# Rotorcraft Aeroelasticity

Prof. Dewey H. Hodges[1]

[1]Daniel Guggenheim School of Aerospace Engineering
Georgia Institute of Technology, Atlanta, Georgia

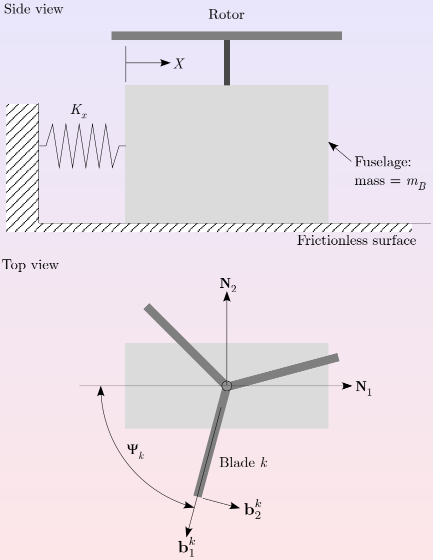
AE 6220, Spring 2017

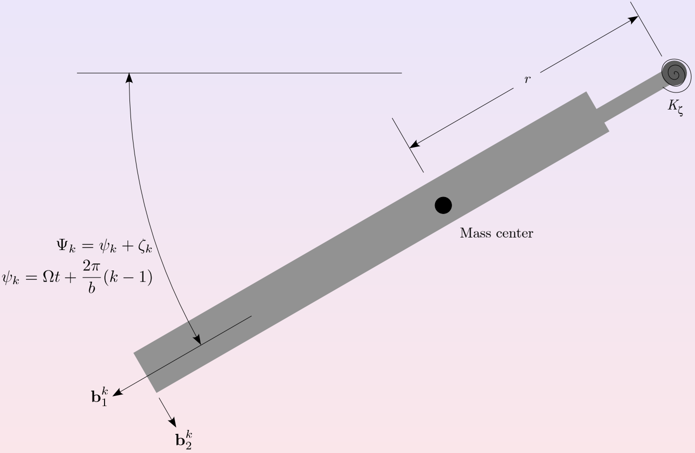
- Helicopters are susceptible to a catastrophic instability called ground resonance
- This instability is not a resonance at all but instead a self-excited mechanical instability
    - Rotor system must be soft-inplane (we'll see why)
    - Aerodynamics is not needed to capture the effect
- The instability was avoided in older articulated rotor systems by incorporation of dampers
    - lead-lag dampers in the rotating system and
    - roll/pitch dampers in the fuselage suspension system

Figure: Catastrophic destruction from ground resonance

Figure: Catastrophic destruction from ground resonance https://www.youtube.com/watch?v=0FeXjhUEXlc

- Here we examine in detail a simple system, using it to introduce key ingredients of rotorcraft dynamics analysis
  - multi-blade coordinate transformation
  - coupling of rotating and non-rotating systems
- Consider a simple spring-mass system with a rotor added:
  - a body of mass $m_B$
  - slides in a frictionless straight track
  - restrained in its translation by a light, linear spring with spring constant $K_X$
  - has a rotor attached to it
  - rotor has $b \geq 3$ identical, equally-spaced blades
  - each blade has mass $m$
  - each blade hinged at the hub in lead-lag rotation
  - lead-lag motion restrained with identical spring constants $K_\zeta$


Side view
Rotor
X
$K_x$
Fuselage:
mass = $m_B$
Frictionless surface
Top view
$\mathbf{N}_2$
$\mathbf{N}_1$
$\Psi_k$
Blade $k$
$\mathbf{b}_2^k$
$\mathbf{b}_1^k$


r
$K_\zeta$
Mass center
$\Psi_k = \psi_k + \zeta_k$
$\psi_k = \Omega t + \frac{2\pi}{b}(k-1)$
$\mathbf{b}_1^k$
$\mathbf{b}_2^k$

