AE 6230: Structural Dynamics

Supplementary Class Notes[1]

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Chapter 1

Basic Elements of Engineering

In Fig. 1.1 the basic elements of engineering are shown. In this course, all these elements will be touched upon, but the primary emphasis will be on developing analysis tools – a sort of "bag of tricks" that you can use to attack a wide class of engineering problems.

We should learn to think of structural dynamics from a number of perspectives. It is at the intersection of the fields of structures and dynamics and is the basis for many aspects of the field of aeroelasticity. It is the generalization of dynamics to include structural flexibility. It is also the generalization of structural mechanics to include inertial forces.

The most general class of systems we will cover is the continuous elastic system; see Fig. 1.2. Such systems are characterized by mathematical models that can be represented in terms of partial differential equations of motion. By means of a host of approximation techniques these can be reduced to discrete, multi-degree-of-freedom systems, mathematical models of which are systems of ordinary differential equations. By modal reduction such systems may in certain cases be reduced to a single-degree-of-freedom system.

The objective of this course can be summarized as follows: to give students the means to analyze the free-vibration and forced response of structures. We will approach Fig. 1.2 from the bottom up, that is, we will first consider single-degree-of-freedom systems. Once we treat that aspect of the field with sufficient depth, we will then proceed to deal with

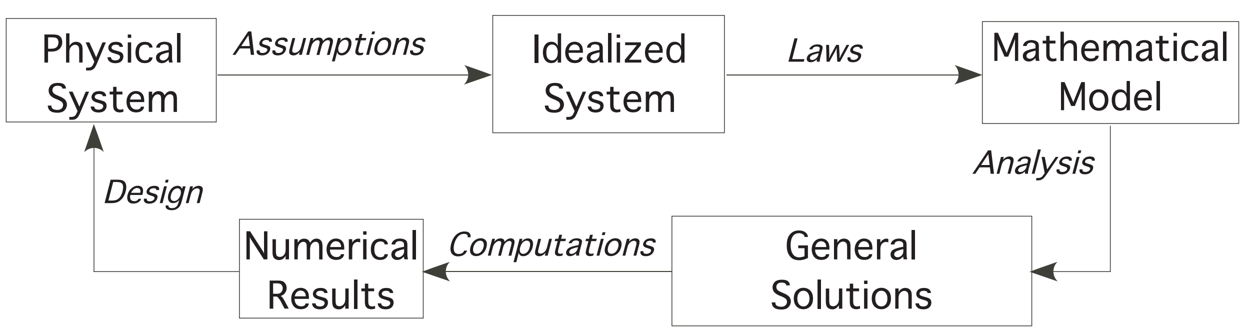

Figure 1.1: Basic Elements of Engineering

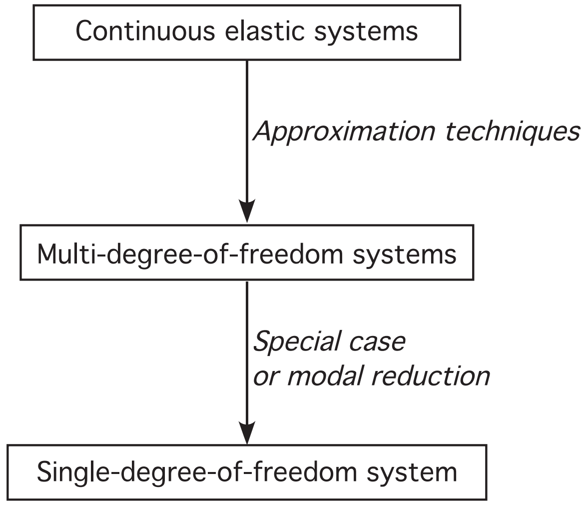

Figure 1.2: Levels of systems under consideration

treatment of multi-degree-of-freedom systems. Finally, we will analyze the most tractable of continuous, 1-D and 2-D elastic systems, namely, 1-D problems associated with strings and beams, and their 2-D analogs, membranes and plates.

Chapter 2

Single-Degree-of-Freedom Systems

2.1 Building Blocks

Three basic building blocks of simple mechanical systems are depicted in Fig. 2.1. First, consider a particle of mass m that is permitted to move in rectilinear motion and subject to a force with magnitude a specified function of time $F(t)$ in the direction motion is permitted, shown at the top of the figure. Thus, according to Newton's second law

$$F = m\ddot{x} \tag{2.1}$$

where x describes the distance moved in an inertial (or Newtonian) frame and the dots indicate differentiation with respect to the time t. A frame of reference is Newtonian if and only if this "law of motion" is satisfied. The suitability of a frame to be regarded as Newtonian can only be ascertained by experiments or by comparison of analytical results obtained with those obtained by a frame that has been verified by experiments to be Newtonian. See Kane and Levinson (1985) for further discussion of this point. Depicted in the middle part of Fig. 2.1 is a damper element. For such elements, the force through the damper is related to the velocity change across the element, so that

$$F = c\dot{x} \tag{2.2}$$

Finally, the bottom part of Fig. 2.1 is a spring element, where the force through the spring is related to the displacement change across the element, so that

$$F = kx \tag{2.3}$$

2.2 Single-degree-of-freedom example 1: applied force

As an illustration, consider a particle of mass m shown in Fig. 2.2 restrained to the ground (assumed to be a Newtonian frame) by a spring and a damper associated with rectilinear

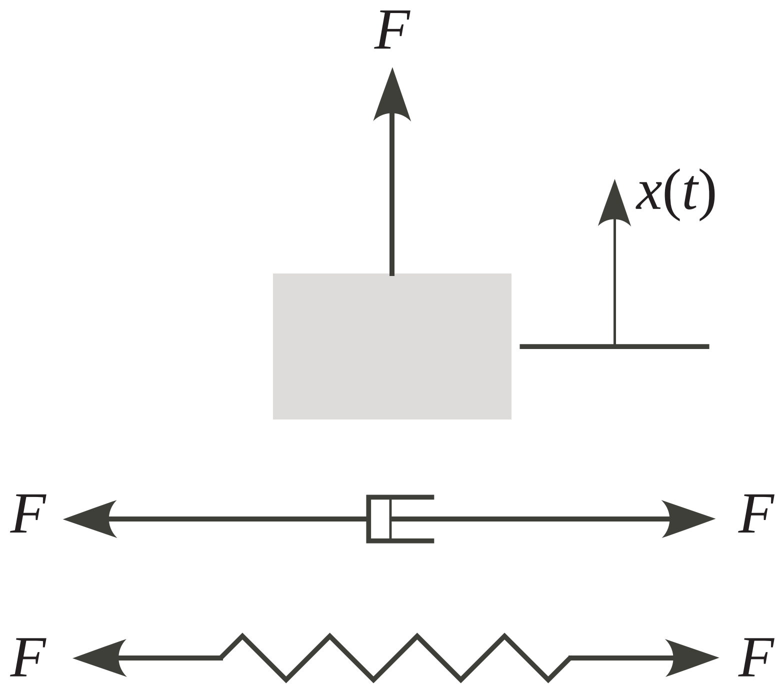

Figure 2.1: Mechanical system components

motion in the x direction. Summing forces and setting the resultant force equal to the mass times acceleration (Newton's second law), one obtains

$$F - c\dot{x} - kx = m\ddot{x} \tag{2.4}$$

or

$$m\ddot{x} + c\dot{x} + kx = F \tag{2.5}$$

2.3 Single-degree-of-freedom example 2: specified displacement

As a second illustration, consider a particle of mass m as shown in Fig. 2.3, with free-body diagrams shown below the system schematic. From left to right, the free-body diagrams yield:

$$\begin{aligned} m\ddot{x}_3 =& F \text{ or } \ddot{x}_3 = \frac{F}{m} \\ k(x_2 - x_3) =& F \text{ or } x_2 - x_3 = \frac{F}{k} \\ c(\dot{x}_1 - \dot{x}_2) =& F \text{ or } \dot{x}_1 - \dot{x}_2 = \frac{F}{c} \end{aligned} \tag{2.6}$$

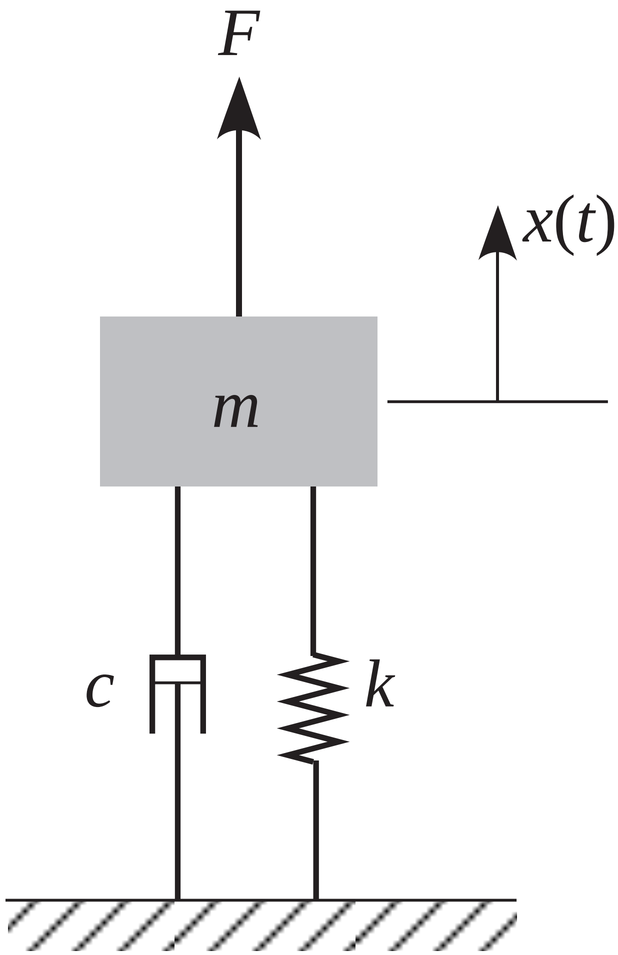

Figure 2.2: Single-degree-of-freedom system – applied force

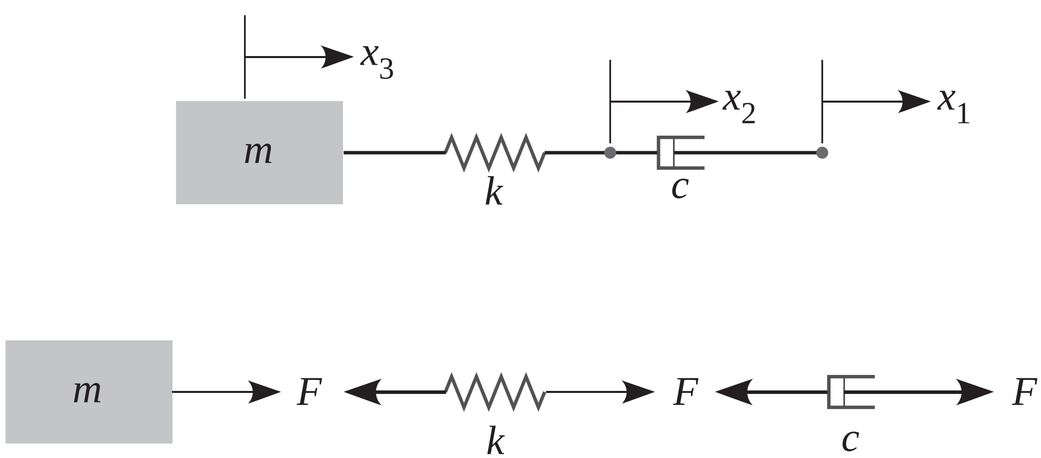

Figure 2.3: Single-degree-of-freedom system – applied displacement

Adding these three equations, one obtains

$$\frac{\ddot{F}}{k} + \frac{\dot{F}}{c} + \frac{F}{m} = \ddot{x}_1 \tag{2.7}$$

where the (given) acceleration $\ddot{x}_1(t)$ serves as the "forcing function" for this equation, and the unknown is the internal force $F(t)$.

2.4 Nonlinear system

It is possible to conduct a linearization of equations for a nonlinear system about an equilibrium point. To illustrate how this provides a linear equation of the same form, we consider a nonlinear system governed by an equation of the form

$$\ddot{x} + Z(x) = F_0 \tag{2.8}$$

where $Z(x)$ is a nonlinear function of x, and F_0 is a constant external force. The nonlinear function can represent a nonlinear spring, for example. The static equilibrium can be represented as $\overline{x}$, a constant, and arbitrary motion can always be written as

$$x(t) = \overline{x} + \hat{x}(t) \tag{2.9}$$

where $\hat{x}$ is the displacement relative to the equilibrium state. Substituting Eq. (2.9) into Eq. (2.8), one obtains

$$\ddot{\hat{x}} + Z(\overline{x} + \hat{x}) = F_0 \tag{2.10}$$

Now, utilizing the Taylor series concept and dropping all terms of second degree and higher of $\hat{x}$, we get

$$\ddot{\hat{x}} + Z(\overline{x}) + \hat{x}Z'(\overline{x}) = F_0 \tag{2.11}$$

To analyze motion relative to the equilibrium state, one must first determine the equilibrium state by setting $\hat{x} = 0$, so that $\overline{x}$ is governed by the nonlinear equation

$$Z(\overline{x}) = F_0 \tag{2.12}$$

Then, dynamics of the system about equilibrium satisfy the equation formed by subtracting Eq. (2.12) from Eq. (2.11), yielding

$$\ddot{\hat{x}} + \hat{x}Z'(\overline{x}) = 0 \tag{2.13}$$

where is it apparent that the (constant) stiffness coefficient $(Z'(\overline{x}))$ is a function of the equilibrium state about which we have linearized.

As an example of such a system, we consider an inverted pendulum shown in Fig. 2.4. The pendulum is comprised of a particle of mass m attached to a rigid, massless rod of length

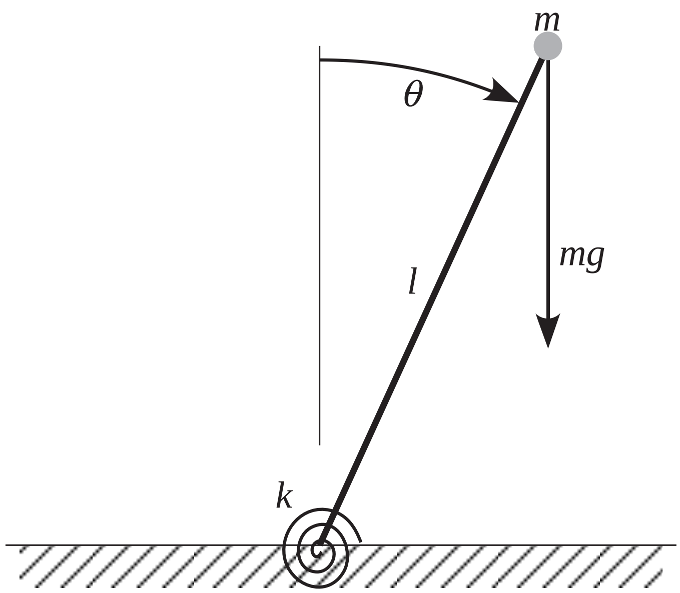

Figure 2.4: Inverted pendulum

l and is restrained by a light rotational spring with spring constant k such that the spring is relaxed when the angle $\theta = \theta_0$. The equation of motion is then obtained by summing moments about the pivot and setting that sum equal to the effective moment of inertia about the pivot times the angular acceleration (Euler's second law), viz.,

$$ml^2\ddot{\theta} + k(\theta - \theta_0) - mgl\sin\theta = 0 \tag{2.14}$$

Letting $\theta = \overline{\theta} + \hat{\theta}(t)$ one may write

$$ml^2\ddot{\hat{\theta}} + k(\overline{\theta} + \hat{\theta} - \theta_0) - mgl\sin(\overline{\theta} + \hat{\theta}) = 0 \tag{2.15}$$

Temporarily setting $\hat{\theta}$ equal to zero, one finds that the static equilibrium state is governed by

$$k(\overline{\theta} - \theta_0) - mgl\sin\overline{\theta} = 0 \tag{2.16}$$

or, after introducing $\kappa = k/(mgl)$

$$\kappa(\overline{\theta} - \theta_0) - \sin\overline{\theta} = 0 \tag{2.17}$$

Now, linearizing the equations of motion, i.e. Eq. (2.15), about the equilibrium state and subtracting Eq. (2.16) from Eq. (2.15), we may write

$$ml^2\ddot{\theta} + k\hat{\theta} - mgl\hat{\theta}\cos\overline{\theta} = 0 \tag{2.18}$$

Introducing $\omega^2 = g/l$, we may write the non-dimensional equation of motion as

$$\ddot{\theta} + \omega^2\left(\kappa - \cos\overline{\theta}\right)\hat{\theta} = 0 \tag{2.19}$$

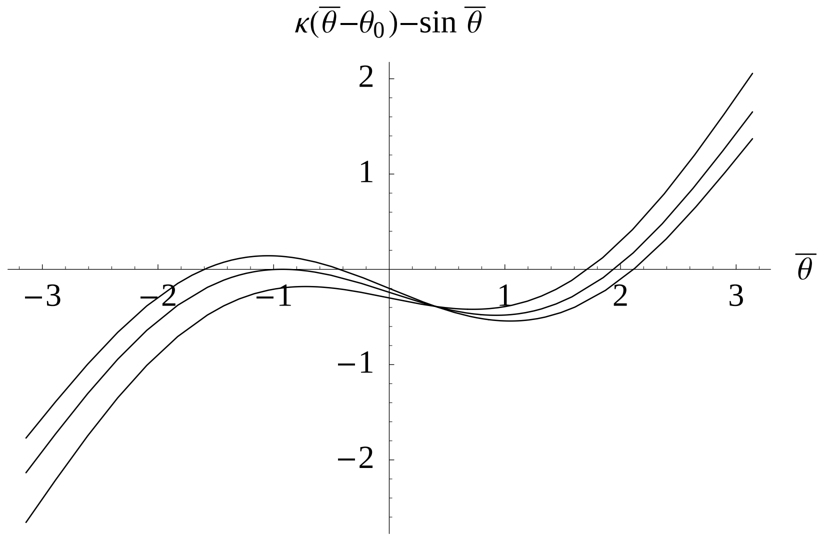

Figure 2.5: Inverted pendulum equilibrium states, $\theta_0 = 0.4$: single-root case, $\kappa = 0.75$; three-root case with double root, $\kappa = 0.602733$; three-distinct-root case, $\kappa = 0.5$

The equation governing the equilibrium state may have one root or three. In the three-root case there are two sub-cases; one has a single root and a double root, and the other has three distinct roots. These three cases are depicted in Fig. 2.5. In the case of the double root, the stiffness term in Eq. (2.19), i.e. $\kappa - \cos\overline{\theta}$, vanishes at the double root and is positive (meaning the system is stable) at the other root (the largest one). In the case of three distinct roots, the stiffness term is negative (meaning the system is unstable) at the middle root and positive (meaning the system is stable) at the smallest and largest roots.

2.5 Standard forms of governing equations

As we've seen, the typical single-degree-of-freedom system is governed by a linear, second-order, ordinary differential equation of the form

$$a_2\ddot{x} + a_1\dot{x} + a_0 x = b_1\dot{y} + b_0 y \tag{2.20}$$

where y is an input and x is the unknown. As seen above, x may be a displacement or a force; it may also be an angle, a moment or a coefficient in a modal expansion. If a_2 is equal to zero, the system is first-order. If the quantities a_2, a_1, and a_0 are constants, then the solution is easily obtained. Otherwise, the solution is much more difficult.

For the usual spring-mass-damper system, $a_2 = m$, $a_1 = c$, $a_0 = k$, $b_1 = 0$, and $b_0 y(t) = F(t) = kf(t)$, so that

$$m\ddot{x} + c\dot{x} + kx = F = kf = mQ \tag{2.21}$$

Dividing by m, we obtain another standard form, viz.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 f = Q \tag{2.22}$$

with the natural frequency $\omega_n = \sqrt{k/m}$ and the viscous damping ratio $\zeta = c/(2\sqrt{km})$.

Introducing non-dimensional time $\psi = \omega_n t$ with $(\ )' = d(\ )/d\psi$, one finally obtains

$$x'' + 2\zeta x' + x = f \tag{2.23}$$

which is the simplest form of the equation.

2.6 Basic responses

2.6.1 1st-order system (zero mass)

The equation of motion assumes the form

$$\dot{x} + ax = 0$$

Note that a is the inverse of the "time constant" τ.

Unit Displacement response: $x(0) = 1$, $x(t) \equiv e(t)$

$$x = e^{-at} \qquad \text{(Note: } \dot{x}(0) \neq 0\text{)}$$

Unit step response: $a_1\dot{x} + a_0 x = 1$, $x(0) = 0$

$$x(t) \equiv g(t) = \frac{1}{a_0}\left(1 - e^{-at}\right) \qquad a = \frac{a_0}{a_1}$$

Standard step response: $\dot{x} + ax = 1$, $x(0) = 0$

$$x(t) \equiv g_s(t) = \frac{1}{a}\left(1 - e^{-at}\right)$$

Normalized step response: $x' + x = 1$, $x(0) = 0$

$$x(t) = \overline{g}(\psi) = 1 - e^{-\psi} \qquad \psi = at$$

2.6.2 2nd-order system

$$x'' + 2\zeta x' + x = f$$

with $(\ )' = d(\ )/d\psi$ and $\psi = \omega_n t$.

