On the Relationship Between the Galerkin and Rayleigh-Ritz Methods

AE 6230 Notes
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1 Extended Galerkin Method

The Galerkin method is derived from the partial differential equation(s) of motion and boundary conditions. It does not require the use of energy at all. Given an equation and boundary conditions, the Galerkin method requires a set of basis functions (note that the trial functions and test functions are the same) which satisfy all the boundary conditions. That is, they must be comparison functions. When defined this way, the relationship between the Galerkin method and the Rayleigh-Ritz method is simple. If the partial differential equations of motion are derived from an energy/virtual work principle and the basis functions are identical, then the two methods will yield exactly the same set of ordinary differential equations.

Given the difficulties associated with finding comparison functions, the generalized Galerkin method, which allows the use of admissible functions, should be of interest to structural dynamicists. Basically, the method requires the boundary conditions to be weighted with the test function (or a derivative thereof). In its most general form for a beam, for example, we can express the standard Galerkin method as

$$\int_0^\ell \left[(EIv'')'' + m\ddot{v} - f\right]\psi_i dx = 0 \tag{1}$$

where the test function is ψ_i. A way to write the extended version is

$$\int_0^\ell \left[(EIv'')'' + m\ddot{v} - f\right]\psi_i dx + M_{\text{bc}}\psi_i'\big|_0^\ell + V_{\text{bc}}\psi_i\big|_0^\ell = 0 \tag{2}$$

where the added terms are of the form of the moment boundary conditions (M_{bc}) and force boundary conditions (V_{bc}) at the ends of the beam. One can, by integration by parts, show that the weak form of the governing equations can be recovered from Eq. (2).

2 Example Showing Equivalence of the Galerkin and Rayleigh-Ritz Methods

Consider a cantilevered beam with its free end restrained by a rotational spring of spring constant k so that the partial differential equation of motion is

$$(EIv'')'' + m\ddot{v} - f = 0 \tag{3}$$

and the boundary conditions are

$$\begin{array}{ll} x = 0 & v = 0 \\ & v' = 0 \\ x = \ell & EIv'' + kv' = 0 = M_{\mathrm{bc}} \\ & -(EIv'')' = 0 = V_{\mathrm{bc}} \end{array} \tag{4}$$

Thus,

$$\int_0^\ell \left[(EIv'')'' + m\ddot{v} - f\right]\psi_i dx + \left(EIv'' + kv'\right)\psi_i'|_\ell - (EIv'')'\psi_i|_\ell = 0 \tag{5}$$

where $v = \sum_j V_j\psi_j$. Integration by parts shows that this is equivalent to the weak form

$$\int_0^\ell \left[EIv''\psi_i'' + (m\ddot{v} - f)\psi_i\right]dx + kv'\psi_i'|_\ell = 0 \tag{6}$$

which, if

$$\delta v = \sum_i \delta V_i \psi_i$$

is exactly that which one gets for the i^{th} equation from the application of the Rayleigh-Ritz method.