- For the $k^{\text{th}}$ of $b$ blades, the azimuth angle for the undeflected blade is $\psi_k = \Omega t + \frac{2\pi}{b}(k-1)$
- For the $k^{\text{th}}$ blade, the azimuth angle for the deflected blade is $\Psi_k = \psi_k + \zeta_k$ where $\zeta_k$ is the lead-lag angle
- The unit vectors are related as

$$\begin{Bmatrix} \boldsymbol{b}_1^k \\ \boldsymbol{b}_2^k \\ \boldsymbol{b}_3^k \end{Bmatrix} = \begin{bmatrix} -\cos\Psi_k & -\sin\Psi_k & 0 \\ \sin\Psi_k & -\cos\Psi_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \\ \boldsymbol{N}_3 \end{Bmatrix}$$

- The angular velocity of the $k^{\text{th}}$ blade is

$$\boldsymbol{\omega}^k = (\Omega + \dot{\zeta}_k)\boldsymbol{b}_3^k$$

- The potential energy is simply

$$P = \frac{K_\zeta}{2}\sum_{k=1}^{b}\zeta_k^2 + \frac{K_X}{2}X^2$$

- The velocity of the body (and of its mass center) is

$$\boldsymbol{v}^B = \dot{X}\boldsymbol{N}_1$$

- The velocity of the mass center of the $k^{\text{th}}$ blade is

$$\begin{aligned}\boldsymbol{v}^k =& \dot{X}\boldsymbol{N}_1 + (\Omega + \dot{\zeta}_k)\boldsymbol{b}_3^k \times r\boldsymbol{b}_1^k \\ =& \dot{X}\boldsymbol{N}_1 + (\Omega + \dot{\zeta}_k)r\boldsymbol{b}_2^k \\ =& [\dot{X} + (\Omega + \dot{\zeta}_k)r\sin\Psi_k]\boldsymbol{N}_1 - [(\Omega + \dot{\zeta}_k)r\cos\Psi_k]\boldsymbol{N}_2\end{aligned}$$

- Thus, the kinetic energy can written as

$$K = \frac{m_B}{2}\dot{X}^2 + \frac{I^*}{2}\sum_{k=1}^{b}(\Omega + \dot{\zeta}_k)^2 +$$

$$\frac{m}{2}\sum_{k=1}^{b}\left\{[\dot{X} + (\Omega + \dot{\zeta}_k) r \sin\Psi_k]^2 + (\Omega + \dot{\zeta}_k)^2 r^2 \cos^2\Psi_k\right\}$$

- Letting the total system mass $M = m_B + bm$ and the moment of inertia about the hub $I = I^* + mr^2$, one finds

$$K = \frac{M\dot{X}^2}{2} + \frac{I}{2}\sum_{k=1}^{b}(2\Omega\dot{\zeta}_k + \dot{\zeta}_k^2) + mr\dot{X}\sum_{k=1}^{b}(\Omega + \dot{\zeta}_k)\sin\Psi_k$$

- Lagrange's equations give us the equations of motion

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{X}} - \frac{\partial K}{\partial X} + \frac{\partial P}{\partial X} = 0$$
$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\zeta}_k} - \frac{\partial K}{\partial \zeta_k} + \frac{\partial P}{\partial \zeta_k} = 0 \qquad k=1,2,\ldots,b$$

- The body equation is

$$M\ddot{X} + mr\sum_{k=1}^{b}\left[\ddot{\zeta}_k \sin\Psi_k + (\Omega + \dot{\zeta}_k)^2 \cos\Psi_k\right] + K_X X = 0$$

- The $k^{\text{th}}$ blade equation is

$$I\ddot{\zeta}_k + mr\ddot{X}\sin\Psi_k + K_\zeta \zeta_k = 0 \qquad k=1,2,\ldots,b$$

- There are $b+1$ unknowns and $b+1$ equations
- The equations are nonlinear and require numerical solution
- Stability of small motions about a static equilibrium state is of practical concern and also amenable to simpler solution
- Since $X=\zeta_k=0$ satisfies the governing equations, we may linearize about the reference state

$$
\begin{aligned}
I\ddot{\zeta}_k + mr\ddot{X}\sin\psi_k + K_\zeta\zeta_k =& 0 \qquad k=1,2,\ldots,b \\
M\ddot{X} + mr\sum_{k=1}^{b}(\ddot{\zeta}_k\sin\psi_k + 2\Omega\dot{\zeta}_k\cos\psi_k & \\
-\Omega^2\zeta_k\sin\psi_k) + K_X X =& 0
\end{aligned}
$$