Initial Displacement Response

$x(0) = 1$, $x'(0) = 0$, $f = 0$, $x(t) \equiv e(\psi)$.
If $\zeta < 1$, then

$$\begin{aligned} x(\psi) &= e^{-\zeta\psi}\left[a\cos\left(\sqrt{1-\zeta^2}\psi\right) + b\sin\left(\sqrt{1-\zeta^2}\psi\right)\right] \\ x'(\psi) &= e^{-\zeta\psi}\left[(b\sqrt{1-\zeta^2} - \zeta a)\cos\left(\sqrt{1-\zeta^2}\psi\right) + (-\zeta b - a\sqrt{1-\zeta^2})\sin\left(\sqrt{1-\zeta^2}\psi\right)\right] \end{aligned}$$

$$\begin{aligned} x(0) &= 1 \implies a = 1 \\ x'(0) &= 0 \implies b\sqrt{1-\zeta^2} = \zeta, \;\; b = \zeta/\sqrt{1-\zeta^2}, \end{aligned}$$

If $\zeta = 1$, then

$$\begin{aligned} x &= ae^{-\psi} + b\psi e^{-\psi} \\ x' &= (-a+b)e^{-\psi} - b\psi e^{-\psi} \end{aligned}$$

$$\begin{aligned} x(0) &= 1 \implies a = 1 \\ x'(0) &= 0 \implies a = b \end{aligned}$$

$$e(\psi) = e^{-\psi} + \psi e^{-\psi}$$

If $\zeta > 1$, then

$$\begin{aligned} x &= ae^{-(\zeta-\sqrt{\zeta^2-1})\psi} + be^{-(\zeta+\sqrt{\zeta^2-1})\psi} \\ x' &= -\left(\zeta - \sqrt{\zeta^2-1}\right)ae^{-(\zeta-\sqrt{\zeta^2-1})\psi} - \left(\zeta + \sqrt{\zeta^2-1}\right)be^{-(\zeta+\sqrt{\zeta^2-1})\psi} \end{aligned}$$

$$\begin{aligned} x(0) &= a + b = 1 \\ x'(0) &= -\zeta(a+b) + \sqrt{\zeta^2-1}(a-b) \\ a - b &= \zeta/\sqrt{\zeta^2-1} \\ a &= \frac{1}{2} + \frac{\zeta}{2\sqrt{\zeta^2-1}} \;\; , \quad b = \frac{1}{2} - \frac{\zeta}{2\sqrt{\zeta^2-1}} \end{aligned}$$

Velocity Responses

Normalized Velocity Response: $x(0) = 0$, $x'(0) = 1$, $x(\psi) \equiv \overline{h}(\psi)$. If $\zeta < 1$

$$x(0) = 0 \implies a = 0$$

$$x'(0) = 1 \implies b = \frac{1}{\sqrt{1-\zeta^2}}$$

If $\zeta = 1$

$$\begin{aligned} x(0) = 0 &\implies a = 0 \\ x'(0) = 1 &\implies b = 1 \end{aligned}$$

If $\zeta > 1$

$$\begin{aligned} x(0) = 0 &\implies a + b = 0 \\ x'(0) = 1 &\implies \sqrt{\zeta^2 - 1}(a - b) = 1 \\ &\quad a - b = \frac{1}{\sqrt{\zeta^2 - 1}} \\ a = \frac{1}{2\sqrt{\zeta^2 - 1}}, &\quad b = \frac{-1}{2\sqrt{\zeta^2 - 1}} \end{aligned}$$

Unit Velocity Response: $x(0) = 0$, $\dot{x}(0) = 1$, $x(\psi) \equiv h_V(\psi)$,

$$h_V(\psi) = \frac{1}{\omega_n}\overline{h}(\psi)$$

Step Responses

Unit Step Response: $x(0) = 0$, $\dot{x}(0) = 0$, $F = 1$, $x(t) \equiv g(t) = \frac{1}{a_0}\overline{g}(\psi)$.
Standard Step Response: $x(0) = 0$, $\dot{x}(0) = 0$, $Q = 1$, $x(t) \equiv g_s(t) = \frac{1}{\omega_n^2}\overline{g}(\psi)$.
Normalized Step Response: $x(0) = 0$, $x'(0) = 0$, $f = 1$, $x(t) \equiv \overline{g}(t) = 1 - e(\psi)$

Impulse Responses

$x(0) = x'(0) = 0$, forcing function $= \delta(t)$.

Nondimensional Impulse Response: $f = \delta(\psi)$, $\int_{-\epsilon}^{+\epsilon} \delta(\psi)d\psi = 1$, $x(t) \equiv \overline{h}(\psi)$ (same as normalized velocity response for 2nd-order)

$$\overline{h}(\psi) = \frac{d}{d\psi}\overline{g}(\psi)$$

Normalized Impulse Response: $f = \delta(t) = \omega_n\delta(\psi)$, $\int_{-\epsilon}^{+\epsilon} \delta(t)dt = 1$

$$x(t) \equiv h_N(t) = \omega_n\overline{h}(\psi)$$

Standard Impulse Response: $Q = \delta(t)$

$$x(t) \equiv h_S(t) = \frac{1}{\omega_n}\overline{h}(\psi) = h_V(t) \quad \text{for 2nd order}$$

Unit Impulse Response: $F(t) = \delta(t)$

$$x(t) \equiv h_i(t) = \frac{1}{\sqrt{km}}\overline{h}(\psi) = \frac{1}{m}h_V(t) = \frac{1}{a_2}h_V(t)$$

2.6.3 Summary for 2nd-order

$\zeta < 1$

$$\begin{aligned}
\text{Unit Displacement Response} &= e(\psi) \\
&= e^{-\zeta\psi}\left[\cos\left(\sqrt{1-\zeta^2}\psi\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\psi\right)\right] \\
&= \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\psi}\cos\left(\sqrt{1-\zeta^2}\psi - \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \\
\text{Normalized Velocity Response} &= \overline{h}(\psi) \\
&= \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\psi}\sin\left(\sqrt{1-\zeta^2}\psi\right) \\
\text{Normalized Step Response} &= \overline{g}(\psi) = 1 - e(\psi)
\end{aligned}$$

Note:

$$\begin{aligned}
\frac{d\overline{g}}{d\psi} &= e^{-\zeta\psi}\left[\zeta\cos\left(\sqrt{1-\zeta^2}\psi\right) + \frac{\zeta^2}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\psi\right)\right] \\
&\quad + \sqrt{1-\zeta^2}\sin\left(\sqrt{1-\zeta^2}\psi\right) - \zeta\cos\left(\sqrt{1-\zeta^2}\psi\right) \\
&= e^{-\zeta\psi}\,\frac{\sin\left(\sqrt{1-\zeta^2}\psi\right)}{\sqrt{1-\zeta^2}} = \overline{h}(\psi)
\end{aligned}$$

2.7 Laplace Transforms

Let $s = \sigma + i\omega$ with units 1/sec (or rad/sec)

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st}f(t)dt$$

Example: $f(t) = 1,\ \ t \geq 0\ \ (t < 0\ \text{ not applicable})$

$$F(s) = \int_0^\infty e^{-st}(1)dt = \frac{-1}{s}e^{-st}|_0^\infty = \frac{1}{s}$$

assuming $\sigma > 0$. (You may assume σ to be as large as you need.) Sometimes, however, there is no σ big enough, e.g. $f(t) = e^{t^2}$. In such situations no Laplace transform exists.

$$\begin{aligned}
\mathcal{L}\left(\frac{d^n f(t)}{dt^n}\right) &= s^n F(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots s f^{(n-2)}(0) - f^{(n-1)}(0) \\
&\Longrightarrow \text{ note } \quad f(0) \Longrightarrow f(0^-) \quad \text{just before } t\text{=}0
\end{aligned}$$

Apply to differential equation

$$a_2\ddot{x} + a_1\dot{x} + a_0x = b_1\dot{y} + b_0y$$

$$a_2[s^2X(s) - sx(0) - \dot{x}(0)] + a_1[sX(s) - x(0)] + a_0X(s) = [b_1s + b_0]Y(s) - b_1y(0)$$

$$X(s) = \frac{(b_1s + b_0)Y(s) - b_1y(0) + (a_2s + a_1)x(0) + a_2\dot{x}(0)}{a_2s^2 + a_1s + a_0}$$

Transfer Function $\equiv \frac{X(s)}{Y(s)} = H(s)$ for all zero initial conditions

Note: for standard form of $y(t)$, $H(s) = \frac{X(s)}{Q(s)}$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega^2x = Q(t)$$

$$\text{or} \quad \dot{x} + \eta x = Q(t)$$

1st order	2nd order
$H(s) = \frac{1}{s+\eta}$	$H(s) = \frac{1}{s^2+2\zeta\omega_ns+\omega_n^2}$
$h(t) = e^{-\eta t}$	$h(t) = h_V(t) = \frac{1}{\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_nt}\sin\sqrt{1-\zeta^2}\omega_nt$

2nd order unit displacement response, $x(0) = 1$, $\dot{x}(0) = 0$

$$\begin{aligned} E(s) &= \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_ns + \omega_n^2} = \frac{(s^2 + 2\zeta\omega_ns + \omega_n^2)}{s(s^2 + 2\zeta\omega_ns + \omega_n^2)} - \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_ns + \omega_n^2)} \\ &= \frac{1}{s} - \frac{1}{s}\,\frac{\omega_n^2}{(s^2 + 2\zeta\omega_ns + \omega_n^2)} \end{aligned}$$

$$E(s) = \mathcal{L}(1) - \overline{G}(s) \quad , \quad \overline{G}(s) = \frac{\omega_n^2}{s}H_V(s)$$

$$e(t) = 1 - \overline{g}(\overline{t}) \quad , \quad \overline{g}(\overline{t}) = \omega_n^2\frac{d}{dt}h_V(t)$$

One of the tools for solving low-order systems of ordinary differential equations of the form considered above is the Laplace Transform. In Table 2.1 are some useful Laplace transforms.

2.7.1 Ways to Invert Laplace Transforms:

1. Ingenious use of table
2. Real partial fractions
3. Complex partial fractions
4. Contour integration
5. Convolution integral

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$\delta(t)$	1
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
te^{at}	$\frac{1}{(s-a)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{at}$	$\frac{1}{(s-a)^n}$
$f(t) = f(t+\tau)$	$\frac{1}{1-e^{-s\tau}} \int_0^\tau e^{-st} f(t) dt$
$f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-\frac{d}{ds}F(s)$
$\frac{d}{dt}f(t)$	$sF(s)$
$\int_0^t f(t)dt$	$\frac{1}{s}F(s)$
$\frac{1}{t}f(t)$	$\int_s^\infty F(s)ds$
$\lim\limits_{t\to 0} f(t)$	$\lim\limits_{s\to\infty} sF(s)$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$

Table 2.1: Some useful Laplace Transforms

2.7.2 Use of Table

Recall $f(0^-) = 0$.

Examples:

a. $F(s) = \frac{1}{(s-a)^2}$

$$\begin{array}{ll} f(t) & F(s) \\ 1 & \frac{1}{s} \\ e^{at}(1) & \frac{1}{s-a} \\ te^{at} & -\frac{d}{ds}(\frac{1}{s-a}) = \frac{1}{(s-a)^2} \end{array}$$

b. $F(s) = \frac{1}{(s^2+1)^2}$

$$\begin{array}{ll} f(t) & F(s) \\ \sin t & \frac{1}{s^2+1} \\ t \sin t & -\frac{d}{ds}(\frac{1}{s^2+1}) = \frac{2s}{(s^2+1)^2} \\ \frac{1}{2}\int_0^t t \sin t dt & \frac{1}{2s}[\frac{2s}{(s^2+1)^2}] \\ = \frac{1}{2}(\sin t - t \cos t) & = \frac{1}{(s^2+1)^2} \end{array}$$

c. $F(s) = \frac{s}{s^2+2\zeta s+1}$, $a = \sqrt{1-\zeta^2}$

$$\begin{array}{ll} f(t) & F(s) \\ \sin at & \frac{a}{s^2+a^2} \\ e^{-\zeta t} \sin at & \frac{a}{(s+\zeta)^2+a^2} \\ \frac{1}{a}\frac{d}{dt}(e^{-\zeta t} \sin at) & \frac{s}{(s+\zeta)^2+a^2} = \frac{s}{s^2+2\zeta s+1} \end{array}$$

2.7.3 Real Partial Fractions

a. $\ddot{x} + x = t$, $x(0) = \dot{x}(0) = 0$

$$(s^2+1)X(s) = 1/s^2, \quad X(s) = \frac{1}{s^2(s^2+1)}$$

$$\frac{1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$1 = (As+B)(s^2+1) + (Cs+D)s^2$$

By equating coefficients

$$A = C = 0, \quad B = 1, \quad D = -1$$

$$X(s) = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$x(t) = t - \sin t$$

b. $\ddot{x} + x = 1$, $x(0) = x_0$, $\dot{x}(0) = v_0$

$$X(s^2+1) = sx(0) + v_0 + \frac{1}{s}$$

$$X = \frac{1}{s(s^2+1)} + \frac{s}{s^2+1}x_0 + \frac{1}{s^2+1}v_0$$
$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$
$$1 = A(s^2+1) + (Bs+C)s$$

By equating coefficients

$$A = 1, \quad B = -1, \quad C = 0$$
$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$
$$x(t) = 1 - \cos t + x_0 \cos t + v_0 \sin t$$

2.7.4 Complex Partial Fractions

a. $X(s) = \frac{s}{s^2+a^2} = \frac{A}{s+a\,i} + \frac{B}{s-a\,i}$, $(s+a\,i)(s-a\,i) = s^2 + a^2$

$$s = A(s - a\,i) + B(s + a\,i)$$

$$1 = A + B, \quad 0 = -a\,i\,A + a\,i\,B$$
$$\implies A = B = \frac{1}{2}$$

$$\begin{aligned} X(s) &= \frac{1}{2}\frac{1}{s+a\,i} + \frac{1}{2}\frac{1}{s-a\,i} \\ x(t) &= \frac{1}{2}e^{-a\,i\,t} + \frac{1}{2}e^{a\,i\,t} = \cos at \end{aligned}$$

b. General case, no repeated roots

$$X(s) = \frac{G(s)}{H(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}$$

$$G(s) = a_m(s - z_1)(s - z_2)\cdots(s - z_m) \qquad z_i = \text{zeros}$$
$$H(s) = b_n(s - p_1)(s - p_2)\cdots(s - p_n) \qquad p_i = \text{poles}$$

If $m \geq n$, $m - n = p$

$$X(s) = A_p s^p + A_{p-1} s^{p-1} + \cdots + A_1 s + A_0 + \frac{B_1}{s-p_1} + \frac{B_2}{s-p_2} + \cdots + \frac{B_n}{s-p_n}$$

If $m < n$, delete A terms.

Multiply by common denominator

$$\begin{array}{rcl}
s^m & // & a_m = A_p b_n \\
s^{m-1} & // & a_{m-1} = A_{p-1} b_n + A_p b_{n-1} \\
\vdots & & \vdots \\
s^n & // & a_n = A_0 b_n + A_1 b_{n-1} + \cdots + A_p b_{n-p} \\
s^{n-1} & // & a_{n-1} = A_0 b_{n-1} + A_p b_{n-1-p} + b_n(B_1 + B_2 + \cdots + B_n)
\end{array}$$

$$x(t) = A_p \delta^{(p)} + A_{p-1}\delta^{(p-1)} + \cdots + A_n \delta + B_1 e^{r_1 t} + \cdots + B_n e^{r_n t}$$

You can always find A_n first, subtract out of $X(s)$, but usually all the A_i are zero anyway. You can also always make $b_n = 1$, by dividing top and bottom by b_n.

$$\begin{aligned}
X(s) &= \frac{G(s)}{H(s)} = \frac{G(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \qquad \left(\lim_{s\to\infty} \frac{G(s)}{H(s)} = 0\right) \\
&= \frac{B_1}{s-p_1} + \frac{B_2}{s-p_2} + \cdots + \frac{B_n}{s-p_n}
\end{aligned}$$

Notice:

$$B_1 = \left[\frac{G(s)}{H(s)}(s-p_1)\right]_{s=p_1}, \quad \ldots \quad B_n = \left[\frac{G(s)}{H(s)}(s-p_n)\right]_{s=p_n}$$

c. Repeated root

$$H(s) = (s-p_1)^2 (s-p_2)(s-p_3)\cdots(s-p_{n-1})$$

$$X(s) = \frac{B_{11}}{s-p_1} + \frac{B_{12}}{(s-p_1)^2} + \frac{B_2}{(s-p_2)} + \cdots$$

$$B_{12} = \left[\frac{G(s)}{H(s)}(s-p_1)^2\right]_{s=p_1}, \qquad B_{11} = \frac{d}{ds}\left[\frac{G(s)}{H(s)}(s-p_1)^2\right]_{s=p_1}$$

$$\begin{aligned}
H(s) &= (s-p_1)^r (s-p_2)\cdots \\
X(s) &= \frac{B_{11}}{s-p_1} + \frac{B_{12}}{(s-p_1)^2} + \cdots + \frac{B_{1r}}{(s-p_1)^r} \\
B_{1j} &= \frac{1}{(r-j)!}\frac{d^{r-j}}{ds^{r-j}}\left[\frac{G(s)}{H(s)}(s-p_1)^r\right]_{s=p_1} \qquad j = 1, r
\end{aligned}$$

In summary of complex partial fractions, r_i= occurrences of p_i

$$\begin{aligned}
X(s) &= \frac{G(s)}{H(s)} = \sum_{i=1}^{n}\sum_{j=1}^{r_i}\frac{B_{ij}}{(s-p_i)^j} \\
x(t) &= \sum_{i=1}^{n}\sum_{j=1}^{r_i}\frac{B_{ij}}{(j-1)!}t^{j-1}e^{p_i t} \\
B_{ij} &= \frac{1}{(r_i-j)!}\,\frac{d^{r_i-j}}{ds^{r_i-j}}\left[\frac{G(s)}{H(s)}(s-p_i)^{r_i}\right]_{s=p_i}
\end{aligned}$$

2.8 Contour Integration

$$F(s) = \int_0^\infty e^{-st} f(t)dt \qquad s = \sigma + i\omega, \quad \sigma > c$$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds$$

"c" must be chosen to the right of all poles of $F(s)$.

2.8.1 Cauchy's Theorem

$$\oint \overline{G}(s)ds = 2\pi i \sum \text{residues}$$

If $(s-p)^r\overline{G}(s) = \text{ finite } \quad \neq 0$, then there is a pole of order r at p and

$$\text{Residue} = \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left[(s-p)^r\overline{G}(s)\right]_{s=p}$$

For use in inverse Laplace, $\overline{G}(s) = e^{st}F(s)$

$$\begin{aligned}
2\pi i \sum \text{ Residues} &= \int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds + \int_\Gamma e^{st}F(s)ds \\
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds &= \sum \text{ Residues} - \underbrace{\frac{1}{2\pi i}\int_\Gamma e^{st}F(s)ds}_{\text{hopefully zero}}
\end{aligned}$$

2.8.2 Checklist for inverse

1. Can appropriate "c" be found? $\begin{cases} \text{No} & \Longrightarrow \text{no inverse} \\ \text{Yes} & \Longrightarrow \text{go to 2} \end{cases}$

2. Does $\lim\limits_{s\to\infty} e^{st}F(s) = 0?$ $\begin{cases} \text{Yes} & \Longrightarrow f(t) = 0 \\ \text{No} & \Longrightarrow \text{go to 3} \end{cases}$

3. Does $\lim\limits_{s\to-\infty} e^{st}F(s) = 0?$ $\begin{cases} \text{Yes} & \Longrightarrow \text{go to 4} \\ \text{No} & \Longrightarrow \text{must do integral directly} \end{cases}$

4. Can all singularities of $e^{st}F(s)$ be represented as

   a repeated pole ? $\begin{cases} \text{Yes} & \Longrightarrow \text{go to 5} \\ \text{No} & \Longrightarrow \text{must do integral directly} \end{cases}$

5. $f(t) = \displaystyle\sum_{i=1}^{n} \frac{1}{(r_i - 1)!} \frac{d^{r_i-1}}{ds^{r_i-1}} \left[(s-p_i)^{r_i} e^{st} F(s)\right]_{s=p_i}$

Note: Reduces to same answer as complex partial fractions method.

2.8.3 Example of Residues

$$F(s) = \frac{s}{(s-2)^2(s+1)} \qquad r_1 = 2, r_2 = 1, p_1 = 2, p_2 = -1$$

Residue checklist: $e^{st}\frac{s}{(s-2)^2(s+1)}$

1. poles at $s = 2, s = -1$, so $c > 2$ will suffice

2. $\lim\limits_{s\to\infty} e^{st} \dfrac{s}{(s-2)^2(s+1)} = \begin{cases} 0 & t \le 0 \\ \infty & t > 0 \end{cases}$
   $\Longrightarrow f(t) = 0$ for negative values of t

3. $\lim\limits_{s\to-\infty} e^{st} \dfrac{s}{(s-2)^2(s+1)} = 0 \quad \text{for } t > 0$

4. All poles simple

5. $f(\tau) = \dfrac{1}{(r_1-1)!} \dfrac{d}{ds} \left[\dfrac{s}{s+1} e^{st}\right]_{s=p_1} + \dfrac{1}{(r_2-1)!} \left[\dfrac{s}{(s-2)^2} e_2^{st}\right]_{s=p_2}$
   $= \left[t\dfrac{s}{s+1}e^{st} + \dfrac{1}{s+1}e^{st} - \dfrac{s}{(s+1)^2}e^{st}\right]_{s=2} + \dfrac{1}{1!}\left[\dfrac{se^{st}}{(s-2)^2}\right]_{s=-1}$
   $= \dfrac{2}{3}te^{2t} + \left(\dfrac{1}{3} - \dfrac{2}{9}\right)e^{2t} + \dfrac{-1}{1}e^{-1}$

$$f(t) = \frac{2}{3}te^{2t} + \frac{1}{9}e^{2t} - e^{-t}/9$$

Compare with Partial Fractions method

$$\begin{aligned} x(t) &= \frac{d}{ds}\left[\frac{s}{s+1}\right]_{s=2} e^{2t} + \left[\frac{s}{s+1}\right]_{s=2} te^{2t} + \left[\frac{s}{(s-2)^2}\right]_{s=-1} e^{-1} \\ &= \left(\frac{1}{3} - \frac{2}{9}\right) e^{2t} + \frac{2}{3}te^{2t} - e^{-t} \end{aligned}$$

2.9 Convolution and Duhamel's Integrals

2.9.1 Convolution Integral

$F(s) = G(s)H(s)$

$$\mathcal{L}^{-1}[G(s)] = g(t), \quad \mathcal{L}^{-1}[H(s)] = h(t)$$

$$\begin{aligned} \mathcal{L}^{-1}[F(s)] &= f(t) = \int_0^t g(\tau)h(t-\tau)d\tau \\ &= \int_0^t g(t-\tau)h(\tau)d\tau \end{aligned}$$

Note $g(t) = h(t) = 0$ for negative arguments.

