bar\phi} c_{\bar\phi}\right)\dot{\hat{\zeta}} + \Omega c_{\bar\beta}\left[c_\theta + \frac{d}{c_{\bar\phi}}\left(1 + s_{\bar\phi}^{2}\right)\right]\dot{\hat{\beta}} + \Omega^2 \frac{c_{\bar\beta} s_{\bar\beta}}{c_{\bar\phi}^{2}}\left[2 c_{\bar\phi}^{2} s_\theta - 3 c_{\bar\phi} s_{\bar\phi} c_\theta - d s_{\bar\phi}\left(3 + s_{\bar\phi}^{2}\right)\right]\hat{\beta}\right\}$$

Nondimensionalization and combination of aerodynamic terms.

$$\omega_\zeta^2 = \frac{K_\zeta}{I\Omega^2} \qquad \frac{K_\beta + I\Omega^2}{I\Omega^2} = \omega_\beta^2 + 1 = p^2$$

$$\frac{\rho a c L^4}{I} = \Gamma \quad \text{Lock number}$$

$$\omega_\zeta^2 \bar{\zeta} = -\frac{\Gamma}{8}\frac{c_{\bar\beta}^3}{c_{\bar\phi}^2}\left(s_{\bar\alpha} s_{\bar\phi} + d\, c_{\bar\phi}\right) \qquad (\bar{\zeta} \text{ never needed})$$

$$(p^2-1)(\bar\beta - \beta_{pc}) + c_{\bar\beta} s_{\bar\beta} = \frac{\Gamma}{8}\frac{c_{\bar\beta}^2}{c_{\bar\phi}^2}\left(s_{\bar\alpha} c_{\bar\phi} - d\, s_{\bar\phi}\right)$$

$$M\hat{q}'' + C\hat{q}' + K\hat{q} = 0 \qquad \hat{q} = \begin{Bmatrix} \hat\zeta \\ \hat\beta \end{Bmatrix} \qquad ()' = \frac{d}{d\psi}() = \frac{1}{\Omega}\frac{d}{dt}()$$

$$M = \begin{bmatrix} c_{\bar\beta}^2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \tan\bar\phi = \frac{v\, c_{\bar\beta}}{\Omega}$$

$$C = 2 c_{\bar\beta} s_{\bar\beta}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$+ \frac{\Gamma}{8}\frac{c_{\bar\beta}}{c_{\bar\phi}}\begin{bmatrix} c_{\bar\beta}^2\left(s_{\bar\phi} s_\theta + 2d - d\, s_{\bar\phi}^2\right) & c_{\bar\beta}\left(c_{\bar\phi} s_\theta - 2 s_{\bar\phi} c_\theta + d\, s_{\bar\phi} c_{\bar\phi}\right) \\ c_{\bar\beta}\left(-2 c_{\bar\phi} s_\theta + s_{\bar\phi} c_\theta + d\, s_{\bar\phi} c_{\bar\phi}\right) & c_{\bar\phi} c_\theta + d\left(1 + s_{\bar\phi}^2\right) \end{bmatrix}$$

$$K = \begin{bmatrix} \omega_\zeta^2 & 0 \\ 0 & p^2 - 2 s_{\bar\beta}^2 \end{bmatrix}$$

$$+ \frac{\Gamma}{8}\frac{c_{\bar\beta} s_{\bar\beta}}{c_{\bar\phi}^2}\begin{bmatrix} 0 & -c_{\bar\beta}\left[\left(5 s_{\bar\phi} c_\theta - 4 c_{\bar\phi} s_\theta\right) s_{\bar\phi} - d\, c_{\bar\phi}\left(3 + s_{\bar\phi}^2\right)\right] \\ 0 & c_{\bar\phi}\left(2 c_{\bar\phi} s_\theta - 3 s_{\bar\phi} c_\theta\right) - d\, s_{\bar\phi}\left(3 + s_{\bar\phi}^2\right) \end{bmatrix}$$

Perturbation equations are constant coefficient. To solve, we first need $\bar{\beta}$. ($\bar{\phi}$ and $\bar{\alpha}$ must be supplied to solve for $\bar{\beta}$.)

Set θ, ω_ζ, p, σ, Γ, β_{pc} (one needs to vary θ for a given configuration defined by ω_ζ, p, Γ, β_{pc}, σ)

Determine $\dfrac{\nu}{\Omega} = \dfrac{\pi\sigma}{6}\left(\sqrt{1+\dfrac{12|\theta|}{\pi\sigma}}-1\right)\mathrm{sgn}(\theta) = \dfrac{\tan\bar{\phi}}{c_{\bar{\beta}}}$

$\sigma = \dfrac{bc}{\pi\ell}$ $\qquad b = $ # blades

Solve $\bar{\beta}$ equation for $\bar{\beta}$

Evaluate M, C, K matrices

Let $\hat{q} = \check{q}\, e^{s\Omega t} = \check{q}\, e^{s\psi}$

Solve for $s = \sigma \pm i\omega$ $\qquad$ $\sigma < 0$ stable; $\sigma = 0$ neutral stability; $\sigma > 0$ unstable

See Ormiston and Hodges (1972) for discussion of stability.

Routh's discriminant (4th order)

$$A s^4 + B s^3 + \cdots + E = 0$$

neutral stability when $D(BC - AD) - B^2 E = 0$

$R = 0$ flexibility in hub

$R = 1$ flexibility in "blade" (springs rotate w/pitch)

elastic blade analogies

pitch-lag / pitch-flap coupling