- Unfortunately, these equations, though linear, have periodic coefficients in time (note $\sin\psi_k$ and $\cos\psi_k$)
- The multi-blade coordinate transformation can be written

$$\zeta_k = \zeta_0 + \zeta_d(-1)^k + \sum_{n=1}^{(b-1)/2} \left(\zeta_{nc}\cos n\psi_k + \zeta_{ns}\sin n\psi_k\right)$$

where

  - the first term, $\zeta_0$, is the collective mode
  - the second term, $\zeta_d(-1)^k$, is the differential collective mode and exists only for even $b$
  - the last two terms are the cyclic modes

- One can derive the following identities

$$\zeta_s = \frac{2}{b}\sum_{k=1}^{b}\zeta_k \sin\psi_k \qquad \dot{\zeta}_s - \Omega\zeta_c = \frac{2}{b}\sum_{k=1}^{b}\dot{\zeta}_k \sin\psi_k$$

$$\zeta_c = \frac{2}{b}\sum_{k=1}^{b}\zeta_k \cos\psi_k \qquad \dot{\zeta}_c + \Omega\zeta_s = \frac{2}{b}\sum_{k=1}^{b}\dot{\zeta}_k \cos\psi_k$$

$$\ddot{\zeta}_c + 2\Omega\dot{\zeta}_s - \Omega^2\zeta_c = \frac{2}{b}\sum_{k=1}^{b}\ddot{\zeta}_k \cos\psi_k$$

$$\ddot{\zeta}_s - 2\Omega\dot{\zeta}_c - \Omega^2\zeta_s = \frac{2}{b}\sum_{k=1}^{b}\ddot{\zeta}_k \sin\psi_k$$

- For our simple system, with $b \geq 3$, it can be shown that only the $n = 1$ cyclic modes couple with $X$
- Thus, $\zeta_k = \zeta_c \cos\psi_k + \zeta_s \sin\psi_k$ and the $X$ equation becomes

$$M\ddot{X} + \frac{mbr}{2}\ddot{\zeta}_s + K_X X = 0$$

- The natural question is, how do we get the other equation(s)?
- Let's look at the answer from the point of view of virtual work

- Let's write the equation for the $k^{\text{th}}$ blade symbolically as $Q_k = I\ddot{\zeta}_k + mr\ddot{X}\sin\psi_k + K_\zeta\zeta_k = 0$
- The virtual rotation of the $k^{\text{th}}$ blade is

$$\delta\zeta_k = \delta\zeta_c\cos\psi_k + \delta\zeta_s\sin\psi_k$$

- Thus, the contribution of blades to the system virtual work is

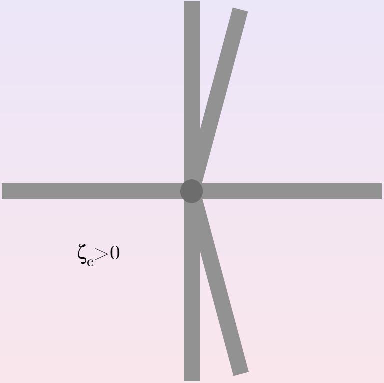
$$\sum_{k=1}^{b} Q_k\delta\zeta_k$$

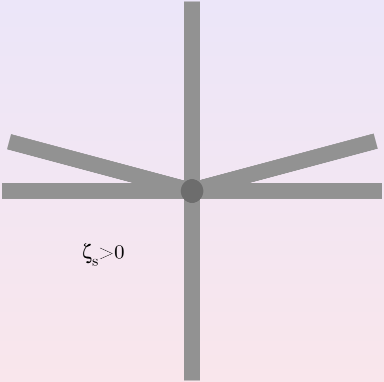
which can be written as $Q_c\delta\zeta_c + Q_s\delta\zeta_s = 0$