2.9.2 Duhamel's Integral

If $h_N(t)$ is impulse response of a system, $g(t)$ = step response, and if $f(t)$ is any forcing function, then the response of $x(t)$ with $x(0) = 0, \dot{x}(0) = 0$ is

$$x(t) = \int_0^t f(\tau)h_N(t-\tau)d\tau = f(0)g(t) + \int_0^t \dot{f}(\tau)g(t-\tau)d\tau$$

if $x(-\infty) = 0$ and $\dot{x}(-\infty) = 0$, then we may write Duhamel's integral

$$x(t) = \int_{-\infty}^t f(\tau)h_N(t-\tau)d\tau$$

If nonzero initial conditions are present, use $e(t)$, $h(t)$ solution.

$$x(t) = x(0)e(t) + \dot{x}(0)h_V(t) + \int_0^t f(\tau)h_N(t-\tau)d\tau$$

2.10 Fourier Transform

2.10.1 Fourier Integral

As discussed earlier, a Fourier Series representation for a periodic function $f(t)$ with period T can be written as

$$f(t) = \sum_{p=-\infty}^{\infty} \alpha_p e^{ip\Omega t}$$

where

$$\alpha_p = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)e^{-ip\Omega t}dt$$

The idea of the Fourier Integral, which leads to the Fourier Transform, is let $T \to \infty$ and $\Omega \to 0$ so that any function of t can be expressed, not just periodic ones.

If $\Omega \to 0$, then $p\Omega$ (i.e. the frequency of the p^{th} harmonic) becomes a continuous variable $u = p\Omega$ with $\Delta u = u_{p+1} - u_p = \Omega$.

Now, $T \to \infty \;\Rightarrow\; \alpha_p \to 0$ while $T\alpha_p$ remains finite. Thus,

$$T\alpha_p = F(p\Omega) = F(u) = \int_{-\infty}^{+\infty} f(t)e^{-iut}dt$$

and

$$f(t) = \sum_{p=-\infty}^{+\infty} \frac{1}{T}(T\alpha_p)e^{ip\Omega t} = \frac{\Omega}{2\pi}\sum_{p=-\infty}^{+\infty} F(u)e^{iut}$$

Now, with u a continuous frequency variable,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(u)e^{iut}du$$

$$F(-u) = F^*(u), \quad f(t) = \frac{1}{\pi}\int_{0}^{+\infty} \Re\left[F(u)e^{iut}\right]du$$

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty}\left\{\Re[F(u)]\cos ut - \Im[F(u)]\sin ut\right\}du$$

Thus, one sees that the Fourier Integral leads to a reciprocal relationship between $f(t)$ and $F(u)$, the latter being the Fourier Transform of the former.

2.10.2 Response using Fourier Transform

Consider now the standard spring-mass-damper system, subjected to an arbitrary force $f(t)$ and thus governed by

$$\ddot{x} + 2\zeta\dot{x} + x = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(u)e^{iut}du$$

Assuming

$$x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(u) e^{iut} du$$

one finds that

$$\dot{x} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} iuX(u) e^{iut} du$$

and

$$\ddot{x} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} -u^2 X(u) e^{iut} du$$

Thus, the governing equation becomes

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[X(u)(1 - u^2 + 2\zeta iu) - F(u) \right] e^{iut} du = 0$$

so that

$$X(u) = \frac{F(u)}{1 - u^2 + 2\zeta iu} = F(u)H(u)$$

Denoting the Fourier Transform of $f(t)$ as $\mathcal{F}[f(t)] = F(u)$

$$\begin{aligned} \mathcal{F}[e^{ait}] =& 2\pi\delta(u - a) \\ \mathcal{F}[e^{-ait}] =& 2\pi\delta(u + a) \end{aligned}$$

so that

$$\begin{aligned} \mathcal{F}[\cos at] =& \pi[\delta(u + a) + \delta(u - a)] \\ \mathcal{F}[\sin at] =& \pi i[-\delta(u - a) + \delta(u + a)] \end{aligned}$$

Also, $\mathcal{F}[\delta(t)] = 1$ and

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(u) e^{iut} du, \quad H(u) = \int_{-\infty}^{+\infty} h(t) e^{-iut} dt$$

2.10.3 Integral Tables Useful in Fourier Transforms

$$\int_0^{\infty} \frac{\beta \cos(ax)}{x^2 + \beta^2} dx = \frac{\pi}{2} e^{-|a\beta|} \mathrm{sgn}(\beta)$$

$$\int_0^{\infty} \frac{x \sin(ax)}{x^2 + \beta^2} dx = \frac{\pi}{2} e^{-|a\beta|} \mathrm{sgn}(a)$$

$$\int_0^{\infty} \frac{\beta \cos(ax)}{\beta^2 - x^2} dx = \frac{\pi}{2} \sin(a\beta) \mathrm{sgn}(a)$$

$$\int_0^{\infty} \frac{x \sin(ax)}{\beta^2 - x^2} dx = -\frac{\pi}{2} \cos(a\beta) \mathrm{sgn}(a)$$

$$\int_0^\infty \frac{\beta\gamma\cos(ax)}{(x^2+\beta^2)(x^2+\gamma^2)}dx = \frac{\pi}{2}\left[\frac{|\beta|e^{-|a\gamma|}-|\gamma|e^{-|a\beta|}}{\beta^2-\gamma^2}\right]\operatorname{sgn}(\beta\gamma)$$

$$\int_0^\infty \frac{x\sin(ax)}{(x^2+\beta^2)(x^2+\gamma^2)}dx = \frac{\pi}{2}\left[\frac{e^{-|a\beta|}-e^{-|a\gamma|}}{\gamma^2-\beta^2}\right]\operatorname{sgn}(a)$$

$$\int_0^\infty \frac{x^2\cos(ax)}{(x^2+\beta^2)(x^2+\gamma^2)}dx = \frac{\pi}{2}\left[\frac{|\beta|e^{-|a\beta|}-|\gamma|e^{-|a\gamma|}}{\beta^2-\gamma^2}\right]$$

$$\int_0^\infty \frac{\beta x^2\cos(ax)}{(x^2+\beta^2)^2}dx = \frac{\pi}{2}\left[e^{-|a\beta|}\right]\operatorname{sgn}(\beta)$$

$$\int_0^\infty \frac{x^3\sin(ax)}{(x^2+\beta^2)(x^2+\gamma^2)}dx = \frac{\pi}{2}\left[\frac{\beta^2e^{-|a\beta|}-\gamma^2e^{-|a\gamma|}}{\beta^2-\gamma^2}\right]\operatorname{sgn}(a)$$

$$\int_0^\infty \frac{\beta^3\cos(ax)}{(x^2+\beta^2)^2}dx = \frac{\pi}{4}(1+|a\beta|)e^{-|a\beta|}\operatorname{sgn}(\beta)$$

$$\int_0^\infty \frac{x\sin(ax)}{(x^2+\beta^2)^2}dx = \frac{\pi}{4}\left|\frac{a}{\beta}\right|e^{-|a\beta|}\operatorname{sgn}(a)$$

$$\int_0^\infty \frac{(1-x^2)\cos(ax)}{(1+x^2)^2}dx = \frac{\pi}{2}|a|e^{-|a|}$$

$$\int_0^\infty \frac{1}{(x^2+\beta)^2+c^2}dx = \frac{\pi}{2c}\frac{B}{\sqrt{\beta^2+c^2}}, \quad B=\frac{1}{2}\left[\sqrt{\beta^2+c^2}-\beta\right]^{1/2}$$

$$\int_0^\infty \frac{x^2+\beta}{(x^2+\beta)^2+c^2}dx = \frac{\pi}{2}\frac{A}{\sqrt{\beta^2+c^2}}, \quad A=\frac{1}{2}\left[\sqrt{\beta^2+c^2}+\beta\right]^{1/2}$$

2.10.4 Relationship Between Laplace and Fourier Transforms

We can derive the Laplace transform from the Fourier transform (and vice versa). First, consider the Fourier transform of a function $f(t)$ which is zero for $t<0$.

$$F(iu)=\int_0^\infty f(t)e^{-iut}dt, \quad f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(iu)e^{iut}du$$

Now, suppose $f(t)$ is such that $\int_0^{+\infty}|f(t)|dt=\infty$. Since no Fourier transform exists for $f(t)$, we try a Fourier transform of a modified function

$$\overline{f}=e^{-ct}f(t)$$

If we can find a c such that

$$\int_0^\infty |e^{-ct}f(t)|dt \text{ is bounded}$$

then we can take a Fourier transform of $\overline{f}(t)$. For example, this would work for $f = t^n$, $f = e^{at}$; but would not work for $f = e^{at^2}$. The transform of $\overline{f}$ will now depend on the original function f, the frequency parameter u, and the constant c that was chosen to limit $\overline{f}(t)$.

$$F(iu, c) = \int_0^\infty f(t)e^{-(c+iu)t}dt$$

We recognize $c + iu$ as a complex variable, s. Thus, our modified transform depends on s; and the Laplace transform of $f(t)$ becomes

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

We now turn to the inverse transform. Since c is held constant, $ds = idu$, and we can write the inverse in terms of ds.

$$\overline{f}(t) = e^{-ct}f(t) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} F(s)e^{iut}ds/i$$

or

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds$$

This is the inverse Laplace transform introduced earlier where c must be chosen large enough so that $e^{-ct}f(t)$ is bounded in the integral (c to the right of all poles).

Thus, for functions that decay on their own with $f(t) = 0$ for $t < 0$, Fourier and Laplace are the same with $s = i\omega$.

2.11 Review of Methods Thus Far

Method	$f(t)$ limitations
Inspection	0 or constant
*Harmonic Balance (Real or Complex)	Simple harmonic, transients decayed
*Fourier Series (Real or Complex)	Periodic, transients decayed
Laplace Transform	$f(t) = 0, t < 0, \int_0^\infty -e^{-st}f(t)dt$ bounded
*Fourier Transform	$\int_{-\infty}^{+\infty} f(t)dt$ bounded, all “transients” decayed. [Includes Random]
Duhamel’s Integral	zero initial conditions or transients decayed ($\Longleftrightarrow f(t) = 0, t < 0$) [Includes Random]

Note all limitations such as “transients decayed” or “zero initial conditions” can be eliminated by superimposing inspection solutions.

*$\lim \zeta \to 0 \quad \zeta > 0$

2.12 Random Functions

1. Let us say several investigators took measurement samples of $f(t)$.

   Ensemble average: pick t_1, average over all samples.

   Time average: pick a sample, average over all t.

   For a "stationary" process, ensemble average is independent of t.

   For an "ergodic" process, time average is independent of sample.

2. Therefore, time average and ensemble average for a "stationary, ergodic" function have same probability distribution $P(x)$ and probability density $p(x)$.

$$\begin{aligned} P(x) &= \text{probability}, x_s < x \\ & p(x) = \frac{dP}{dx} \\ \text{Probability} &= P(x_2) - P(x_1) = \int_{x_1}^{x_2} p(x)dx \\ \text{that} &\quad x_1 < x < x_2 \end{aligned}$$

3. The relevant statistics of a random function

$$\text{average} = \mu = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \text{mean}$$

$$R_x(\tau) \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)d\tau = \text{correlation}$$

$$R_x(-\tau) = R_x(\tau), \quad |R_x(\tau)| \leq R_x(0)$$

4. Relation between probability and statistics:

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} p(x)dx \\ \mu &= \int_{-\infty}^{+\infty} xp(x)dx \end{aligned}$$

$$\psi^2 = \int_{-\infty}^{+\infty} x^2 p(x)dx = \text{mean square value} = R(0)$$

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 p(x)dx = \psi^2 - \mu^2 = \text{variance}$$

5. For a "Gaussian" or "normal process"

$$\begin{aligned} p(x) &= \tfrac{1}{\sqrt{2\pi}\sigma} \exp\left[-\tfrac{1}{2}\left(\tfrac{x-u}{\sigma}\right)^2\right], \qquad t = \tfrac{x-\mu}{\sqrt{2}\sigma} \\ P &= [\mathrm{erf}(t)+1]/2 \end{aligned}$$

6. Power Spectral Density

$$S_x(\omega) \equiv \int_{-\infty}^{+\infty} R_x(\tau)e^{i\omega\tau}d\tau = \text{Fourier Transform of Autocorrelation (Real)}$$

$$\Longrightarrow R_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_x(\omega)e^{i\omega\tau}d\omega$$

$$\int_{-\infty}^{+\infty} S_x(\omega)d\omega = 2\pi R_x(0) = 2\pi\psi^2$$

$$S_x(\omega) = S_x(-\omega), \quad S_x(\omega) \geq 0$$

$$S(\omega) = 2\int_0^\infty R(\tau)\cos\omega\tau d\tau, \quad R(\tau) = \frac{1}{\pi}\int_0^\infty S(\omega)\cos\omega\tau d\omega$$

$$R(0) = \psi^2 = \frac{1}{\pi}\int_0^\infty S(\omega)d\omega, \quad S(\omega) \text{ is power at each frequency}$$

7. Relationships between x and $\dot{x}$

$$R_x(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t+\tau)dt$$

$$\begin{aligned} \frac{d^2R_x}{d\tau^2} &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)\frac{d^2}{d\tau^2}x(t+\tau)dt \\ &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)\frac{d^2}{dt^2}x(t+\tau)dt \\ &= -\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} \frac{d}{dt}x(t)\frac{d}{dt}x(t+\tau)dt \end{aligned}$$

Therefore,

$$\ddot{R}_x(\tau) = R_{\dot{x}}(\tau)$$

$$-\ddot{R}_x(0) = R_{\dot{x}}(0) = \sigma_{\dot{x}}^2 \quad (\mu = 0)$$

$$R_{\dot{x}}(\tau) = -\ddot{R}_x(\tau) = -\frac{d^2}{d\tau^2}\left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} S(\omega)e^{i\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\omega^2 S(\omega)e^{i\omega\tau}d\omega$$

Therefore, $S_{\dot{x}}(\omega) = \omega^2 S_x(\omega)$

8. Number of threshold crossings (Gaussian process only)

   (to be proved later)

   Number of positive crossings of threshold $\overline{x}$ per unit time (average)

$$E[N^+(\overline{x})] = \frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_x}e^{-\frac{1}{2}(\overline{x}/\sigma_x)^2},\ \text{if Gaussian } \mu = 0.$$

9. Distribution of peaks

   Zero Mean ($\mu = 0$), Gaussian, Positive Peaks only locate peaks, record number of peaks above certain values of x

   $P(x)$ = probability that a given peak will be smaller than a given value x.

   Percentage number of points at $x_1 = p(x_1)$

   Percentage number of points at $x_2 = p(x_2)$

   Difference must be peaks

$$\text{Number of Peaks} = [p_x(x_1) - p_x(x_2)] = -\frac{dp_x}{dx}dx$$

$$\text{Probability peak} < x = \frac{\text{total peaks} < x}{\text{total peaks}} = P(x)_{peaks}$$

$$P(x)_{peaks} = \frac{\int_0^x -p'dx}{\int_0^\infty -p'dx} = \frac{p(0) - p(x)}{p(0) - p(\infty)} = 1 - e^{-x^2/2\sigma^2}$$

$$p(x)_{peaks} = \text{probability density} = \frac{dP}{dx} = \frac{x}{\sigma^2}e^{-x^2/2\sigma^2}$$

   (Rayleigh Distribution)

2.12.1 Examples

Sinusoid with phase random from sample to sample

Here we let $x(t) = a\sin(\omega t + \phi)$ where ϕ random from sample to sample:

$$p(x_s < x) \quad = 1 - \frac{\lambda}{2\pi} = 1 - \frac{1}{2\pi}\left[\pi - 2\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$P(x) = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left(\frac{x}{a}\right)$$

$$p(x) = \frac{dP}{dx} = \frac{1}{\pi a}\frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}$$

$$\mu = \frac{1}{2\pi}\int_{-\pi}^{+\pi} a\sin(\theta)d\theta = 0, \quad \theta = \omega t + \phi$$

$$\begin{aligned} R(\tau) &= \frac{a^2}{2\pi}\int_{-\pi}^{+\pi} \sin(\theta)\sin(\theta+\omega\tau)d\theta \\ &= \frac{a^2}{2\pi}\cos\omega\tau\int_{-\pi}^{+\pi}\sin^2\theta d\theta + \frac{a^2}{2\pi}\sin\omega\tau\int_{-\pi}^{+\pi}\sin\theta\cos\theta d\theta \end{aligned}$$

$$R(\tau) = \frac{a^2}{2}\cos\omega\tau, \quad R(0) = \frac{a^2}{2} = \psi^2$$

$$S(\omega) = \frac{a^2}{2}\mathcal{F}(\cos\omega\tau) = \frac{\pi}{2}a^2[\delta(\mu - a) + \delta(\mu + a)]$$

All power in one frequency!

This is a non-Gaussian process, but we can define probability of threshold crossings and peaks.

$$E[N + (\overline{x})](\text{per unit time}) = \begin{cases} \frac{\omega}{2\pi} & -a < \overline{x} < +a \\ 0 & |\overline{x}| \geq a \end{cases}$$

$$p \text{ peaks} = \qquad p(x) = \delta(a)$$

White noise

$$S(\omega) = S_0 = \text{constant (infinite power)}$$

$$R(\tau) = S_0\delta(\tau)$$

Band limited white noise

$$R(\tau) = \frac{1}{\pi}\int_0^{\omega_c} S_0\cos\omega\tau d\omega = \frac{S_0}{\pi\tau}\sin(\omega_c\tau)$$

2.12.2 Two Random Functions

1. Probability

   probability $x_1 < x < x_2$ and $y_1 < y < y_2$ simultaneously

$$+P(x_1, y_1) + P(x_2, y_2) - P(x_1, y_2) - P(x_2, y_1) = \int_{x_1}^{x_2}\int_{y_1}^{y_2}[p(x,y)dydx$$

$$p = \frac{\partial^2 P}{\partial x \partial y}, \text{ Independent (not synonymous)}^* \Longrightarrow p(x,y) = p(x)p(y)$$

2. Statistics (Cross-correlation)

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)y(t+\tau)dt$$

$$\Longrightarrow R_{yx}(\tau) = R_{xy}(-\tau), \quad |R_{xy}(0)| \le \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$$

3. Relation between probability and statistics

$$\text{Covariance} = C_{xy}(\tau) = R_{xy}(\tau) - \mu_x\mu_y$$

$$\begin{gathered} C_{xx}(0) = R_{xx}(0) - \mu_x^2 = \psi_x^2 - \mu_x^2 = \sigma_x^2(\text{variance}) \\ C_{yy}(0) = R_{yy}(0) - \mu_y^2 = \psi_y^2 - \mu_y^2 = \sigma_y^2 \\ C_{xy}(0) = R_{xy}(0) - \mu^x\mu_y \equiv \rho_{xy}\sigma_x\sigma_y \\ |\rho_{xy}| \le 1 \end{gathered}$$

$\rho =$ correlation coefficient, $\rho = 0 \Longrightarrow$ uncorrelated (not synonymous)*

$$R_{x\dot{x}} = \frac{d}{d\tau}R_{xx}(\tau) \Longrightarrow R_{x\dot{x}}(0) = 0 \begin{cases} \Longrightarrow & \text{uncorrelated if} \\ \Longrightarrow & \text{not so if} \end{cases}$$

4. Gaussian system

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y}{\sigma_x\sigma_y}\right]\right\}$$

For Gaussian uncorrelated $\Longleftrightarrow$ independent

For non-Gaussian independent $\Longrightarrow$ uncorrelated

But *not* vice versa

5. Cross-spectral density

$$\begin{aligned} S_{xy}(\omega) &= \int_{-\infty}^{+\infty} R_{xy}(\tau)e^{-i\omega\tau}d\tau \\ R_{xy}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{xy}(\omega)e^{+i\omega\tau}d\omega \\ S_{xy}(-\omega) &= S_{yx}(\omega)S_{xy}^*(\omega) \end{aligned}$$

S_{xx}, S_{yy} are real.

6. Superposition

$$\begin{aligned} R_{ax+by} &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}[ax(t)+by(t)][ax(t+\tau)+by(t+\tau)dt \\ &= a^2R_{xx}+b^2R_{yy}+abR_{xy}+abR_{yx}=\langle ab\rangle\begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy}\end{bmatrix}\begin{Bmatrix} a \\ b\end{Bmatrix}\end{aligned}$$

$$S_{ax+by}=\langle ab\rangle\begin{bmatrix} S_{xx} & X_{xy} \\ S_{yx} & S_{yy}\end{bmatrix}\begin{Bmatrix} a \\ b\end{Bmatrix}$$
$$S_{xy}=S^*_{yx}\Longrightarrow S_{ax+by}\text{ Real}$$

7. We are now ready for proof of the Threshold-Crossing Theorem

At a given time t, there is a magnitude x and velocity $\dot{x}$. There will be a positive crossing of threshold value $\overline{x}$ in Δt if:

a. $x<\overline{x}$ b. $\dot{x}\Delta t>\overline{x}-x$ c. $\dot{x}>0$

Therefore, the expected number of crossings (positive) of $\overline{x}$ within the time Δt is

$$E[N^+(\overline{x})]\Delta t=\int_{\dot{x}=0}^{\dot{x}=\infty}\int_{x=\overline{x}-\dot{x}\Delta t}^{x=\overline{x}}p(x,\dot{x})dxd\dot{x}$$
$$\lim_{\Delta t\to 0}\int_{\overline{x}-\dot{x}\Delta t}^{\overline{x}}p(x,\dot{x})dx=p(\overline{x},\dot{x})\dot{x}\Delta t$$
$$\Longrightarrow E[N^+(\overline{x})]=\int_0^\infty \dot{x}p(\overline{x},\dot{x})d\dot{x}$$

If x and $\dot{x}$ are independent $\Longrightarrow$ $p(x,\dot{x})=p(x)p(\dot{x})$

$$E[N^+(\overline{x})]=p(\overline{x})\int_0^\infty \dot{x}p(\dot{x})d\dot{x}=p(\overline{x})\cdot[\text{average of positive }\dot{x}\text{'s}]$$

For Gaussian process (independent $\Longleftrightarrow$ uncorrelated)

$$p(\overline{x})=\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_x}e^{-\frac{1}{2}\overline{x}^2/\sigma_x^2}$$
$$p(\dot{x})=p(v)=\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_v}e^{-\frac{1}{2}v^2/\sigma_v^2}$$

$$\begin{aligned} E[N^+(\overline{x})] &= p(\overline{x})\int_0^\infty\frac{v}{\sqrt{2\pi}\sigma_v}e^{-\frac{1}{2}v^2/\sigma_v^2}dv \\ &= p(\overline{x})=\left|\left[-\frac{\sigma_v}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2/\sigma_v^2}\right]\right|_0^\infty=\frac{p(\overline{x})\sigma_v}{\sqrt{2\pi}}\end{aligned}$$
$$E[N^+(\overline{x})]=\frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_x}e^{-\frac{1}{2}(\overline{x}/\sigma_x)^2}$$

2.12.3 Response to Random Excitation

Correlation Function

$$\ddot{x} + 2\zeta\dot{x} + x = f(t) = \text{random}$$

By convolution

$$x(t) = \int_{-\infty}^{t} f(\lambda)h(t-\lambda)d\lambda = \int_{-\infty}^{+\infty} f(\lambda)h(t-\lambda)d\lambda$$

or

$$x(t) = \int_{-\infty}^{+\infty} h(\lambda)f(t-\lambda)d\lambda = \int_{0}^{\infty} h(\lambda)f(t-\lambda)d\lambda$$

$$x(t+\tau) = \int_{0}^{\infty} h(\lambda)f(t+\tau-\lambda)d\lambda$$

$$R_x(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \left[\int_0^\infty h(\lambda_1)f(t-\lambda_1)d\lambda_1\right]\left[\int_0^\infty h(\lambda_2)f(t+\tau-\lambda_2)d\lambda_2\right]dt$$

$$\Longrightarrow R_x(\tau) = \int_0^\infty\int_0^\infty h(\lambda_1)h(\lambda_2)R_f(\tau+\lambda_1-\lambda_2)d\lambda_1 d\lambda_2$$

Power Spectral Density

$$\begin{aligned} S_x(\omega) &= \int_{-\infty}^{+\infty} R(\tau)e^{-i\omega\tau}d\tau = \int_0^\infty\int_0^\infty h(\lambda_1)h(\lambda_2)\left[\int_{-\infty}^{+\infty} R_f(\tau+\lambda_1-\lambda_2)e^{-i\omega\tau}d\tau\right]d\lambda_1 d\lambda_2 \\ &= \int_0^\infty\int_0^\infty h(\lambda_1)h(\lambda_2)e^{-i\omega\lambda_2}e^{i\omega\lambda_1}S_f(\omega)d\lambda_1 d\lambda_2 \qquad \overline{\tau} = \tau+\lambda_1-\lambda_2 \end{aligned}$$

$$S_x(\omega) = H(\omega)H^*(\omega)S_f(\omega) = |H(\omega)|^2 S_f(\omega)$$

Example

$$\ddot{x} + 2\zeta\dot{x} + x = f(t)\ , \quad f(t) = \text{Gaussian, White noise}$$

$$H(\omega) = \frac{1}{1-\omega^2+2\zeta i\omega}\ , \quad h(t) = \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta t}\sin\sqrt{1-\zeta^2 t}$$

$$|H(\omega)|^2 = \frac{1}{(1-\omega^2)^2+(2\zeta\omega)^2}$$

$$S_x(\omega) = \frac{1}{(1-\omega^2)^2+(2\zeta\omega)^2}$$

$$R_x(\tau) = \int_0^\infty\int_0^\infty \frac{1}{1-\zeta^2}e^{-\zeta\lambda_1=\zeta\lambda_2}\sin\sqrt{1-\zeta^2}\lambda_1\sin\sqrt{1-\zeta^2}\lambda_2\delta(\tau+\lambda_1-\lambda_2)d\lambda_2 d\lambda_1$$

area under $\delta = 1$ when $\lambda_2 = \tau + \lambda_1$

$$R_x(\tau) = \int_0^\infty \frac{1}{1-\zeta^2} e^{-\zeta(\tau+\lambda_1)} e^{-\zeta\lambda_1} \sin\sqrt{1-\zeta^2}(\tau+\lambda_1) \sin\sqrt{1-\zeta^2}\lambda_1 d\lambda_1$$

$$\lambda_2 > 0 \Longrightarrow \tau + \lambda_1 > 0 \Longrightarrow \tau > 0$$

$$R_x(\tau) = \frac{1}{4\zeta} e^{-\zeta\tau} \left[\cos\sqrt{1-\zeta^2}\tau + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\tau \right] \ , \ \tau > 0$$

$$R_x(-\tau) = R_x(+\tau) \Longrightarrow |\tau| \text{ in formula}$$

$$\sigma_x^2 = R_x(0) = \frac{1}{4\zeta} \ , \ \sigma_{\dot{x}}^2 = -\ddot{R}_x(0) = \frac{1}{4\zeta}$$

$$E[N^+(x_m)] = \frac{1}{2\pi} e^{-\frac{1}{2}(x_m^2 4\zeta)} = \frac{1}{2\pi} e^{-2\zeta x_m^2}$$

$$\text{Zero crossings} = \left(\frac{1}{2\pi}\right) / \sec \ \text{ independent of } \zeta$$

$$x = 1 \text{ crossings, } \zeta = 1$$

$$\frac{1}{2\pi} e^{-2} = 0.02/\sec \qquad (.0215)$$

$$\eta = 1 \quad \text{probability} \quad |x| > 1 \left[1 - \text{erf}\sqrt{2_2}\right]/2 = .023$$

$$\eta = 1 \quad \text{probability peak} \quad e^{-2} = 0.135$$

$$> 1$$

$$\dot{R}_x(\tau) = \frac{1}{4\eta} e^{-h\tau} \left[\frac{1}{\sqrt{1-\eta^2}} \sin\sqrt{1-\eta^2}\tau \right]$$

Chapter 3

Multi-Degree-of-Freedom Systems

3.1 Equations of Motion

3.1.1 Newton-Euler method

A. Definitions

Number of variables (i.e. displacement and rotation measures) = M
Number of spring units, damper units and sliding friction interfaces = P
Number of constraints = L
Number of unknowns = $N + P + 2L \; = \; M + L + P$
Number of degrees of freedom = $N = M - L$ (the number of coordinates required to uniquely define the configuration of the system)

1. Draw free-body diagrams for all particles having unknown displacements and rigid bodies having unknown mass-center displacements and rotations, and for springs and dampers.

2. Give names and define positive direction for all these unknown displacements and rotations, x_n, $n = 1, 2, \ldots, M$.

3. Give names and define positive directions for all spring, damper and sliding friction forces F_i, $i = 1, 2, \ldots, P$.

4. Give names and define positive directions for all constraint forces G_j, $j = 1, 2, \ldots, L$.

B. Equations

1. Write Newton's and Euler's laws for each variable from the free-body diagrams (M equations).

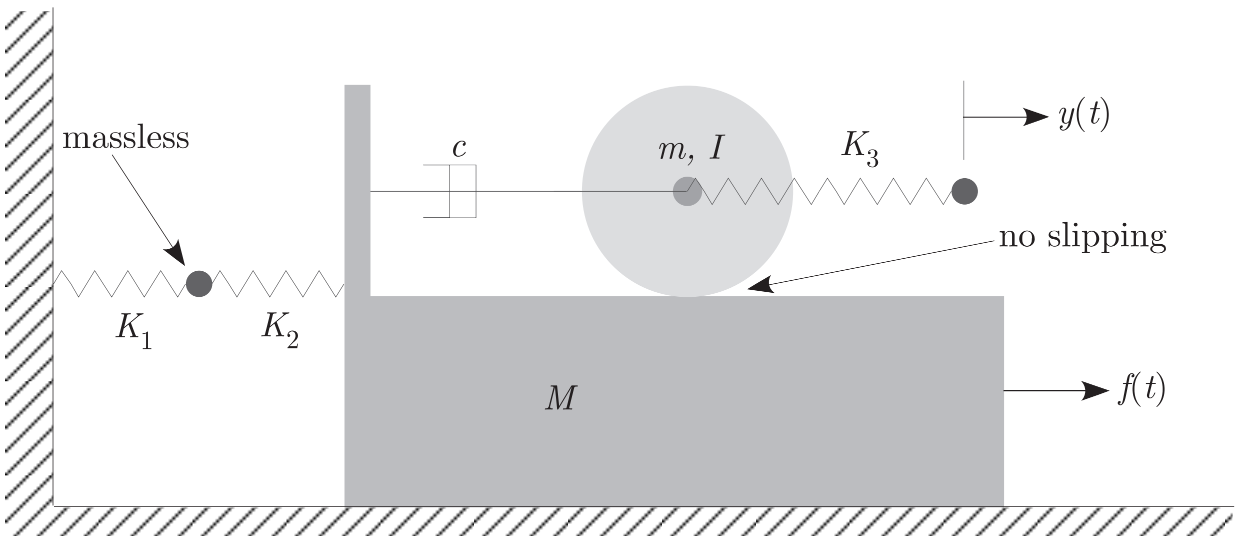

Figure 3.1: Schematic of example problem

2. Write constraint equations for each geometric constraint between variables (L equations).

3. Write spring, damper or friction law for each mechanical element (P equations):

$$\Longrightarrow \quad M+L+P \text{ equations in } M+L+P \text{ unknowns}$$

C. Simplification of Equations

1. Eliminate constraint forces from the Newton-Euler equations by simple algebra:

$$(\Longrightarrow \quad M+P=N+P+L \text{ equations in } M+P=N+P+L \text{ unknowns })$$

2. Eliminate extra variables from the Newton-Euler equations by use of constraint equations.

$$(\Longrightarrow \quad N+P \text{ equations in } N+P \text{ unknowns })$$

3. Eliminate spring, damper and sliding friction forces by use of spring, damper and friction laws:

$$(\Longrightarrow \quad N \text{ equations in } N \text{ unknowns })$$

4. If any of the equations are algebraic in terms of one of the degrees of freedom or its time derivative (say x or $\dot{x}$), then that quantity may be eliminated from the equations.

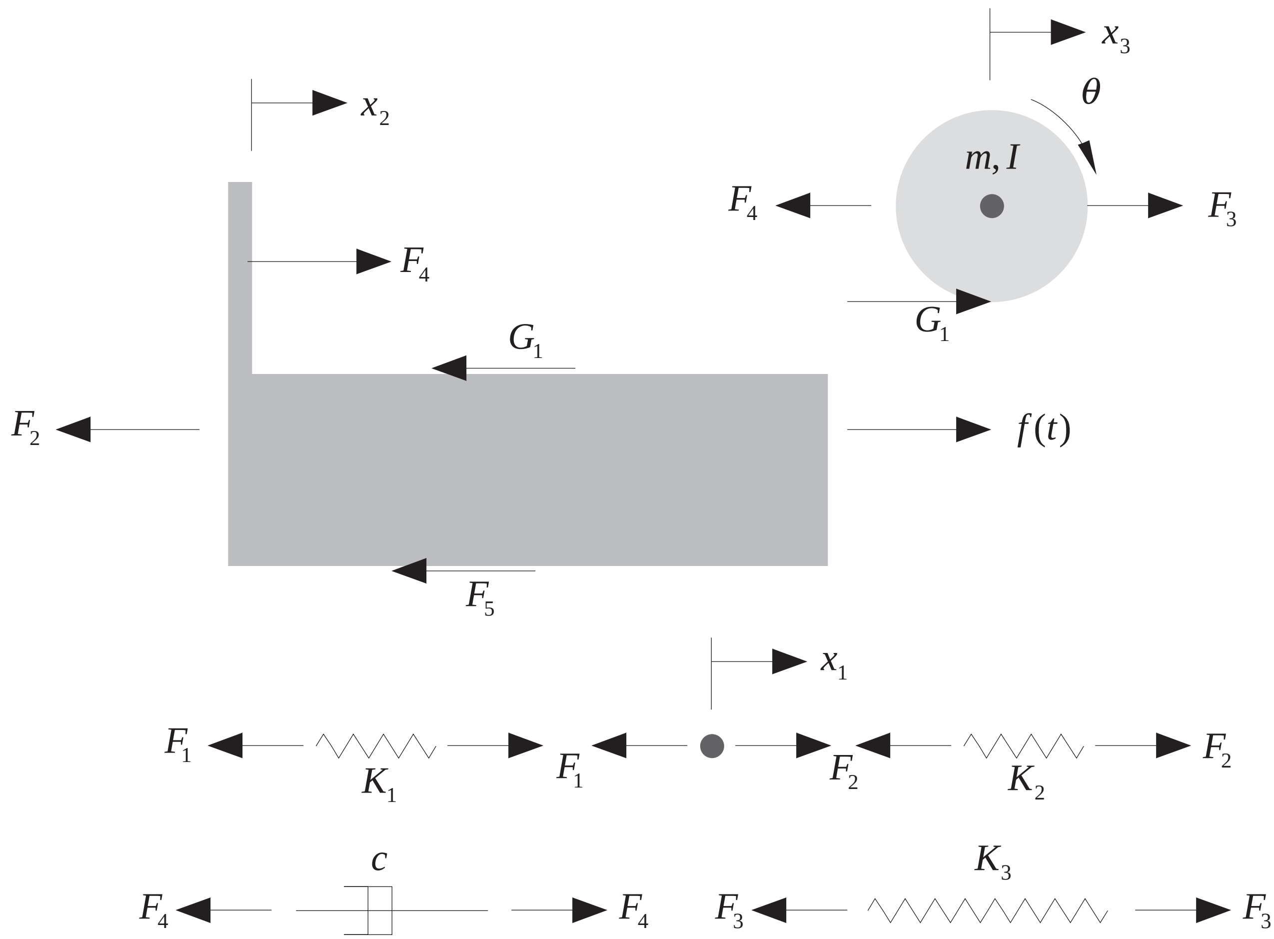

Figure 3.2: Free-body diagrams

Example (see Figs. 3.1 and 3.2)

A1 – A4

$$\Longrightarrow \quad M = 4$$

$$\Longrightarrow \quad P = 5\ , \quad L = 1$$

$$\Longrightarrow \quad \text{3 degrees of freedom}$$

B1

$$x_1 \quad \Longrightarrow \quad 0\ (\ddot{x}_1) = F_2 - F_1$$

$$x_2 \quad \Longrightarrow \quad M\ddot{x}_2 = f + F_4 - F_2 - F_5 - G_1$$

$$x_3 \quad \Longrightarrow \quad m\ddot{x}_3 = F_3 - F_4 + G_1$$

$$\theta \quad \Longrightarrow \quad I\ddot{\theta} = -G_1 r$$

B2

$$x_3 - x_2 = r\theta \quad \text{(no slipping)}$$

B3

$$F_1 = K_1 x_1$$

$$F_2 = K_2(x_2 - x_1)$$

$$F_3 = K_3(y - x_3)$$

$$F_4 = c(\dot{x}_3 - \dot{x}_2)$$

$$F_5 = \mu\ (M + m)\ g\ \mathrm{sgn}(\dot{x}_2)$$

C1

$$0 = F_2 - F_1$$

$$M\ddot{x}_2 = f + F_4 - F_2 - F_5 + \frac{I}{r}\ddot{\theta}$$

$$m\ddot{x}_3 = F_3 - F_4 - \frac{I}{r}\ddot{\theta}$$

C2

$$0 = F_2 - F_1$$

$$M\ddot{x}_2 = f + F_4 - F_2 - F_5 + \frac{I}{r^2}(\ddot{x}_3 - \ddot{x}_2)$$

$$m\ddot{x}_3 = F_3 - F_4 - \frac{I}{r^2}(\ddot{x}_3 - \ddot{x}_2)$$

C3

$$0 = -(K_1 + K_2)x_1 + K_2 x_2$$

$$M\ddot{x}_2 = f + c(\dot{x}_3 - \dot{x}_2) - K_2(x_2 - x_1) - \mu\,(M + m)\,g\,\mathrm{sgn}(\dot{x}_2) + \frac{I}{r^2}(\ddot{x}_3 - \ddot{x}_2)$$

$$m\ddot{x}_3 = K_3(y - x_3) - c(\dot{x}_3 - \dot{x}_2) - \frac{I}{r^2}(\ddot{x}_3 - \ddot{x}_2)$$

C4

$$x_1 = \frac{K_2}{K_1 + K_2} x_2$$

$$M\ddot{x}_2 = f + c(\dot{x}_3 - \dot{x}_2) - K_2 x_2\left(1 - \frac{K_2}{K_1 + K_2}\right) - \mu\,(M + m)\,g\,\mathrm{sgn}(\dot{x}_2) + \frac{I}{r^2}(\ddot{x}_3 - \ddot{x}_2)$$

$$m\ddot{x}_3 = K_3(y - x_3) - c(\dot{x}_3 - \dot{x}_2) - \frac{I}{r^2}(\ddot{x}_3 - \ddot{x}_2)$$

Standard Form

$$\left(M + \frac{I}{r^2}\right)\ddot{x}_2 - \frac{I}{r^2}\ddot{x}_3 + c\dot{x}_2 - c\dot{x}_3 + \frac{K_1 K_2}{K_1 + K_2} x_2 = f(t) - \mu\,(M + m)\,g\,\mathrm{sgn}(\dot{x}_2)$$

$$\left(m + \frac{I}{r^2}\right)\ddot{x}_3 - \frac{I}{r^2}\ddot{x}_2 + c\dot{x}_3 - c\dot{x}_2 + K_3 x_3 = K_3 y$$

Matrix Form

$$\begin{bmatrix} M + \frac{I}{r^2} & -\frac{I}{r^2} \\ -\frac{I}{r^2} & m + \frac{I}{r^2} \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \begin{Bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix}$$

$$+ \begin{bmatrix} \frac{K_1 K_2}{K_1 + K_2} & 0 \\ 0 & K_3 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} f(t) - \mu\,(M + m)\,g\,\mathrm{sgn}(\dot{x}_2) \\ k_3 y(t) \end{Bmatrix}$$

3.1.2 D'Alembert's Principle

Newton's law $\Longrightarrow \sum F = m\ddot{x}$, $\quad \ddot{x}$ must be inertial acceleration

$$\sum M = I\ddot{\theta}$$

θ must be about principal axis through c.g. or about pivot point (pinned).

To get D'Alembert's Principle, replace acceleration of every component with imaginary force $= m\ddot{x}$ in opposite direction of positive x. Replace every angular acceleration with imaginary moment $I\ddot{\theta}$ (about c.g. or pivot) in opposite direction to θ. Then, $\sum F + F_{\text{imag}} = 0$ and $\sum M + M_{\text{imag}} = 0$ (about any point). Finally, apply Newton's method.

Example

$$\sum F = 0\ , \implies M(\ddot{x}_1 + l\ddot{\theta}) + \mu l(\ddot{x}_1 + \frac{l}{2}\ddot{\theta}) + Kx_1 + K(x_1 + a\theta) = 0$$

$$\sum M = 0\ , \implies M(\ddot{x}_1 + l\ddot{\theta})\frac{l}{2} + F_2(a - \frac{l}{2}) + \frac{1}{12}\mu l^3\ddot{\theta} - F_1\frac{l}{2} = 0$$

$$\begin{bmatrix} M + \mu l & Ml + \frac{\mu l^2}{2} \\ \frac{Ml}{2} & M\frac{l^2}{2} + \frac{1}{12}\mu l^3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 2K & Ka \\ Ka - K\frac{l}{1} & Ka^2 - K\frac{al}{2} \end{bmatrix} \begin{Bmatrix} x_1 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The above matrix is not symmetric, to make symmetric, first let θ changed to $l\theta$ (dimensional)

$$\begin{bmatrix} M + \mu l & M + \frac{\mu l}{2} \\ \frac{M}{2} & \frac{M}{2} + \frac{1}{12}\mu l \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ l\ddot{\theta} \end{Bmatrix} + K \begin{bmatrix} 2 & \frac{a}{l} \\ \frac{a}{l} - 1 & \frac{a^2}{l^2} - \frac{1}{2}\frac{a}{l} \end{bmatrix} \begin{Bmatrix} x_1 \\ l\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\frac{1}{2}$ of the first equation and then added to the second equation

$$\begin{bmatrix} M + \mu l & M + \frac{\mu l}{2} \\ M + \frac{\mu l}{2} & M + \frac{1}{3}\mu l \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ l\ddot{\theta} \end{Bmatrix} + K \begin{bmatrix} 2 & \frac{a}{l} \\ \frac{a}{l} & \frac{a^2}{l^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ l\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or, alternately, just add $\frac{l}{2} \times$ the first equation to the second equation

$$\begin{bmatrix} M + \mu l & Ml + \frac{\mu l^2}{2} \\ Ml + \frac{\mu l^2}{2} & Ml^2 + \frac{1}{3}\mu l^3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{Bmatrix} + K \begin{bmatrix} 2 & a \\ a & a^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

3.1.3 Virtual Work

1. Use D'Alembert's principle.

2. Move each degree of freedom a small amount δx_1 and calculate the work done δW_1.

3. Set $\delta W_i = 0\ ,\ i = 1, N$ to get equations.

Example Let $x_2 = x_3 = \theta = 0$ at same instant in time. We have four variables: x_1, x_2, x_3, θ

2 Constraints $\begin{cases} \sin\theta = \frac{x_3}{l} \\ \cos\theta = 1 - \frac{x_2}{l} \end{cases}$

$\Longrightarrow$ 2 degrees of freedom x_1 and θ

$$\frac{x_3}{l} = \sqrt{\frac{x_2}{l}\left(2 - \frac{x_2}{l}\right)}$$

$$\begin{aligned} \dot{x}_3 &= l\dot{\theta}\cos\theta \\ \dot{x}_2 &= l\dot{\theta}\sin\theta \\ \ddot{x}_3 &= -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta \\ \ddot{x}_2 &= l\dot{\theta}^2\cos\theta + l\ddot{\theta}\sin\theta \end{aligned}$$

a. Hold θ, move tiny x_1

$$\begin{aligned} \delta W_1 &= \delta x_1[K(x_2 - x_1) - Kx_1 - m\ddot{x}_1] = 0 \\ &\Longrightarrow m\ddot{x}_1 + 2Kx_1 - Kx_2 = 0 \end{aligned}$$

b. Hold x_1, move $\delta\theta$

$$\delta W_\theta = \delta x_2[-m\ddot{x}_2 - K(x_2 - x_1)] + \delta x_3[-Kx_3 + mg - m\ddot{x}_3]$$

$$\delta x_2 = l\sin\theta\delta\theta \ , \quad \delta x_3 = l\cos\theta\delta\theta$$

$$\delta W_\theta = -\delta\theta[m\ddot{x}_2 l\sin\theta + K(x_2 - x_1)l\sin\theta + Kx_3 l\cos\theta + m\ddot{x}_3 l\cos\theta - mgl\cos\theta]$$

$$ml^2\ddot{\theta} + K[\sin\theta - \sin\theta\cos\theta]l^2 - Kx_1 l\sin\theta + Kl\sin\theta\cos\theta x_1 = mgl\cos\theta$$

$$\Longrightarrow \begin{cases} m\ddot{x}_1 + 2Kx_1 = Kl(1 - \cos\theta) \\ ml^2\ddot{\theta} - Kx_1 l\sin\theta + Kl^2\sin\theta = mgl\cos\theta \end{cases}$$

3.1.4 Lagrange's Equations

1. Write down system kinetic energy, T.
2. Write down potential energy of every force that you know has a potential, V.
3. Write down the Rayleigh dissipation function R for all viscous dampers; R is just like V for a spring except we replace x by $\dot{x}$ and K by c.
4. Write down virtual work for all forces that are neither included in V and R nor are constraint forces. We do *not* include centripetal or Coriolis forces, as these are included automatically.

$$\delta W = F_1\delta x_1 + F_2\delta x_2 + \ldots F_n\delta x_n$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} - \frac{\partial T}{\partial x_i} + \frac{\partial R}{\partial \dot{x}_i} + \frac{\partial V}{\partial x_i} = F_i \ , \quad i = 1, n$$

The x_i must be independent degrees of freedom.