- Equations that multiply $\delta\zeta_0$, $\delta\zeta_d$, etc., may be found in the same way, but they are uncoupled from $X$, $\zeta_c$ or $\zeta_s$
- Since $\delta\zeta_c$ and $\delta\zeta_s$ are arbitrary, we may conclude that $Q_c = Q_s = 0$ or

$$I(\ddot{\zeta}_c + 2\Omega\dot{\zeta}_s - \Omega^2\zeta_c) + K_\zeta\zeta_c = 0$$
$$I(\ddot{\zeta}_s - 2\Omega\dot{\zeta}_c - \Omega^2\zeta_s) + mr\ddot{X} + K_\zeta\zeta_s = 0$$

- So, what are the meanings of $\zeta_s$ and $\zeta_c$?
  - positive $\zeta_c$ shifts the rotor mass centroid laterally to the right
  - positive $\zeta_s$ shifts the rotor mass centroid longitudinally to the front


$\zeta_c > 0$


$\zeta_s > 0$

- To explain how the phenomenon got its name is now easy
- The rotor mass centroid moves longitudinal harmonic motion caused by time varying $\zeta_s$
- When that motion has a frequency that coincides with that of the body motion $X$, this suggests resonance
- It is not a true resonance, however, as we are not talking about an external driving frequency
- Consider uniform blades of length $R$, so that $r = \frac{R}{2}$ and $I = \frac{mR^2}{3}$
- Introduce $\Omega_0$ as a nominal angular speed and dimensionless frequencies $\omega_X = \sqrt{\frac{K_X}{M\Omega_0^2}}$ and $\omega_\zeta = \sqrt{\frac{K_\zeta}{I\Omega_0^2}}$

- Finally, with non-dimensional displacement $\overline{X} = \frac{X}{R}$ and mass ratio $\mu = \frac{bm}{M}$, the governing equations become

$$\begin{bmatrix} \frac{6}{\mu} & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\overline{X}} \\ \ddot{\zeta}_c \\ \ddot{\zeta}_s \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\overline{\Omega} \\ 0 & -2\overline{\Omega} & 0 \end{bmatrix} \begin{Bmatrix} \dot{\overline{X}} \\ \dot{\zeta}_c \\ \dot{\zeta}_s \end{Bmatrix}$$

$$+ \begin{bmatrix} \frac{6}{\mu}\omega_X^2 & 0 & 0 \\ 0 & \omega_\zeta^2 - \overline{\Omega}^2 & 0 \\ 0 & 0 & \omega_\zeta^2 - \overline{\Omega}^2 \end{bmatrix} \begin{Bmatrix} \overline{X} \\ \zeta_c \\ \zeta_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- There four things about this matrix equation that need to be noted
  - The gyroscopic/Coriolis matrix is antisymmetric
  - The gyroscopic/Coriolis terms do no work
  - Gyroscopic/Coriolis terms cannot destabilize a system
  - The stiffness matrix is not positive definite
- Thus, we expect possible problems when $\omega_\zeta \leq \overline{\Omega}$
- Rotors that operate thus are called soft-inplane rotors
- Articulated rotors are generally in this category and typically have problems with ground resonance

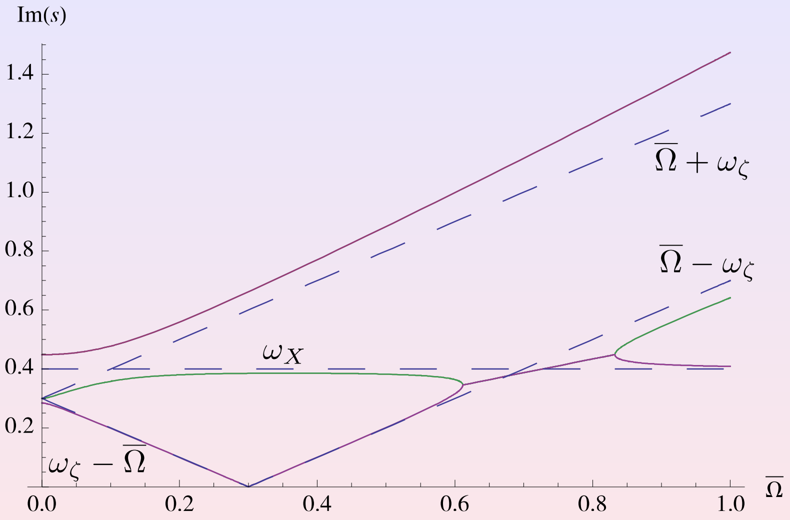



Figure: Modal frequencies for $\mu = 0.3$, $\omega_\zeta = 0.3$, $\omega_X = 0.4$

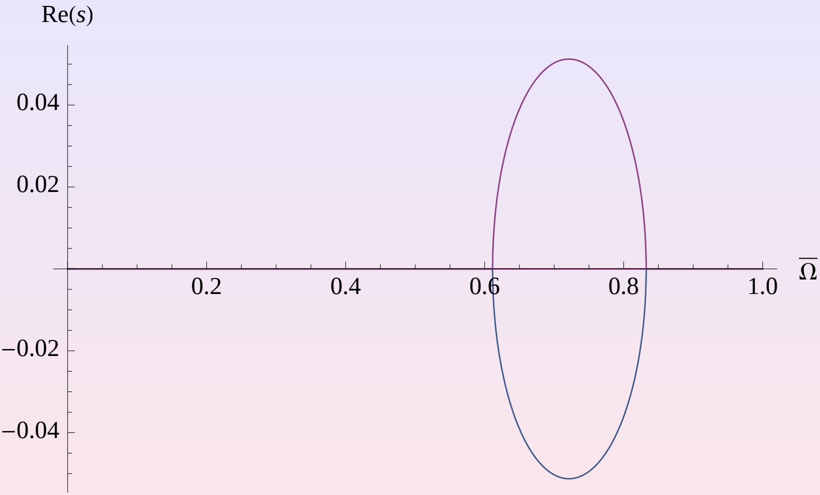



Figure: Modal frequency for $\mu = 0.3$, $\omega_\zeta = 0.3$, $\omega_X = 0.4$

- The frequency of motions dominated by $X$ is right around $\omega_X$
- The leag-lag motions are the most interesting
  - The largest values on the plot of $\Im(s)$ are close to $\overline{\Omega} + \omega_\zeta$, referred to as the progressing mode
  - The line $\omega_\zeta - \overline{\Omega}$ decreases to zero at $\overline{\Omega} = \omega_\zeta$
  - The line $\overline{\Omega} - \omega_\zeta$ increases from zero at $\overline{\Omega} = \omega_\zeta$, referred to as the regressing mode
  - Lead-lag is hardly affected by $X$ for stiff-inplane rotors (i.e. to the left of the point where $\overline{\Omega} = \omega_\zeta$
  - Lead-lag locks onto $X$ for soft-inplane rotors (i.e. to the right of the point where $\overline{\Omega} = \omega_\zeta$
  - The plot of the $\Re(s)$ shows a sudden instability coincident with the region where $X$ and $\zeta$ motions are locked onto each other

- The strength of the instability is a function of the system parameters
- The instability may be eliminated by including damping in both the fixed system, i.e. on $X$, and in the rotating system, i.e. on each blade
- Classical work of Coleman and Feingold showed that the *product of the two damping constants* must exceed a certain threshold value determined by other system parameters to overpower the instability
- Articulated rotor helicopters are soft-inplane and thus must have both lead-lag and fixed-system dampers
- Rotor modes are referred to as progressing and regressing (or backward whirl and forward whirl)

- The equations with damping can be shown to be

$$\begin{bmatrix} \frac{6}{\mu} & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\overline{X}} \\ \ddot{\zeta}_c \\ \ddot{\zeta}_s \end{Bmatrix} + \begin{bmatrix} \frac{6}{\mu} c_X & 0 & 0 \\ 0 & c_\zeta & 2\overline{\Omega} \\ 0 & -2\overline{\Omega} & c_\zeta \end{bmatrix} \begin{Bmatrix} \dot{\overline{X}} \\ \dot{\zeta}_c \\ \dot{\zeta}_s \end{Bmatrix}$$

$$+ \begin{bmatrix} \frac{6}{\mu}\omega_X^2 & 0 & 0 \\ 0 & \omega_\zeta^2 - \overline{\Omega}^2 & \overline{\Omega} c_\zeta \\ 0 & -\overline{\Omega} c_\zeta & \omega_\zeta^2 - \overline{\Omega}^2 \end{bmatrix} \begin{Bmatrix} \overline{X} \\ \zeta_c \\ \zeta_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- We now consider two cases with the same product $c_X c_\zeta = 0.0125$