Examples

1. Let's go back to Newton example, noting that $F_5 = \mu(m+M)g\,\mathrm{sgn}(\dot{x}_2)$

$$T = \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 + \frac{1}{2}I\left(\frac{\dot{x}_3-\dot{x}_2}{r}\right)^2$$
$$V = \frac{1}{2}K_1x_1^2 + \frac{1}{2}K_2(x_1-x_2)^2 + \frac{1}{2}K_3(y-x_3)^2$$
$$\frac{\partial V}{\partial x_1} = K_1x_1 + K_2(x_1-x_2) \quad R = \frac{1}{2}c(\dot{x}_3-\dot{x}_2)^2 \quad \frac{\partial R}{\partial \dot{x}_1} = 0 \quad \frac{\partial R}{\partial \dot{x}_2} = c(\dot{x}_2-\dot{x}_3)$$
$$\frac{\partial V}{\partial x_2} = K_2(x_2-x_1) \quad \delta W = f(t)\delta x_2 - \mu(m+M)g\,\mathrm{sgn}(\dot{x}_2)\delta x_2 \quad \frac{\partial R}{\partial \dot{x}_3} = c(\dot{x}_3-\dot{x}_2)$$
$$\frac{\partial V}{\partial x_3} = K_3(x_3-y) \quad \frac{\partial T}{\partial \dot{x}_1} = 0\ , \quad \frac{\partial T}{\partial \dot{x}_2} = M\dot{x}_2 + \frac{I}{r^2}(\dot{x}_2-\dot{x}_3)$$
$$\frac{\partial T}{\partial \dot{x}_3} = m\dot{x}_3 + \frac{I}{r^2}(\dot{x}_3-\dot{x}_2)\ , \quad \frac{\partial T}{\partial \dot{x}_i} = 0\ , \quad i = 1,3$$
$$K_1x_1 + K_2(x_1-x_2) = 0$$
$$M\ddot{x}_2 + \frac{I}{r^2}(\ddot{x}_2-\ddot{x}_3) + K_2(x_2-x_1) + c(\dot{x}_2-\dot{x}_3) = f(t) - \mu(m+M)g\,\mathrm{sgn}(\dot{x}_2)$$
$$m\ddot{x}_3 + \frac{I}{r^2}(\ddot{x}_3-\ddot{x}_2) + K_3(x_3-y) + c(\dot{x}_3-\dot{x}_2) = 0$$

We note two things here. First, a potential energy may not be a function of a time derivative of any generalized coordinate. Second, if V is an explicit function of time, then conservation of mechanical energy does not apply.

2.

$$T = \frac{1}{2}m(\dot{x}+l\dot{\theta})^2 + \frac{1}{2}\mu\ell(\dot{x}+\frac{l}{2}\dot{\theta})^2 + \frac{1}{2}\left(\frac{1}{12}\mu l^3\right)(\dot{\theta})^2$$
$$V = \frac{1}{2}Kx^2 + \frac{1}{2}K(x+a\theta)^2$$
$$R = 0,\ \delta W = 0$$
$$\frac{\partial T}{\partial \dot{x}} = m(\dot{x}+l\dot{\theta}) + \mu l(\dot{x}+\frac{l}{2}\dot{\theta})$$
$$\frac{\partial T}{\partial \dot{\theta}} = ml(\dot{x}+l\dot{\theta}) + \frac{\mu l^2}{2}(\dot{x}+\frac{l}{2}\dot{\theta}) + \frac{1}{12}\mu l^3\dot{\theta}$$
$$(m+\mu l)\ddot{x} + (ml+\frac{\mu l^2}{2})\ddot{\theta} + 2Kx_1 + Ka\theta = 0$$
$$ml^2 + \frac{1}{3} = \mu l^3)\ddot{\theta} + (ml+\frac{\mu l^2}{2})\dot{x} + K(x+a\theta)^{a'} = 0$$

same as D'Alembert's

3.

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_2^2 + \dot{x}_3^2)$$
$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m(\cos^2\theta + \sin^2\theta)\dot{\theta}^2 l^2$$
$$V = \frac{1}{2}Kx_1^2 + \frac{1}{2}K(x_1 - l + l\cos\theta)^2 - mgl\sin\theta + \frac{1}{2}K(l\sin\theta)^2$$
$$R = 0, \quad \delta W = 0$$
$$m\ddot{x}_1 + Kx_1 + K(x_1 - l + l\cos\theta) = 0$$
$$ml^2\ddot{\theta} + Kl(x_1 - l + l\cos\theta)(-\sin\theta) - mgl\cos\theta + Kl^2\sin\theta\cos\theta = 0$$

same as Virtual Work

General Lagrange's Equation for Linear System

In matrix form:

$$T = \frac{1}{2}\langle\dot{x}\rangle[M]\{\dot{x}\} + \langle\dot{x}\rangle[Q]\{x\} + \frac{1}{2}\langle x\rangle[P]\{x\}$$
$$R = \frac{1}{2}\langle\dot{x}\rangle[C]\{\dot{x}\} + \langle\dot{x}\rangle[N]\{x\}$$
$$V = \frac{1}{2}\langle x\rangle[K]\{x\} + \langle x\rangle[G]$$
$$\delta W = \langle\delta x\rangle\{F\}$$
$$M = M^T\ , \quad P = P^T\ , \quad C = C^T\ , \quad K = K^T$$

In index notation

$$
\begin{gathered}
T = \frac{1}{2} M_{ij} \dot{x}_i \dot{x}_j + Q_{ij} \dot{x}_i x_j + \frac{1}{2} P_{ij} x_i x_j \\
R = \frac{1}{2} C_{ij} \dot{x}_i \dot{x}_j + N_{ij} \dot{x}_i x_j \\
V = \frac{1}{2} K_{ij} x_i x_j + x_i G_i \\
\delta W = \delta x_i F_i \\
\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_i} = \frac{d}{dt} \left(\frac{1}{2} M_{ij} \dot{x}_j + \frac{1}{2} M_{ji} \dot{x}_j + Q_{ij} x_j \right) \quad M_{ij} = M_{ji} \\
\frac{\partial T}{\partial x_i} = Q_{ji} \dot{x}_j + \frac{1}{2} P_{ij} x_j + \frac{1}{2} P_{ji} x_j \quad P_{ij} = P_{ji} \\
\frac{\partial R}{\partial \dot{x}_i} = \frac{1}{2} C_{ij} \dot{x}_j + \frac{1}{2} C_{ji} \dot{x}_j + N_{ij} x_j \quad C_{ij} = C_{ji} \\
\frac{\partial V}{\partial x_i} = \frac{1}{2} K_{ij} x_j + \frac{1}{2} K_{ji} x_j + G_i \quad K_{ij} = K_{ji} \\
M_{ij} \ddot{x}_j + (Q_{ij} - Q_{ji}) \dot{x}_j - P_{ij} x_j + C_{ij} \dot{x}_j + N_{ij} x_j + K_{ij} x_j = F_i - G_i \\
[M]\{\ddot{x}\} + [C + Q - Q^T]\{\dot{x}\} + [K + N - P]\{x\} = \{F\} - \{G\}
\end{gathered}
$$

Special Case Consider a systems consisting only of springs, dampers, and masses (i.e. no inertias, pulleys, levers, etc.). Then, Lagrange's equations take the particularly simple form

$$
[M] = \begin{bmatrix} \ddots & & \\ & m_i & \\ & & \ddots \end{bmatrix} \quad m_i = \text{mass at } x_i
$$

$$
[K] = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \quad K_{ij} = \text{spring constant between } x_i \text{ and } x_j
$$

Example – see Fig. 3.3

$$
\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 + K_5 + K_6 & -K_2 & -K_6 \\ -K_2 & K_2 + K_3 & -K_3 \\ -K_6 & -K_3 & K_3 + K_4 + K_6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}
$$

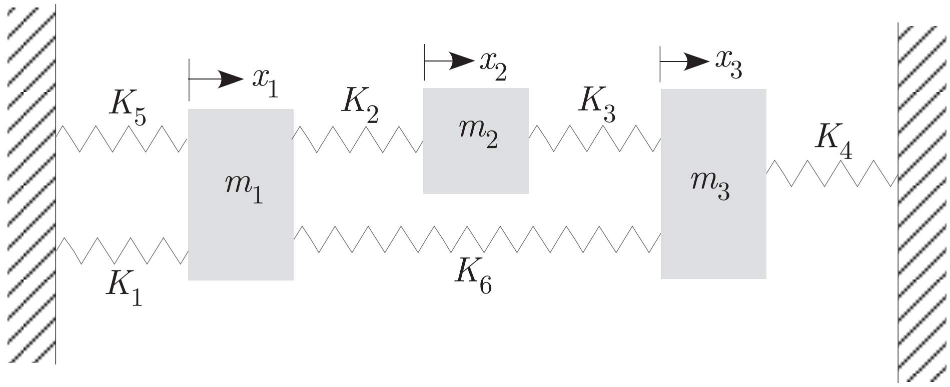

Figure 3.3: Schematic for system of particles connected by springs

Example with non-symmetric matrix

$$T = \frac{1}{2}m(\dot{y} + \epsilon\dot{\theta})^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

$$V = \frac{1}{2}(K_y y^2 + K_\theta \theta^2)$$

$$\delta W = -\left(\frac{\rho A v^2}{2}C_L\right)\delta y = -\frac{\rho A v^2}{2}(2\pi)\left[\theta + \frac{\dot{y}}{v}\right]\delta y$$

$$m\begin{bmatrix} 1 & \epsilon \\ \epsilon & r^2+\epsilon^2 \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} K_y & 0 \\ 0 & K_\theta \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = -\begin{Bmatrix} \theta + (\dot{y}/v) \\ 0 \end{Bmatrix}\frac{\rho A v^2}{2}(2\pi)$$

$$m\begin{bmatrix} 1 & \epsilon \\ \epsilon & r^2+\epsilon^2 \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \pi\rho A v & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} K_y & \pi\rho A v^2 \\ 0 & K_\theta \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

non-symmetric stiffness

aerodynamic damping

Stiffness and Flexibility Matrix

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F\}$$

K_{ij} = Force at points i $(i = 1, N)$ necessary for
$x_j = 1$ all other x's $= 0$
$[K]^{-1} = [G] =$ flexibility matrix

$$G_{ij} = \text{displacement at points } i \ (i = 1, N) \text{ due to unit load at point } j$$

$$\{x\} = [G]\{F\} - [D]\{\ddot{x}\}$$

G is measurable and K is hard to measure.

3.2 Setting up the Eigen-analysis

3.2.1 General Case

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\}$$

First we will look at transients, $f = 0$, and the following special cases:

a. $M = M^T$, $K = K^T$, $C = 0$: conservative, undamped

b. $M \neq M^T$ or $K \neq K^T$, $C = 0$: nonconservative, undamped

c. $M = M^T$, $K = K^T$, $C = C^T$: viscous damping

d. $M = M^T$, $K = K^T$, $C = \alpha M + \beta K$: proportional viscous damping

e. $M = M^T$, $K = K^T$, $C = -C^T$: conservative, gyroscopic

f. $M = M^T$, $K = K^T$, general C: conservative mass and stiffness

These special cases come up quite often in pratice and are *much* easier to handle than the general case.

3.2.2 Matrix Notation

$$a_{ij} \implies i^{\text{th}} \text{ row, } j^{\text{th}} \text{ column of } [a]_{m\times n} \quad i = 1, \ldots, m \ ; \ j = 1, \ldots, n$$

$$[a] + [b] = [c] \quad \longrightarrow \quad a_{ij} + b_{ij} = c_{ij}$$

$$[a]_{m\times n}[b]_{n\times p} = [c]_{m\times p} \quad \implies c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$[a]^T = [b] \quad \implies \quad b_{ij} = a_{ji} \qquad [[a][b]]^T = [b]^T[a]^T$$

$$[I] \quad \implies \quad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$[0] \quad \implies \quad a_{ij} = 0$$

Inverse

$$[a]^{-1}[a] = [a][a]^{-1} = [I] \qquad [[a][b]]^{-1} = [b]^{-1}[a]^{-1}$$

Eigenvalues

$$[a]\{\phi\}_i = \lambda_i\{\phi\}_i \qquad i = 1, \ldots, n$$

$$\implies [a][\phi] = [\phi][{}^{\backslash}\lambda_{\backslash}]$$

Note that each column of $[\phi]$ is an eigenvector. We can now write $[a]$ as

$$[a] = [\phi][{}^{\backslash}\lambda_{\backslash}][\phi]^{-1}$$

Alternatively,

$$[\phi]^{-1}[a][\phi] = [{}^{\backslash}\lambda_{\backslash}] \tag{3.1}$$

Taking the transpose, one finds

$$[\phi]^T[a]^T][\phi]^{-T} = [{}^{\backslash}\lambda_{\backslash}]$$

Thus, $[a]^T$ has the same eigenvalues as $[a]$. Their eigenvectors are not the same, however. Denoting $[\psi]$ as the eigenvectors of $[a]^T$, we obtain

$$[a]^T\{\psi\}_i = \lambda_i\{\psi\}_i \qquad i = 1, \ldots, n$$

or

$$[\psi]^{-1}[a]^T[\psi] = [{}^{\backslash}\lambda_{\backslash}] \tag{3.2}$$

Comparing Eqs. (3.1) and (3.2), one sees that

$$[\psi]^{-1} = [\phi]^T \text{ or } [\phi]^{-1} = [\psi]^T$$

As a side note, it is now possible to define any function of a matrix $[a]$:

$$f\left([a]\right) = [\phi][{}^{\backslash}f(\lambda)_{\backslash}][\phi]^{-1}$$

Now, we consider a symmetric matrix $[a] = [a]^T$. Then $\{\psi\}_i$ can only differ by a constant from $\{\phi\}_i$. Let this constant be α_i for the i^{th} eigenvector (i.e. the i^{th} column of $[\phi]$). Thus,

$$[\psi] = [\phi][{}^{\backslash}\alpha_{\backslash}]$$

so that

$$[\phi]^{-1} = [\psi]^T = [{}^{\backslash}\alpha_{\backslash}][\phi]^T$$

and

$$[\phi]^{-1}[\phi] = [I] = [{}^{\backslash}\alpha_{\backslash}][\phi]^T[\phi]$$

Thus, it is clear that the original eigenvectors $[\phi]$ can always be normalized so that

$$[\overline{\phi}] = [\phi][\lceil \alpha^{\frac{1}{2}} \rfloor]$$

yielding

$$[\overline{\phi}]^T[\overline{\phi}] = [I]$$

or, in other words, $[\overline{\phi}]^{-1} = [\overline{\phi}]^T$. Finally, with the normalized eigenvectors we obtain for symmetric $[a]$

$$[\overline{\phi}]^T[a][\overline{\phi}] = [\lceil \lambda \rfloor]$$

3.2.3 Conservative, Undamped

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$[M] = [M]^T \qquad [K] = [K]^T$$

Look for solution of the form $\{x\} = \{\phi\}e^{i\omega t}$. Thus,

$$\big[[K] - \omega^2[M]\big]\,\{\phi\} = 0$$

Solution approaches

a. $|[K] - \omega^2[M]| = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots\ a_1\lambda + a_0 = 0$ with $\lambda = \omega^2$

Given solutions for ω, all but any one of the scalar equations represented by the matrix equation $[K] - \omega^2[M] = 0$ will yield $[\phi]$.

b. If $[M]$ nonsingular (no reducible degree of freedom)

$$\{\ddot{x}\} + [M]^{-1}[K]\{x\} = \{0\}$$

$$\big[[D] - \omega^2[I]\big]\,\{\phi\} = 0 \qquad D = M^{-1}K \qquad \lambda = \omega^2$$

c. If $[K]$ nonsingular (structure supported – i.e. no rigid-body motion)

$$[K]^{-1}[M]\{\ddot{x}\} + \{x\} = \{0\}$$

$$\left[[F] - \frac{1}{\omega^2}[I]\right]\{\phi\} = \{0\} \qquad F = K^{-1}M \qquad \lambda = \frac{1}{\omega^2}$$

d. If $[M] = [I]$

$$\big[[K] - \omega^2[I]\big]\,\{\phi\} = 0 \implies [\phi]^T[K][\phi] = [\lceil \omega^2 \rfloor]$$

e. If $[M]$ positive definite (all positive eigenvalues)

1. Find $[M]^{-1/2}$:

   (a) Find ϕ_M, λ_M, $\phi_M^T[M]\phi_M = \lambda_M$

   (b) $[M]^{-1/2} = [\phi_M][\,\grave{}\,\frac{1}{\sqrt{\lambda_m}}\,\grave{}\,][\phi_M]^T$

   Note: $[M]^{-1/2}[M][M]^{-1/2} = [I]$

2. Make change of variable $\{x\} = [M]^{-1/2}\{y\}$

$$[M][M]^{-1/2}\{\ddot{y}\} + [K][M]^{-1/2}\{y\} = \{0\}$$

3. Premultiply by $[M]^{-1/2}$

$$\{\ddot{y}\} + \underbrace{[M]^{-1/2}[K][M]^{-1/2}}_{\text{symmetric } \overline{K}}\{y\} = \{0\} \qquad \text{(same frequencies)}$$

4. Find $[\phi_{\overline{K}}]$, $\lambda_{\overline{K}}$

$$[\phi_{\overline{K}}^T][\overline{K}][\phi_{\overline{K}}] = [\,\grave{}\,\lambda_{\overline{K}}\,\grave{}\,] \qquad \omega^2 = \lambda_{\overline{K}} \qquad [\phi] = [M]^{-1/2}[\phi_{\overline{K}}]$$

Note:

$$[\phi]^T[M][\phi] = [\phi_{\overline{K}}]^T[M]^{-1/2}[M][M]^{-1/2}[\phi_{\overline{K}}] = [I]$$

$$[\phi]^T[K][\phi] = [\phi_{\overline{K}}]^T[M]^{-1/2}[K][M]^{-1/2}[\phi_{\overline{K}}] = [\phi_{\overline{K}}]^T[\overline{K}][\phi_{\overline{K}}] = [\,\grave{}\,\omega^2\,\grave{}\,]$$

The i^{th} mode can be multiplied by a constant α_i, so we could have $[\overline{\phi}] = [\phi][\,\grave{}\,\alpha\,\grave{}\,]$, and

$$\overline{\phi}^T M\overline{\phi} = [\,\grave{}\,\alpha^2\,\grave{}\,] = [\,\grave{}\,m\,\grave{}\,] \qquad \overline{\phi}^T K\overline{\phi} = [\,\grave{}\,\omega^2\alpha^2\,\grave{}\,] = [\,\grave{}\,k\,\grave{}\,]$$

$$\alpha_i^2 = m_i \qquad \text{generalized masses}$$

$$\omega_i^2\alpha_i^2 = m_i\omega_i^2 = k_i \qquad \text{generalized stiffnesses} \qquad \frac{k_i}{m_i} = \omega_i^2$$

f. Another approach for the case of $[M]$ positive definite (all positive eigenvalues) is to use the Cholesky decomposition

$$[M] = [L][L]^T$$

where $[L]$ is lower-triangular. Recall that

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F\}$$

We write this instead as

$$[L][L]^T\{\ddot{x}\} + [K]\{x\} = \{F\}$$

and make the transformation $\{y\} = [L]^T\{x\}$ or $\{x\} = [L]^{-T}\{y\}$ so that

$$[L]\{\ddot{y}\} + [K][L]^{-T}\{y\} = \{F\}$$

Premultiplying by $[L]^{-1}$ gives

$$\{\ddot{y}\} + [L]^{-1}[K][L]^{-T}\{y\} = [L]^{-1}\{F\}$$

or

$$\{\ddot{y}\} + [\overline{K}]\{y\} = [L]^{-1}\{F\}$$

To distinguish the modal matrices, introduce $[\phi]$ for the original degrees of freedom x and $[\phi_{\overline{K}}]$ for the transformed degrees of freedom y, so that

$$\{y\} = \{\phi_{\overline{K}}\}e^{i\omega t}$$

and

$$\{x\} = \{\phi\}e^{i\omega t}$$

Thus,

$$\begin{aligned}[\phi_{\overline{K}}] &= [L]^T[\phi] \\ [\phi] &= [L]^{-T}[\phi_{\overline{K}}]\end{aligned}$$

the former of which satisfies

$$[\phi_{\overline{K}}]^T[\overline{K}][\phi_{\overline{K}}] = [\lceil \omega^2 \rfloor]$$

and the normalization on $[\phi_{\overline{K}}]$, viz.,

$$[\phi_{\overline{K}}]^T[\phi_{\overline{K}}] = [I]$$

implies also that

$$[\phi_{\overline{K}}]^T[\phi_{\overline{K}}] = [\phi]^T[L][L]^T[\phi] = [\phi]^T[M][\phi] = [\lceil m \rfloor]$$

which we know must be equal to $[I]$! In other words, if we normalize the columns of $[\phi_{\overline{K}}]$ such that $[\phi_{\overline{K}}]^T[\phi_{\overline{K}}] = [I]$, then the generalized masses m_i are unity for all i, and

$$[\phi]^T[M][\phi] = [\lceil m \rfloor] = [I]$$

Note that this normalization also yields the result that $[\phi]^{-1} = [\phi]^T[M]$ and shows that only when the modes are normalized with respect to generalized masses of unity and the original mass matrix is identity does one find that $[\phi]^{-1} = [\phi]^T$.

The original problem is now solved by taking the modal transformation

$$[\phi]\{q\} = \{x\}$$

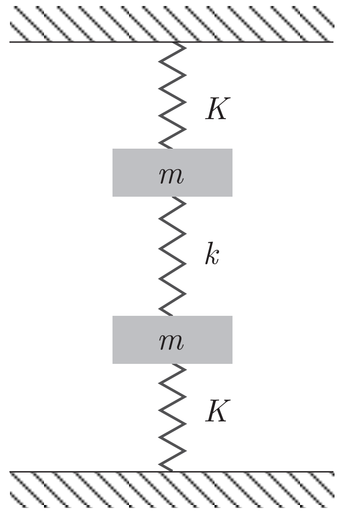

Figure 3.4: Simple system schematic

or

$$\{q\} = [\phi]^T[M]\{x\}$$

so that

$$[\,^{\backslash} m _{\backslash}]\{\ddot{q}\} + [\,^{\backslash} m\omega^2 _{\backslash}]\{q\} = [\phi]^T\{F\}$$

where with normalization such $m_i = 1$ for all i we get instead

$$\{\ddot{q}\} + [\,^{\backslash} \omega^2 _{\backslash}]\{q\} = [\phi]^T\{F\} = \{Q\}$$

This leads to uncoupled scalar equations

$$\ddot{q}_i + \omega_i^2 q_i = \frac{Q_i}{m_i}$$

or

$$\ddot{q}_i + \omega_i^2 q_i = Q_i$$

when $m_i = 1$.

Physical Meaning of Modes Consider a simple system, pictured in Fig. 3.4, the equations of motion are

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{[M]} \left\{ \begin{array}{c} \ddot{x}_1 \\ \ddot{x}_2 \end{array} \right\} + \underbrace{\begin{bmatrix} K+k & -k \\ -k & K+k \end{bmatrix}}_{[K]} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}$$

$$[M]^{-1/2}[K][M]^{-1/2} = \begin{bmatrix} \Omega_1^2 + \Omega_2^2 & -\Omega_2^2 \\ -\Omega_2^2 & \Omega_1^2 + \Omega_2^2 \end{bmatrix} = [\overline{K}]$$

with

$$\Omega_1^2 \equiv \frac{K}{m} \qquad \Omega_2^2 = \frac{k}{m}$$

Note that $\overline{K}$ is here also equal to $L^{-1}KL^{-T}$. Set $|[\overline{K}] - \lambda[I]| = 0$ for eigenvalues

$$\begin{bmatrix} \Omega_1^2 + \Omega_2^2 - \lambda & -\Omega_2^2 \\ -\Omega_2^2 & \Omega_1^2 + \Omega_2^2 - \lambda \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Longrightarrow \quad (\Omega_1^2 + \Omega_2^2 - \lambda)^2 - \Omega_2^4 = 0$$

$$\Omega_1^2 + \Omega_2^2 - \lambda = \pm\Omega_2^2$$

$$\lambda = \lambda_1 = \omega^2 = \Omega_1^2 \quad \text{and} \quad \lambda = \lambda_2 = \Omega_1^2 + 2\Omega_2^2 = \frac{K + 2k}{m}$$

For $\lambda_1 = \Omega_1^2$

$$\begin{bmatrix} \Omega_1^2 + \Omega_2^2 - \lambda_1 & -\Omega_2^2 \\ -\Omega_2^2 & \Omega_1^2 + \Omega_2^2 - \lambda_1 \end{bmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$\begin{bmatrix} \Omega_2^2 & -\Omega_2^2 \\ -\Omega_2^2 & \Omega_2^2 \end{bmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\phi_{11} = 1 \qquad \phi_{21} = 1 \quad \text{(or times any constant)}$$

For $\lambda_2 = \Omega_1^2 + 2\Omega_2^2$

$$\begin{bmatrix} \Omega_1^2 + \Omega_2^2 - \lambda_2 & -\Omega_2^2 \\ -\Omega_2^2 & \Omega_1^2 + \Omega_2^2 - \lambda_2 \end{bmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$\begin{bmatrix} -\Omega_2^2 & -\Omega_2^2 \\ -\Omega_2^2 & -\Omega_2^2 \end{bmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\phi_{12} = 1 \qquad \phi_{22} = -1 \quad \text{(or times any constant)}$$

Thus,

$$[\phi] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

or we normalize so that $[\overline{\phi}]^T[M][\overline{\phi}] = [I]$

$$[\overline{\phi}] = \begin{bmatrix} \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & -\frac{1}{\sqrt{2m}} \end{bmatrix}$$

The first mode has the masses moving together, while the second mode has them moving out of phase (i.e. toward each other or apart from each other).

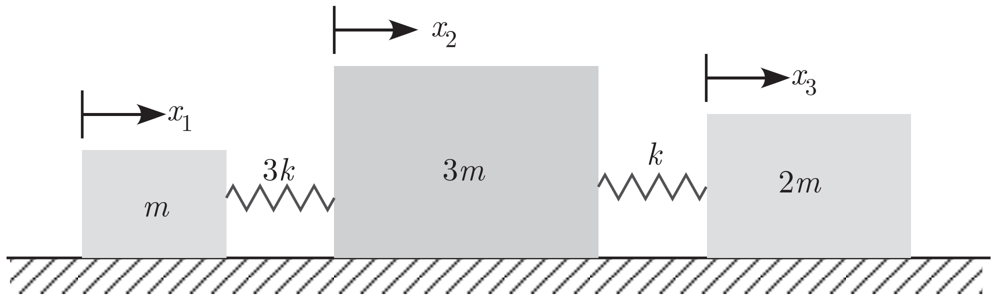

Figure 3.5: System to illustrate removal of rigid-body modes

Rigid-body Modes

If a structure is not restrained, so that it has one or more rigid-body modes, then solution procedure **c** cannot be applied because $[K]$ is singular. However, it is possible to undertake a systematic removal of the rigid-body modes. This procedure is illustrated below. Consider a system depicted in Fig. 3.5 with three masses connected by two springs on a frictionless surface.

The equations of motion are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3k & -3k & 0 \\ -3k & 4k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.3}$$

We can easily answer the question of whether or not a rigid-body mode exists for this system. We could find the eigenvalues and show that one of them is zero. However, this is unnecessary. Recalling that a rigid-body mode has zero frequencies, this means that we only need to see if a rigid-body mode has zero potential energy. To undertake this check, we pre- and post-multiply $[K]$ by a rigid-body mode $\{x_\mathrm{r}\} = \overline{x}\lfloor 1 \;\; 1 \;\; 1 \rfloor^T$, with $\overline{x}$ being arbitrary, so that

$$\overline{x}^2 \{x_\mathrm{r}\}^T [K] \{x_\mathrm{r}\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}^T \begin{bmatrix} 3k & -3k & 0 \\ -3k & 4k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 0 \tag{3.4}$$

Because this is exactly zero, this proves the existence of a rigid-body mode. Another approach is to see if there is at least one null-space mode for the matrix $[K]$. Taking the three rows of $[K]$, we first see that the third row shows $x_2 = x_3$. Substitution of this into the first and second rows gives $x_1 = x_2$ and $x_1 = x_2$, which has the solution $x_1 = x_2 = x_3 = \overline{x}$, an arbitrary constant. This is, of course, a rigid-body mode.

What we would like to know is the nature of all other mode shapes, i.e. the ones that are orthogonal to the rigid-body mode. Denoting these mode shapes by $\{x_\mathrm{e}\}$, then it is clear that

$$\{x_\mathrm{r}\}^T [M] \{x_\mathrm{e}\} = 0 \tag{3.5}$$

Thus,

$$\{x_{\rm r}\}^T[M]\{x\} = \begin{Bmatrix}1\\1\\1\end{Bmatrix}^T \begin{bmatrix}m & 0 & 0\\0 & 3m & 0\\0 & 0 & 2m\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix} = m(x_1+3x_2+2x_3)=0 \tag{3.6}$$

Thus, we may pick

$$x_1 = -3x_2 - 2x_3 \tag{3.7}$$

for example (not the only way!). Therefore, all elastic modes have the property that

$$\{x_{\rm e}\} = \begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix} = \begin{bmatrix}-3 & -2\\1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}x_2\\x_3\end{Bmatrix} = [\chi]\begin{Bmatrix}x_2\\x_3\end{Bmatrix} \tag{3.8}$$

where $[\chi]$ is the so-called constraint matrix. Substituting the right-hand side of Eq. (3.8) for $\{x\}$ in the equations of motion, Eq. (3.3), and premultiplying by $[\chi]$, one obtains

$$[\chi]^T[M][\chi]\{\ddot{\hat{x}}\} + [\chi]^T[K][\chi]\{\hat{x}\} = 0 \tag{3.9}$$

where $\{\hat{x}\} = \lfloor x_2 \quad x_3\rfloor^T$, or for this problem

$$m\begin{bmatrix}12 & 6\\6 & 6\end{bmatrix}\begin{Bmatrix}\ddot{x}_2\\\ddot{x}_3\end{Bmatrix} + k\begin{bmatrix}49 & 23\\23 & 13\end{bmatrix}\begin{Bmatrix}x_2\\x_3\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix} \tag{3.10}$$

This set of equations governs the elastic modes and has a positive definite $[K]$ matrix.

3.2.4 Nonconservative, Undamped

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \qquad [M] \neq [M]^T \qquad [K] \neq [K]^T$$

$$\{x\} = \{\phi\}e^{\eta t}$$

$$\big[\, [M]\eta^2 + [K] \,\big]\{\phi\} = \{0\}$$

$$\eta = \lambda + i\omega \qquad \{\phi\} = \{u\} + i\{v\}$$

Solution methods

Methods a, b, c from above apply. The physical meaning of complex roots shows that motion is of the form $e^{\lambda t}\sin\omega t$ or $(\cos\omega t)$, convergent or divergent sinusoidal oscillations depending on the sign of λ. The physical meaning of complex modes can be explored by normalizing the modes by x_1, effectively setting $x_1 = 1$ and all other degrees of freedom then are in general complex. Consider any other arbitrary degree of freedom of the form, say, $x_2 = a + ib$. Then it can be seen that

$$\frac{|x_2|_{\max}}{|x_1|_{\max}} = \sqrt{a^2+b^2}$$

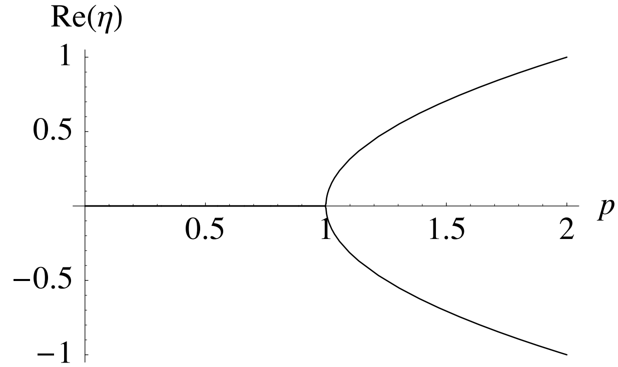

Figure 3.6: Real part of η for inverted pendulum (see Fig. 2.4) subjected to a nondimensional dead force p

and also that x_2 leads x_1 by

$$\tan^{-1}\left(\frac{b}{a}\right)$$

A classification of loads and reactions, when dealing with all mechanical systems, is given by Ziegler (1968). A system is conservative when subjected only to conservative forces. Let us recall the simple mechanical model discussed earlier, i.e. Fig. 2.4. For $\theta_0 = 0$ and θ restricted to be a small angle, the differential equation is

$$ml^2\ddot{\theta} + (k - mgl)\theta = 0 \tag{3.11}$$

Assuming a solution of the form $\theta = \check{\theta}\exp(\overline{\eta}t)$ we find the characteristic equation to be

$$\eta^2 + 1 - p = 0 \tag{3.12}$$

where $\eta^2 = ml^2\overline{\eta}^2/k$ and $p = P/P_{\text{cr}} = mgl/k$. In Figs. 3.6 and 3.7 one finds the real and imaginary parts of η, respectively, versus p, showing that the real parts of both roots become nonzero when the applied force exceeds the critical load. Since the real part for one of the two roots is positive when $p > 1$, the perturbations about the static equilibrium state grow in amplitude. However, it is also interesting to note that for $p \geq 1$ the imaginary part is identically zero. This is characteristic of all conservative systems that lose their stability by buckling: one of the natural frequencies of oscillations about the static equilibrium state becomes zero as the critical load is approached. This is not the case for nonconservatively loaded systems; this difference is one of several that will be seen.

Example of nonconservative system

One example of nonconservative forces is the follower force.[1] A follower force follows the deformations of the body in some manner such that the work done by the force is path-

[1]Here we gratefully acknowledge a correction by Mr. Eliot Quon in Nov. 2009.

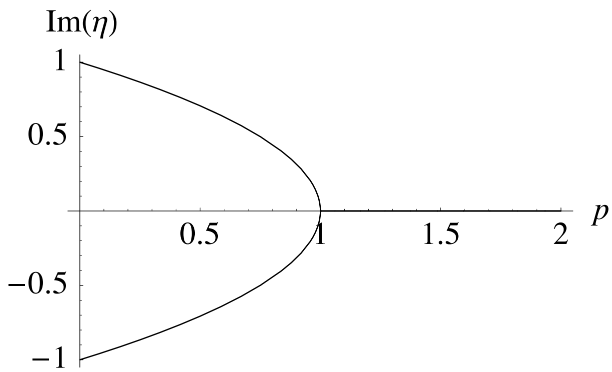

Figure 3.7: Imaginary part of η for inverted pendulum (see Fig. 2.4) subjected to a nondimensional dead force p

dependent. Consider the system in Fig. 3.8. It is easily seen that the applied force P, which follows the orientation of the upper rod, is nonconservative. Let us consider two different sequences of deflection away from the starting point when $q_1 = q_2 = 0$. First, the load is applied when $q_1 = q_2 = 0$, where zero work is done. Then the system moves so that $q_1 = \hat{q}_1$ so that the work done is $P\ell(1-\cos\hat{q}_1) \approx P\ell\hat{q}_1^2/2$. Finally, the system moves so that $q_2 = \hat{q}_2$, for which the work done is zero. So, the total work done to get in this first way from $q_1 = q_2 = 0$ to $q_1 = \hat{q}_1$ and $q_2 = \hat{q}_2$ is approximately $P\ell\hat{q}_1^2/2$. Now, consider a second path in which the load is again applied when $q_1 = q_2 = 0$, where zero work is done. Then, let q_2 move from zero to $q_2 = \hat{q}_2$. In this motion zero work is done. Then let q_1 move from zero to $q_1 = \hat{q}_1$. During this motion, the work done by P is $P\ell(1-\cos\hat{q}_1)\cos\hat{q}_2 - P\ell\sin\hat{q}_1\sin\hat{q}_2 \approx P\ell(\hat{q}_1^2/2 - \hat{q}_1\hat{q}_2)$. So, the total work done to get in this second way from $q_1 = q_2 = 0$ to $q_1 = \hat{q}_1$ and $q_2 = \hat{q}_2$ is approximately $P\ell(\hat{q}_1^2/2 - \hat{q}_1\hat{q}_2)$. Now the second scenario is very similar but simply reversed in order. Additional scenarios with still different answers for the work done are not hard to conceive. Thus, it is quite clear that the follower force in Fig. 3.8 is nonconservative. Another aspect of the properties of such a force is that it does not possess a potential energy function which, when varied, will give the negative of the force's virtual work. To put it another way, the virtual work of the forces cannot be "integrated" to provide the negative of the force's potential energy. Follower forces are typically nonconservative in this sense.

As pointed out by Bolotin (1963, 1964), the study of the stability of structures under follower force systems apparently started with work by Nikolai in the late 1920s. In addition to the books by Bolotin and others, there are also many papers devoted to this subject; see, for example, the work of Leipholz (1978), Celep (1979), Park (1987), Chen and Ku (1992), and Higuchi (1994). Much of the analytical research to date has focused on the stability of beams subjected to various types of follower forces and examination of the effects of various physical phenomena, such as damping and transverse shear deformation.

Analytical examples of solved follower force problems help to clarify the nature of these systems and their analysis. For example, it is now commonly understood that static analysis

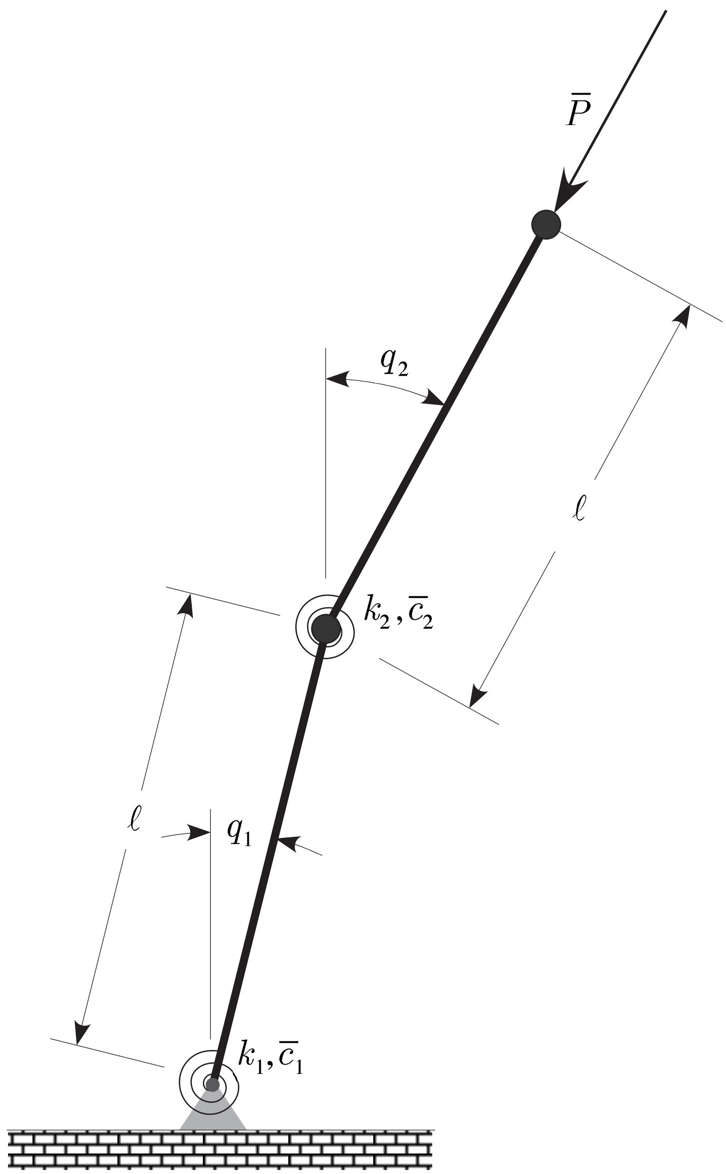

Figure 3.8: Schematic of mechanical model subjected to a follower force

of elastic systems subjected to follower forces may erroneously show that the system is free of instability. In order to ascertain whether a system subjected to follower forces is stable requires a kinetic analysis. For problems that do in fact lose their stability by buckling, the kinetic method will predict that one of the system natural frequencies will tend to zero as the critical load is approached. However, for nonconservative systems one may also find flutter instabilities in addition to possible buckling instabilities. By this we mean that small perturbations about the static equilibrium state oscillate with increasing amplitude.

In Fig. 3.8 a mechanical model of a simple system loaded by a follower force is depicted. The system is comprised of two particles of mass m joined together with massless rigid rods of length ℓ. The rods are joined to each other with a rotational hinge, and one of the rods is also joined to the ground with a rotational hinge. The motion of the system takes place in a plane, and the hinges are spring-restrained and damper restrained with elastic and damping constants equal to k_α and $\overline{c}_\alpha$, respectively, with α=1 and 2. This system is a mechanical model that behaves in a manner similar to Beck's column.

Here we will use Lagrange's equations to derive equations of motion for this system. For small angles q_1 and q_2, the kinetic and potential energies are

$$\begin{aligned} T =& m\ell^2\dot{q}_1^2 + \frac{m\ell^2}{2}\left(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right) \\ V =& \frac{k}{2}q_1^2 + \frac{k}{2}\left(q_2 - q_1\right)^2 \end{aligned} \tag{3.13}$$

The virtual work of the nonconservative applied and damping forces is

$$\overline{\delta W} = -\overline{P}\ell\left(q_2 - q_1\right)\delta q_1 - \left[\left(\overline{c}_1 + \overline{c}_2\right)\dot{q}_1 - \overline{c}_2\dot{q}_2\right]\delta q_1 - \overline{c}_2\left(\dot{q}_2 - \dot{q}_1\right)\delta q_2 \tag{3.14}$$

Thus, the equations of motion are

$$\begin{bmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \overline{c}_1 + \overline{c}_2 & -\overline{c}_2 \\ -\overline{c}_2 & \overline{c}_2 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2k - \overline{P}\ell & \overline{P}\ell - k \\ -k & k \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.15}$$

First, we consider only the static terms in the equation, viz.,

$$\begin{bmatrix} 2k - \overline{P}\ell & \overline{P}\ell - k \\ -k & k \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.16}$$

From this one sees that a non-trivial solution can only exist when $2k^2 - \overline{P}k\ell + \overline{P}k\ell - k^2 = k^2 = 0$, which cannot happen for nonzero k. Thus, no matter how large a force $\overline{P}$ is applied, the mechanism does not exhibit a static buckling instability.