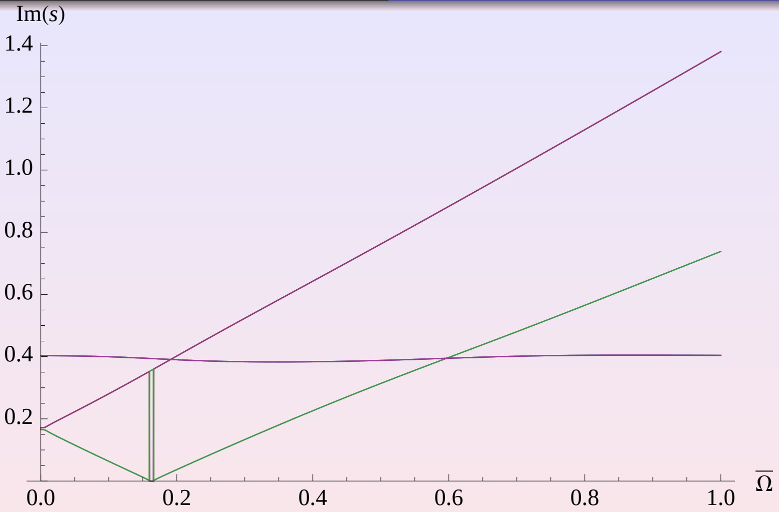



Figure: Modal frequency for $\mu = 0.3$, $\omega_\zeta = 0.3$, $\omega_X = 0.4$, $c_X = 0.025$, $c_\zeta = 0.5$

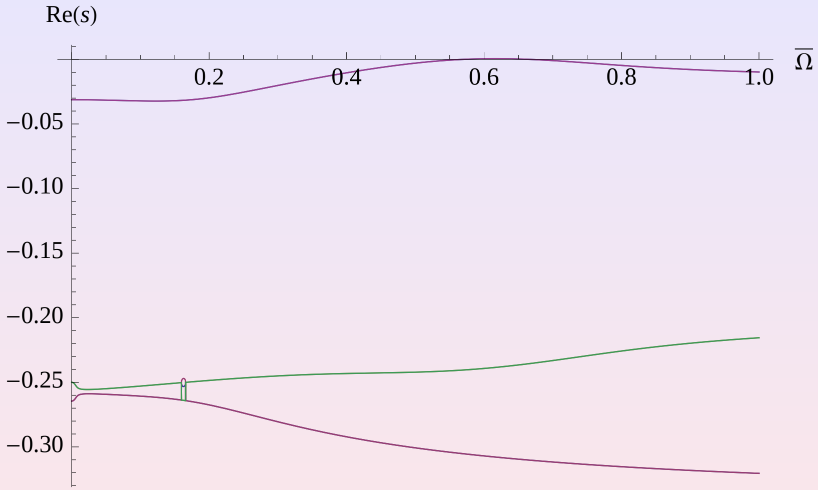



Figure: Modal damping for $\mu = 0.3$, $\omega_\zeta = 0.3$, $\omega_X = 0.4$, $c_X = 0.025$, $c_\zeta = 0.5$