To better treat the dynamic case, we introduce nondimensional variables for time, $\tau = \sqrt{k/(m\ell^2)}t$; force, $P = \overline{P}\ell/k$; and damping parameters, $c_\alpha = \overline{c}_\alpha/\sqrt{km}$. Then one can write the equations of motion more simply as

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} q_1'' \\ q_2'' \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix} + \begin{bmatrix} 2 - P & P - 1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.17}$$

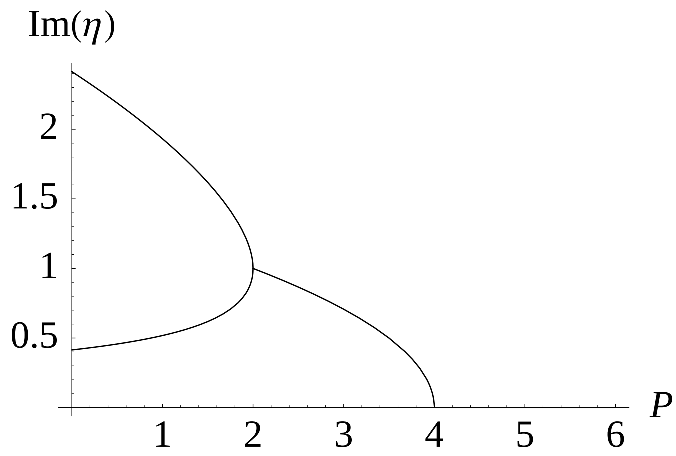

Figure 3.9: Imaginary part of nondimensional eigenvalue η versus P for double mechanical pendulum model without damping

where $(\ )'$ represents the derivative with respect to τ. Letting $q_\alpha = \check{q}_\alpha \exp(\eta\tau)$, we find that a non-trivial solution only exists when

$$\begin{vmatrix} 2\eta^2 + (c_1 + c_2)\eta + 2 - P & \eta^2 - c_2\eta + P - 1 \\ \eta^2 - c_2\eta - 1 & \eta^2 + c_2\eta + 1 \end{vmatrix} \tag{3.18}$$

Ignoring the damping for now, the characteristic equation becomes

$$\eta^4 + 2(3 - P)\eta^2 + 1 = 0 \tag{3.19}$$

Notice that $\eta = 0$ is not a root, so a loss of stability by buckling (i.e., passing from a stable system directly to a buckled one) is not possible. The quartic equation has four roots such that

$$\eta^2 = P - 3 \pm \sqrt{(P - 4)(P - 2)} \tag{3.20}$$

The first sign change of the radicand is at $P = 2$. If $P \leq 2$ the real parts of all roots are zero, as shown in Fig. 3.10; the real part of one root becomes positive when $P > 2$, which means that there is a loss of stability. Since $\Im(\eta) \neq 0$ when $P > 2$, the unstable motion is oscillatory with increasing amplitude. This type of instability is usually referred to as flutter in the mechanics literature and is closely related mathematically to the flutter instability of aeroelasticity.[2] When $P > 4$ all roots are real and there is a strong buckling instability; but the system always first loses stability by flutter.

Let's now look at the modeshapes for the case with the damping set equal to zero. The governing equations are

$$\begin{bmatrix} \eta^2 + 2 - P & \eta^2 - P - 1 \\ \eta^2 - 1 & \eta^2 + 1 \end{bmatrix} \begin{Bmatrix} \check{q}_1 \\ \check{q}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

[2]The physical connection is weak, however, in that unsteady aerodynamics are involved in the aeroelasticity problem.

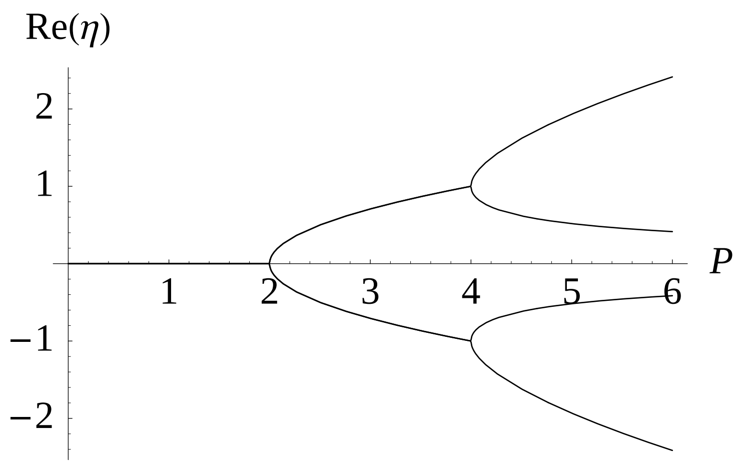

Figure 3.10: Real part of nondimensional eigenvalue η versus P for double mechanical pendulum model without damping

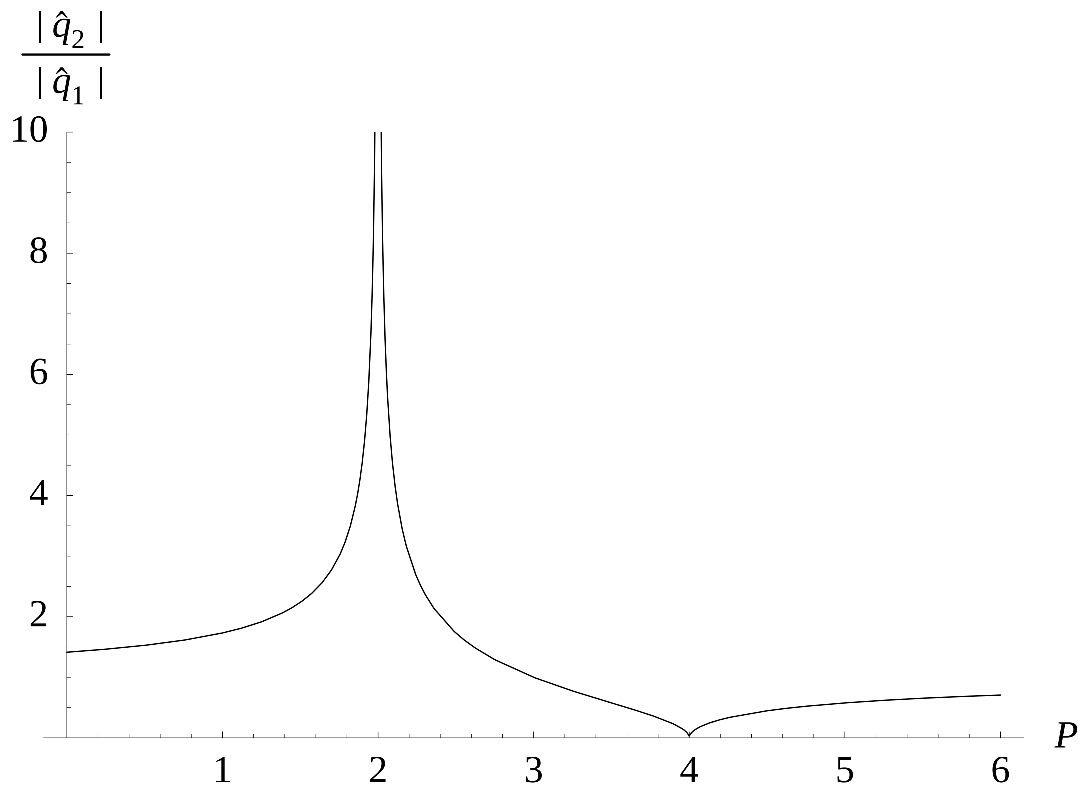

Figure 3.11: Ratio of $\check{q}_2$ to $\check{q}_1$ versus P

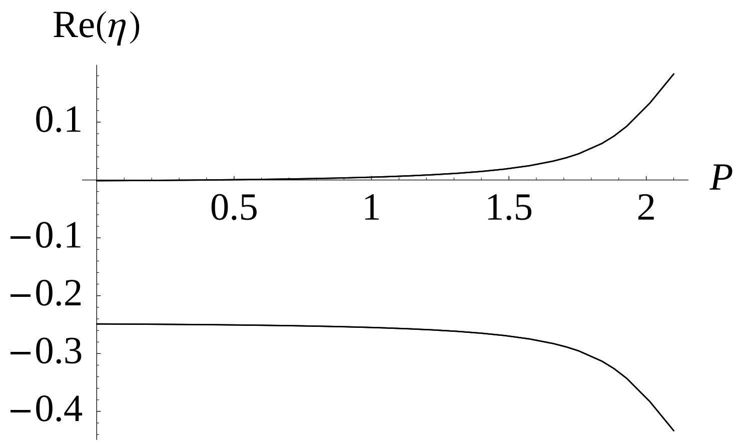

Figure 3.12: Real part of nondimensional eigenvalue η versus P for double mechanical pendulum model with damping parameters $c_1 = 0.0001$ and $c_2 = 0.1$

so that

$$(\eta^2 - 1)\check{q}_1 + (\eta^2 + 1)\check{q}_2 = 0$$

or

$$\left[P - 4 \pm \sqrt{(P-4)(P-2)}\right]\check{q}_1 + \left[P - 2 \pm \sqrt{(P-4)(P-2)}\right]\check{q}_2 = 0$$

Letting $\eta_1^2 = P - 3 - \sqrt{(P-4)(P-2)}$ and $\eta_2^2 = P - 3 + \sqrt{(P-4)(P-2)}$, the modal matrix then assumes the form

$$[\phi] = \begin{bmatrix} 1 & 1 \\ \sqrt{\frac{P-4}{P-2}} & -\sqrt{\frac{P-4}{P-2}} \end{bmatrix}$$

When $2 < P < 4$, the second row is pure imaginary; otherwise it's real. Thus, when flutter occurs, $\check{q}_2$ leads or lags $\check{q}_1$ by 90°. The magnitude of $\check{q}_2$ holding $\check{q}_1 = 1$ is plotted in Fig. 3.11.

The addition of damping forces to the model of a nongyrosopic conservative system will generally stabilize the system. Such is not the case with either gyroscopic conservative systems or with nonconservative systems. For example, in Fig. 3.12 the real part of η is plotted versus P and a loss of stability is observed for $P > 0.401928$. Such a dramatic change in the stability boundary can lead to catastrophic failure if not properly accounted for in the design of a system undergoing nonconservative forces. See Herrmann (1967) for further discussion of this point.

Method of Obtaining Eigenvalues and Eigenvectors

$$[\, A - \lambda I \,]\{\phi\} = \{0\}$$

1. Determinant polynomial This method is best suited for very low-order systems.

$$| A - \lambda I | = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

Solve this characteristic polynomial equation for its roots. (For symmetric case, the λ's are real.) For each λ, $\phi_1 \equiv 1$ leaves n equations in $n-1$ unknowns ϕ_2, ϕ_3,..., ϕ_n; but one of these equations is redundant.

2. Matrix Iteration As we will see, this method converges to the largest eigenvalues first. Because we are generally interested in the lowest frequencies, it is best suited for finding the eigenvalues of $K^{-1}M$ (which are $1/\omega^2$). If K is singular, the rigid-body modes must first be removed (see below).

a. Guess a $\{\phi\} = \{u_0\}$ with $\{u_0\} = [\phi]\{\beta\}$, $\{\beta\} = [\phi]^{-1}\{u_0\}$ but with $\{\beta\}$ and $[\phi]$ unknown. This is based on the so-called expansion theorem, which shows that *any* vector can be written as a linear combination of the modes. Let $\{u_1\} = [A]\{u_0\}$, $\{u_2\} = [A]\{u_1\}$, etc. Then,

$$\begin{aligned} \{u_n\} &= [A]^n\{u_0\} = [\phi][\grave{}\,\lambda^n\,\acute{}\,][\phi]^{-1}[\phi]\{\beta\} \\ &= [\phi][\grave{}\,\lambda^n\,\acute{}\,]\{\beta\} \implies \{\phi_1\}\lambda_1^n\beta_1 \end{aligned}$$

where λ_1 is the largest eigenvalue (so that λ_1^n dominates all other eigenvalues λ_i^n with $i > 1$). We thus converge on $\{\phi_1\}$ and λ_1 simultaneously, with

$$\{\phi_1\}_{n+1} \div \{\phi_1\}_n = \lambda_1$$

Similarly, if $[A]$ is not symmetric, we let $\{v_n\} = \left[A^T\right]^n\{u_0\}$, $\{v_n\}$ converges to $\{\psi_1\}$, the 1^{st} row of $[\phi]^{-1}$.

b. To get the second mode, we must sweep out all of the first mode from our guess. Here are two methods:

(1) Method # 1: Let $\{\overline{u}\}$ be the guessed mode with all ϕ_1 removed. Then,

$$\{\overline{u}\} = [\phi]\{\beta\} \quad \text{no } \phi_1 \implies \beta_1 = 0$$

Thus,

$$\{\beta\} = [\phi]^{-1}\{\overline{u}\} = [\psi]^T\{\overline{u}\} \implies \beta_1 = \{\psi_1\}^T\{\overline{u}\} = 0$$

so that

$$\{\psi_1\}^T\{\overline{u}\} = \sum_{i=1}^{n}\psi_{1i}\overline{u}_i = 0$$

We can solve for any element of $\{\overline{u}\}$ in terms of the others. For example, let

$$\overline{u}_1 = -\sum_{i=2}^{n}\overline{u}_i\frac{\psi_i}{\psi_1}$$

so that

$$\{\overline{u}\} = \begin{bmatrix} 0 & -\frac{\psi_2}{\psi_1} & -\frac{\psi_3}{\psi_1} & \cdots & \cdots & -\frac{\psi_n}{\psi_1} \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \{u\} \equiv [s]\{u\}$$

We must sweep out mode 1 from every iterate of mode 2. This can be generalized to sweep out higher modes as well, and all modes less than the i^{th} from every iterate of the i^{th} mode must be swept out.

(2) Method # 2: Here again we let

$$\{u\} = [\phi]\{\beta\} \qquad \{\overline{u}\} = [\phi] \begin{Bmatrix} 0 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_n \end{Bmatrix}$$

$$\overline{u} = u - [\phi] \begin{Bmatrix} \beta_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} = \{u\} - \{\phi_1\}\{\psi_1\}^T\{u\}$$

$$\{\overline{u}\} = [I - \phi_i\phi_j]\{u\} \equiv [s]\{u\}$$

with sweeping matrix $[s]$,

$$\{u_{n+1}\} = [A]\{\overline{u}\} = [A][s]\{u_n\}$$

Here $[A][s]$ is the sweeping matrix. Note that $A - \lambda I$ is also a sweeping matrix. QR, LR algorithms are based on similar ideas; see Wilkinson's book, *The Algebraic Eigenvalue Problem*.

3. Successive Rotations, $[A]$ Symmetric For a simple 2×2 example, one can see the eigenvalue problem as seeking a rotation (i.e. a change of coordinate system) that diagonalizes the matrix. Let

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

and let

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{Bmatrix} \equiv [\phi]\{\overline{x}\} \qquad [\phi]^T[\phi] = [I]$$

$\phi^T A\phi$ clearly reflects a change of coordinate system by rotation, viz.,

$$\phi^T A\phi = \begin{bmatrix} a_{11}\cos^2\theta + a_{22}\sin^2\theta + 2a_{12}\sin\theta\cos\theta & a_{12}(\cos^2\theta - \sin^2\theta) - (a_{11} - a_{22})\sin\theta\cos\theta \\ a_{12}(\cos^2\theta - \sin^2\theta) - (a_{11} - a_{22})\sin\theta\cos\theta & a_{22}\cos^2\theta + a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta \end{bmatrix}$$

Now, we can see that this is just like looking for principal axes. We set the off-diagonal term to zero, yielding

$$a_{12}\cos 2\theta = \frac{\sin 2\theta(a_{11} - a_{22})}{2}$$

or

$$\theta = \frac{1}{2}\tan^{-1}\frac{2a_{12}}{(a_{11} - a_{22})}$$

The method can be expanded for $n - 1$ rotations for an n^{th} order system; see chapter 6 of Meirovitch (1997) for exposition of several popular techniques.

3.2.5 Proportional Viscous Damping

Proportional viscous damping involves the standard form of the equations of motion, given by

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\}$$

However, the matrix $[C]$ is restricted to be a linear combination of the mass and stiffness matrices, viz.,

$$[C] = \alpha[M] + \beta[K]$$

where $[M]$ and $[K]$ are both symmetric and positive definite. This restriction will result in complex eigenvalues and real eigenvectors.

To proceed, we first find the natural frequencies ω and eigenvector matrix $[\phi]$ of the undamped system. As has been shown before, when $[\phi]$ is suitably normalized, then

$$[\phi]^T[M][\phi] = [I] \qquad [\phi]^T[K][\phi] = [\grave{}\,\omega^2\,\acute{}]$$

Now, make the change of variable $\{x\} = [\phi]\{y\}$ and premultiply the equations of motion by $[\phi]^T$ to obtain

$$\{\ddot{y}\} + [\grave{}\,\alpha + \beta\omega^2\,\acute{}]\{\dot{y}\} + [\grave{}\,\omega^2\,\acute{}]\{y\} = \{0\}$$

Thus, each individual scalar equation will have the form

$$\ddot{y}_i + \left(\alpha + \beta\omega_i^2\right)\dot{y}_i + \omega_i^2 y_i = 0$$

Special cases include:

1. mass proportional damping: $\beta = 0$ (e.g. concrete); $\zeta_i \downarrow$ with ω_i

2. stiffness proportional damping: $\alpha = 0$ (e.g. structural damping); $\zeta_i \uparrow$ with ω_i

3. modal damping can be added after the fact by picking the ζ's as you please (e.g. take them 1% for all modes) and setting

$$[C] = [\phi][\,\grave{}\,2\zeta_i\omega_i\,\acute{}\,][\phi^T]$$

Note that any system such that $[\phi]^T[C][\phi]$ is diagonal will also give real eigenvectors. See Adhikari (2006) for new developments in this field.

Free-Decay Solution

The free-decay solution will then be of the form

$$y_i = e^{-\zeta_i\omega_i t}\cos\left(\sqrt{1-\zeta_i^2}\,\omega_i t\right) \text{ or } e^{-\zeta_i\omega_i t}\sin\left(\sqrt{1-\zeta_i^2}\,\omega_i t\right)$$

where

$$\zeta_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right)$$

Response of Systems with Linear, Otherwise Arbitrary Damping

The standard form of the equations is

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$

Laplace Transform Taking the Laplace transform of both sides, one obtains

$$\left[[M]s^2 + [C]s + [K]\right]\{X(s)\} = \{F(s)\} - [M]\{\dot{x}(0)\} - [M]\{x(0)\}s - [C]\{x(0)\}$$

so that

$$\{X(s)\} = \left[Ms^2 + Cs + K\right]^{-1}\{\{F(s)\} - [M]\{\dot{x}(0)\} - [M]\{x(0)\}s - [C]\{x(0)\}\}$$

This is quite a cumbersome approach. The inverse cannot be found analytically unless the problem is very low-order, and even then computerized symbolic manipulation may be required.

Harmonic Excitation

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} = \{\overline{F}\}e^{i\Omega t}$$

Let

$$\{x\} = \{\overline{x}\}e^{i\Omega t}$$

which gives

$$\{\overline{x}\} = \left[[K] - [M]\Omega^2 + i\Omega[C]\right]^{-1}\{\overline{F}\}$$

Structural damping can be accommodated by multiplying the $[K]$ term by $(1 + ig)$.

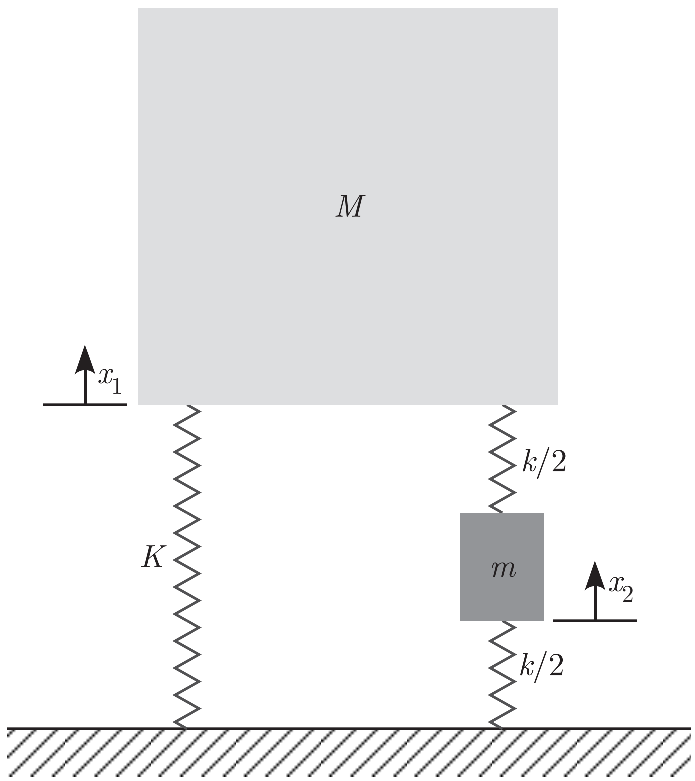

Figure 3.13: Schematic of vibration absorber

Example Consider a vibration absorber as depicted in Fig. 3.13. The main part of the subsystem is of mass M, and the absorber is a smaller body of mass m. The main part of the subsystem is subjected to a harmonic force $F_0 e^{i\Omega t}$, so that the equations of motion are

$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K + \frac{k}{2} & -\frac{k}{2} \\ -\frac{k}{2} & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} e^{i\Omega t}$$

Now assume $\{x\} = \{\overline{x}\} e^{i\Omega t}$

$$\begin{aligned} \begin{Bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{Bmatrix} &= \begin{bmatrix} K + \frac{k}{2} - M\Omega^2 & -\frac{k}{2} \\ -\frac{k}{2} & k - m\Omega^2 \end{bmatrix}^{-1} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \\ &= \frac{F_0}{(K + \frac{k}{2} - M\Omega^2)(k - m\Omega^2) - \frac{k^2}{4}} \begin{Bmatrix} k - m\Omega^2 \\ \frac{k}{2} \end{Bmatrix} \end{aligned}$$

The force transmitted to the base is then

$$\overline{F}_B = K\overline{x}_1 + \frac{k}{2}\overline{x}_2 \quad \Longrightarrow \quad \frac{\overline{F}_B}{F_0} = \frac{K(k - m\Omega^2) + \frac{k^2}{4}}{(K + \frac{k}{2} - M\Omega^2)(k - m\Omega^2) - \frac{k^2}{4}}$$

Introducing the following nondimensional parameters:

$$\frac{\Omega^2}{k/m} = \omega^2 \qquad \frac{k/m}{K/M} = \mu \qquad k/K = \rho$$

$$\Omega^2 = \frac{k}{m}\left(1 + \frac{1}{4}\frac{k}{K}\right)$$

one finds that the transmitted force can be written more simply as

$$\frac{\overline{F}_B}{F_0} = \frac{1 - \omega^2 + \frac{1}{4}\rho}{(1 + \frac{1}{2}\rho - \mu\omega^2)(1 - \omega^2) - \frac{1}{4}\rho}$$

We can now plot $k|\overline{x}_1|/F_0$, $k|\overline{x}_2|/F_0$, and $|\overline{F}_B|/F_0$ for typical values of μ and ρ versus ω^2. These plots are shown, respectively, in Figs. 3.14 – 3.16. All three quantities exhibit infinite response at the natural frequencies of the two-degree-of-freedom system. The plots of $|x_1|$ and $|F_B|$ show zero responses at values of $\omega^2 = 1$ and $\omega^2 = 1 + \rho/4$, respectively. These points are called antiresonances. When a small amount of damping is added to the problem, the response at the resonances becomes finite, and the response at the antiresonances becomes nonzero. Thus, it behooves the designer of vibration absorbers to limit the amount of damping.