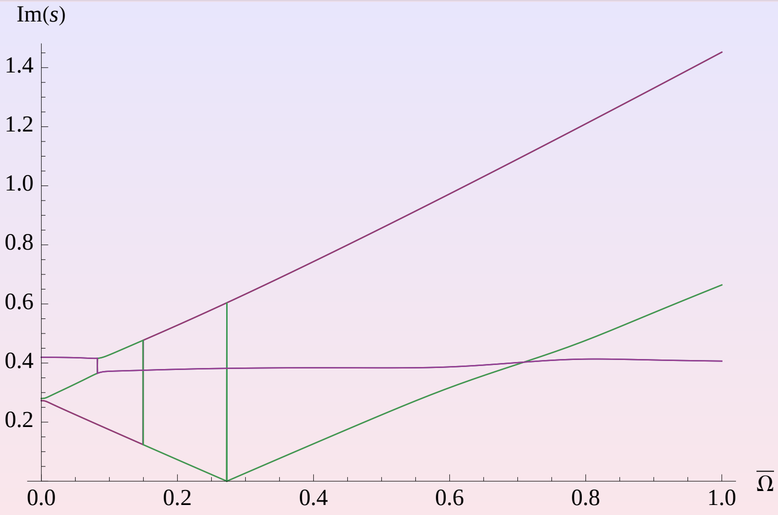



Figure: Modal frequency for $\mu = 0.3$, $\omega_\zeta = 0.3$, $\omega_X = 0.4$, $c_X = 0.05$, $c_\zeta = 0.25$

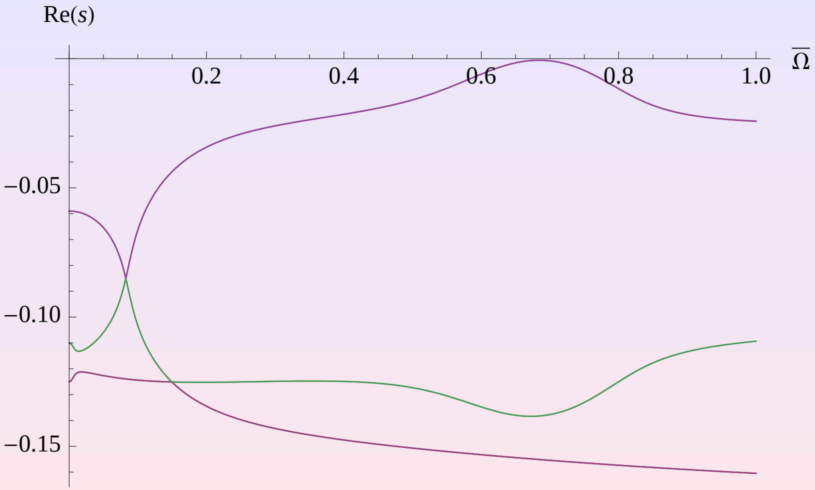



Figure: Modal damping for $\mu = 0.3$, $\omega_\zeta = 0.3$, $\omega_X = 0.4$, $c_X = 0.05$, $c_\zeta = 0.25$

- Two-bladed rotors have a fundamentally different behavior
- They possess periodic coefficients in time
- There is no transformation that will eliminate them as long as the support is nonisotropic
- Rotor modes are $\zeta_c = \zeta_1 + \zeta_2$ and $\zeta_s = \zeta_1 - \zeta_2$
  - $\zeta_c$ is called the collective mode
    - does not shift the rotor center of mass
    - would couple with dynamics of the drive train, were it modeled, and rotor shaft torsion
  - $\zeta_s$ is called the differential collective mode
    - shifts the rotor center of mass laterally and longitudinally
    - would couple with any degree of freedom involving lateral or longitudinal hub motion (such as lateral translations, pitch or roll rotations)

- Therefore, for our simple model
  - $\zeta_c$ is uncoupled
  - $\zeta_s$ is inertially coupled to $X$
- The two equations of motion are then

$$I\ddot{\zeta}_s + 2mr\dot{X}\sin\Omega t + K_\zeta \zeta_s = 0$$

$$M\ddot{X} + mr[(\ddot{\zeta}_s - \Omega^2\zeta_s)\sin\Omega t + 2\Omega\dot{\zeta}_s\cos\Omega t] + K_X X = 0$$

- These equations may be solved by Floquet theory (to be covered later by Prof. Prasad)
  - We first find a numerical solution for periodic motion such that $X(0) = X(T)$, $\dot{X}(0) = \dot{X}(T)$, $\zeta_s(0) = \zeta_s(T)$ and $\dot{\zeta}_s(0) = \dot{\zeta}_s(T)$ with $T = 2\pi/\Omega$
  - We then find the transition matrix at the end of one period when $t = T$
  - Then we find its (complex) eigenvalues
  - The system is unstable if its eigenvalues lie outside the unit circle in the complex plane