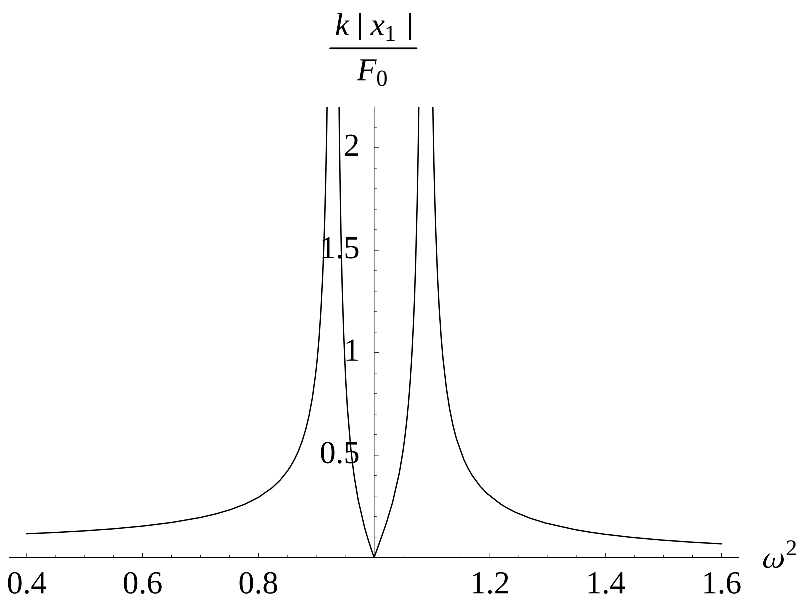

Figure 3.14: $k|\overline{x}_1|/F_0$ versus ω^2 for vibration absorber with $\mu = 0.01$, $\rho = 0.1$

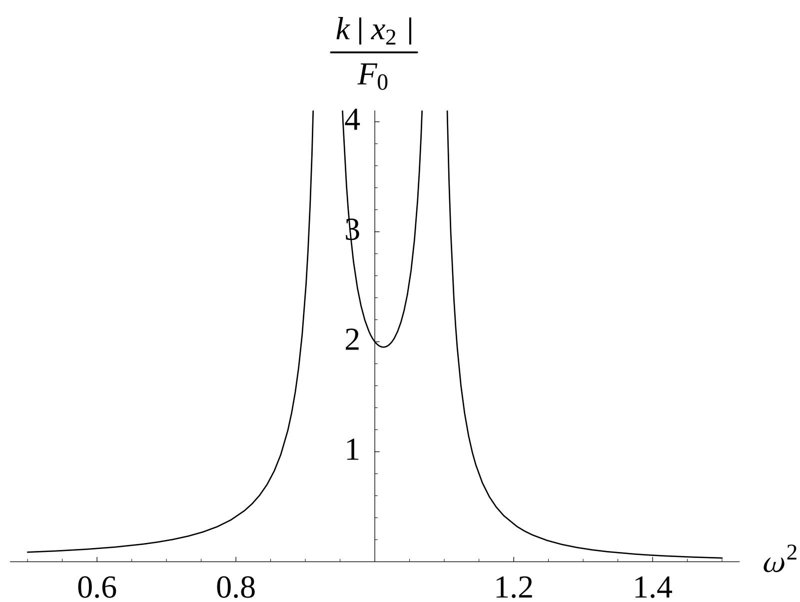

Figure 3.15: $k|\overline{x}_2|/F_0$ versus ω^2 for vibration absorber with $\mu = 0.01$, $\rho = 0.1$

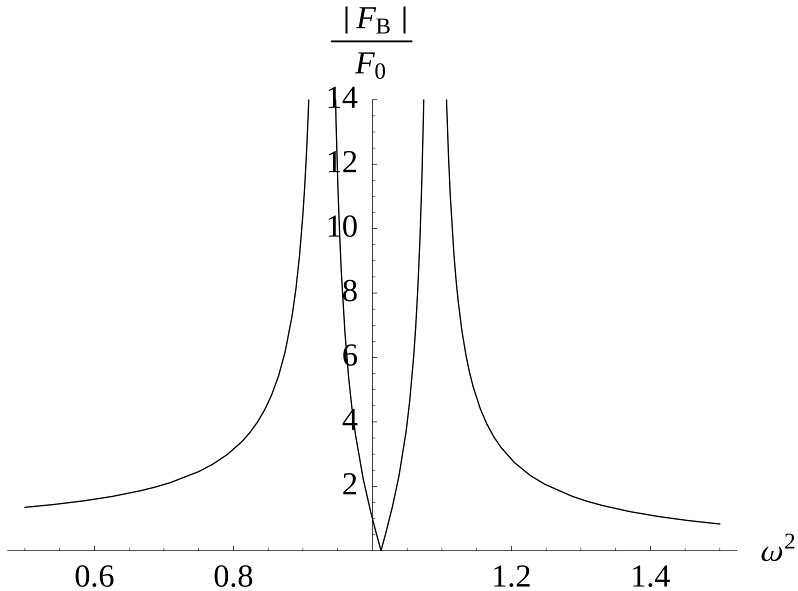

Figure 3.16: $|\overline{F}_B|/F_0$ versus ω^2 for vibration absorber with $\mu = 0.01$, $\rho = 0.1$

Modal Decoupling

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$
$$\{x(0)\} = \{x_0\} \qquad \{\dot{x}(0)\} = \{v_0\}$$

Let $\{x\} = [\phi]\{y\}$ where the columns of $[\phi]$ are the mode shapes and

$$[\phi]^T[M][\phi] = [\lceil m_j \rfloor] \qquad \text{generalized masses, often normalized to 1}$$
$$[\phi]^T[C][\phi] = [\lceil c_j \rfloor] \qquad \text{if we have proportional or modal damping}$$
$$[\phi]^T[K][\phi] = [\lceil k_j \rfloor] \qquad \text{generalized stiffnesses}$$

Thus,

$$\omega_j^2 = \frac{k_j}{m_j} \qquad \zeta_j = \frac{c_j}{2\sqrt{k_j m_j}}$$

Premultiplying equation by $[\phi]^T$ and defining $[\phi^T]\{F\} = \{G\}$ as the generalized forces, one finds

$$m_j\ddot{y}_j + c_j\dot{y}_j + k_j y_j = G_j \qquad j = 1, \ldots, n$$

or

$$\ddot{y}_j + 2\zeta_j\omega_j\dot{y}_j + \omega_j^2 y_j = \frac{G_j}{m_j} \qquad j = 1, \ldots, n$$

The initial conditions are thus expressed as

$$\{y(0)\} = [\phi]^{-1}\{x_0\} \qquad \{\dot{y}(0)\} = [\phi]^{-1}\{v_0\}$$

Note that not all modes are required. If we truncate the displacement to m modes with $m << n$, then

$$\{x\} = [\phi]\{y\} = \{\phi_1\}y_1 + \{\phi_2\}y_2 + \dots \{\phi_m\}y_m$$

The elastic forces can be written as

$$[K]\{x\} = [K][\phi]\{y\} = [\phi]^{-T}[\lceil m\omega^2 \rfloor]\{y\} = [M][\phi][\lceil \omega^2 \rfloor]\{y\}$$

or

$$[K]\{x\} = [M]\left\{\{\phi_1\}\omega_1^2 y_1 + \{\phi_2\}\omega_2^2 y_2 + \dots \{\phi_m\}\omega_m^2 y_m\right\}$$

Because the frequencies are arranged from lowest to highest, one can expect to need more modes to get accurate elastic forces than are required to get accurate displacements. When a subset of the modes is used, the calculation of initial conditions must be approximated as well. One way to approach this is to note that $[\phi]^{-1} = [\lceil m \rfloor]^{-1}[\phi]^T[M]$ so that

$$\{y(0)\} = [\lceil m \rfloor]^{-1}[\phi]^T[M]\{x_0\} \approx \{\{\phi_1\}/m_1 + \{\phi_2\}/m_2 + \dots + \{\phi_m\}/m_m\}\,[M]\{x_0\}$$

and

$$\{\dot{y}(0)\} = [\lceil m \rfloor]^{-1}[\phi]^T[M]\{v_0\} \approx \{\{\phi_1\}/m_1 + \{\phi_2\}/m_2 + \dots + \{\phi_m\}/m_m\}\,[M]\{v_0\}$$

a. Transient Response Example First, we consider a transient response example with $F(t) = 0$. Superposing the basic responses of section 2.6, we obtain

$$\begin{aligned}
y_j(t) &= y_j(0)e^{-\zeta_j\omega_j t}\left[\cos\left(\sqrt{1-\zeta_j^2}\,\omega_j t\right) + \frac{\zeta_j}{\sqrt{1-\zeta_j^2}}\sin\left(\sqrt{1-\zeta_j^2}\,\omega_j t\right)\right] \\
&+ \frac{\dot{y}_j(0)}{\omega_j\sqrt{1-\zeta_j^2}}e^{-\zeta_j\omega_j t}\sin\left(\sqrt{1-\zeta_j^2}\,\omega_j t\right) \\
&= y_j(0)e_j(t) + \dot{y}_j(0)h_j(t) \quad \text{(dimensional)}
\end{aligned}$$

$$\{y\} = [\lceil e(t) \rfloor]\{y(0)\} + [\lceil h(t) \rfloor]\{\dot{y}(0)\}$$

$$\{x(t)\} = [\phi][\lceil e(t) \rfloor][\phi]^{-1}\{x_0\} + [\phi][\lceil h(t) \rfloor][\phi]^{-1}\{v_0\}$$

b. Harmonic Excitation Example Letting $\{F\} = \{\overline{F}\}e^{i\Omega t}$, and $\{\overline{g}\} = [\phi]^T\{\overline{F}\}$, one finds the individual uncoupled scalar equations of motion as

$$m_j\ddot{y}_j + c_j\dot{y}_j + k_j y_j = \overline{g}_j e^{i\Omega t}$$

$$y_j = \overline{y}_j e^{i\Omega t} \qquad \overline{y}_j = \frac{\overline{g}_j}{k_j - m_j\Omega^2 + i\Omega c_j} = \overline{g}_j H_j(\Omega)$$

$$\{\overline{x}\} = [\phi][\grave{}\, H(\Omega) \acute{}\,][\phi]^T\{\overline{F}\}$$

Note, for structural damping

$$[M]\{\ddot{x}\} + (1 + ig)[K]\{x\} = \{\overline{F}\}e^{i\Omega t}$$

and

$$H_j(\Omega) = \frac{1}{k_j(1 + ig) - m_j\Omega^2}$$

c. Convolution Integral Example

$$m_i\ddot{y}_i + c_i\dot{y}_i + k_i y_i = \delta(t) \Longrightarrow y_i(t) = h_i(t)$$

$$m_i\ddot{y}_i + c_i\dot{y}_i + k_i y_i = g_i(t) \qquad \underbrace{y_i = \int_0^t h_i(t - \tau)g_i(\tau)d\tau}_{i^{\text{th}}\text{ degree of freedom}}$$

$$\{x(t)\} = [\phi]\int_0^t [\grave{}\, h(t - \tau) \acute{}\,]\{g(\tau)\}d\tau \qquad \underbrace{\{g(\tau)\} = [\phi]^T\{F(\tau)\}}_{\text{multi-degree of freedom}}$$

d. Periodic Excitation A specific example of this is not needed, as recall response to periodic excitation is most easily treated by making use of Fourier series and superimposing results for harmonic excitation.

e. Random Excitation

$$\{f\} \Longrightarrow R_{f_i f_j}(\tau) \text{ known } \ , S_{f_i f_j}(\omega) \text{ known}$$
$$\left[R_{g_i g_j}\right] = [\phi]^T\left[R_{f_i f_j}\right][\phi]\ , \quad \left[S_{g_i g_j}\right] = [\phi]^T\left[S_{f_i f_j}\right][\phi]$$

$$\begin{aligned}\Longrightarrow \left[S_{x_i x_j}\right] &= [\phi][H^*][\phi]^T\left[S_{f_i f_j}\right][\phi][H][\phi]^T \\ &= \left[\mathcal{H}^*_{ij}\right][S_{ff}][\mathcal{H}_{ij}]\end{aligned}$$

$$R_{y_i y_j} = \int_0^{+\infty}\int_0^{+\infty} h_i(\lambda_i)h_j(\lambda_j)R_{g_i g_j}(\tau + \lambda_i - \lambda_j)d\lambda_i d\lambda_j$$
$$\left[R_{x_i, x_j}\right] = [\phi]\left[R_{y_i y_j}\right][\phi]^T$$
$$h_i(t) = \frac{1}{\omega_i m_i}\frac{1}{\sqrt{1 - \zeta_i^2}}e^{-\zeta_i\omega_i t}\sin\sqrt{1 - \zeta_i^2}\,\omega_i t$$
$$\mathcal{H}_{ij} = [K - \omega^2 M + I\omega G]^{-1}$$

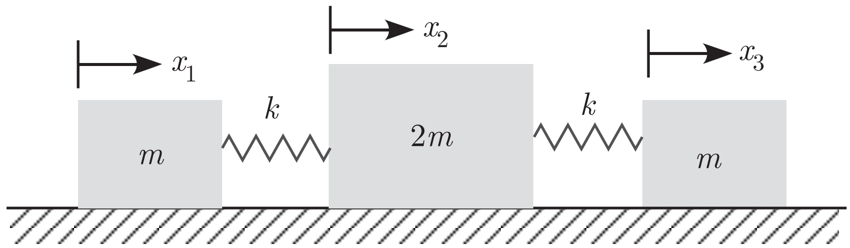

Figure 3.17: Schematic of simple system for numerical example

Numerical Example Consider a simple, unrestrained system as depicted in Fig. 3.17 with initial conditions given as

$$x_1(0) = x_2(0) = x_3(0) = 0 \qquad \dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = -v_0$$

The equations of motion are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_0 \end{Bmatrix}$$

Let $\lambda = \omega^2 m/k$ and $\{x\} = \{\phi\}e^{i\omega t}$. The governing equations then become an eigenvalue problem of the form

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-2\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$(1-\lambda)\left[(2-2\lambda)(1-\lambda)-1\right] - (1-\lambda) = 0$$

$$(1-\lambda)\left[2-2\lambda-2\lambda+2\lambda^2-1-1\right] = 0$$

$$(1-\lambda)(2\lambda^2-4\lambda) = 0 \quad \lambda(1-\lambda)(\lambda-2) = 0$$

Thus, $\lambda = 0$, 1, and 2, and

$$\omega^2 = 0,\ \frac{k}{m},\ \frac{2k}{m}$$

The first eigenvector is a rigid-body mode, governed by the null space of $[K]$:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\phi_{11} - \phi_{21} = 0 \implies \phi_{11} = \phi_{21}$$

$$-\phi_{11} + 2\phi_{21} - \phi_{31} = 0 \implies \phi_{11} = \phi_{31}$$

$$-\phi_{21} + \phi_{31} = 0 \implies \phi_{21} = \phi_{31} \quad \text{(redundant)}$$

$$\implies \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

The second eigenvector is governed by

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$-\phi_{22} = 0 \qquad \phi_{12} + \phi_{32} = 0 \qquad \phi_{12} = -\phi_{32} \qquad \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$$

The third eigenvector is governed by

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\phi_{13} + \phi_{23} = 0 \qquad \phi_{23} + \phi_{33} = 0 \qquad \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

Thus,

$$[\phi] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now we transform the equations of motion into the uncoupled systems:

$$[\phi]^T[M][\phi] = m \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[\phi]^T[K][\phi] = k \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$[\phi]^T\{F\} = F_0 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$[\phi]^{-1}\{\dot{x}(0)\} = -\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} v_0 = -\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} v_0$$

Regarding the force as a step function at $t = 0$, one finds the transformed, uncoupled equations as

$$\begin{aligned} 4m\ddot{y}_1 &= F_0 & y_1(0) &= 0 & \dot{y}_1(0) &= -v_0 \\ 2m\ddot{y}_2 + 2ky_2 &= F_0 & y_2(0) &= 0 & \dot{y}_2(0) &= 0 \\ 4m\ddot{y}_3 + 8ky_3 &= F_0 & y_3(0) &= 0 & \dot{y}_3(0) &= 0 \end{aligned}$$

Take $v_0 = m = k = F_0 = 1$ for convenience, so that the solutions for y_i and x_i are given by

$$y_1 = \frac{1}{8}t^2 - t$$

$$y_2 = \frac{1}{2} - \frac{1}{2}\cos t$$

$$y_3 = \frac{1}{8} - \frac{1}{8}\cos\sqrt{2}t$$

$$x_1 = \frac{1}{8}t^2 - t - \frac{3}{8} + \frac{1}{2}\cos t - \frac{1}{8}\cos\sqrt{2}t$$

$$x_2 = \frac{1}{8}t^2 - t - \frac{1}{8} + \frac{1}{8}\cos\sqrt{2}t$$

$$x_3 = \frac{1}{8}t^2 - t + \frac{5}{8} + \frac{1}{2}\cos t - \frac{1}{8}\cos\sqrt{2}t$$

3.2.6 General Gyroscopic/Damping Matrix

State-Variable Approach

The standard form of the governing equations now takes on the form

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{Q(t)\}$$

with the column matrix of generalized forces denoted here by $\{Q\}$. Now, define the $2n\times1$ column matrix of state variables as

$$\{y\} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$$

Then the equations can be written in various ways:

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}\{\dot{y}\} + \begin{bmatrix} 0 & -I \\ K & C \end{bmatrix}\{y\} = \begin{Bmatrix} 0 \\ Q \end{Bmatrix} \tag{3.21}$$

$$\begin{bmatrix} C & M \\ I & 0 \end{bmatrix}\{\dot{y}\} + \begin{bmatrix} K & 0 \\ 0 & -I \end{bmatrix}\{y\} = \begin{Bmatrix} Q \\ 0 \end{Bmatrix} \tag{3.22}$$

and so forth; altogether there are four ways to write the equations. In any case they can be expressed as

$$[J]\{\dot{y}\} + [A]\{y\} = \{F\} \qquad \text{general form} \tag{3.23}$$

Now, for casting this as an eigenvalue problem, we set

$$\{y\} = \{\phi\}e^{\eta t}$$

so that

$$[\,[A] + \eta[J]\,]\,\{\phi\} = \{0\}$$

As a check on the answer, one should find that

$$\{\phi\} = \left\{ \begin{array}{c} \phi_x \\ \eta\phi_x \end{array} \right\}$$

where $\{x\} = \{\phi_x\}e^{\eta t}$.

Special Cases of Eigenvalue Problem

a. First we consider an unforced, conservative system, so that $\{Q\} = 0$, $[M] = [M]^T$ and $[K] = [K]^T$, with $[M]$ and $[K]$ positive definite. Then we add a viscous damping model, such that $[C] = [C]^T$. Then, Eq. (3.22) can be rewritten as

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \{y\} = \{0\}$$

or, in the form of Eq. (3.23) with $[J] = [J]^T$ and $[A] = [A]^T$. If $[M]$ has no zero eigenvalues, then J has no zero eigenvalues. The proof is simple: just note that

$$[J]^{-1} = \begin{bmatrix} 0 & M^{-1} \\ M^{-1} & -M^{-1}CM^{-1} \end{bmatrix}$$

exists if and only if $[M]^{-1}$ exists. Thus, we can set

$$\{y\} = J^{-1/2}\{z\}$$

which allows us to write the equations of motion in the form

$$\{\dot{z}\} + [\overline{A}]\{z\} = \{0\}$$

where $[\overline{A}] = [J]^{-1/2}[A][J]^{-1/2}$ is symmetric and complex, i.e. $[\overline{A}]^T = [\overline{A}]$. Therefore,

$$[\overline{\phi}]^T[\overline{A}][\overline{\phi}] = [\,\grave{}\,\lambda\,\acute{}\,]$$

where

$$\eta = -\lambda \qquad [\phi] = [J]^{-1/2}[\overline{\phi}]$$

and

$$[\phi]^T[J][\phi] = [I] \qquad [\phi]^T[A][\phi] = -[^\backslash\eta_\backslash]$$

Like before, the modes are orthogonal, but with a different weighting, and unlike before both $[^\backslash\eta_\backslash]$ and $[\phi]$ are complex.

It should be noted that the problem can also be set up such that

$$[\phi]^H[J][\phi] = I \qquad [\phi]^H[A][\phi] = -[^\backslash\eta_\backslash]$$

where the superscripted H is the Hermitian transpose (i.e. the transpose of the complex conjugate).

b. Next we consider an unforced, conservative system, so that $\{Q\} = 0$, $[M] = [M]^T$ and $[K] = [K]^T$, with $[M]$ and $[K]$ positive definite. Then we add a gyroscopic matrix $[C] = -[C]^T$. Then, Eq. (3.21) can be rewritten as

$$\begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} 0 & -K \\ K & C \end{bmatrix} \{y\} = \{0\}$$

1. From Eq. (3.2.6) $[J] = [J]^T$ and $[A] = -[A]^T$. Now, let

$$\{y\} = [J]^{-1/2}\{z\} = \begin{bmatrix} K^{-\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} \{z\}$$

$$\{\dot{z}\} + [\overline{A}]\{z\} = \{0\}$$

2. Calculate $[J]^{-1/2}$

3. Form $[\overline{A}] = [J]^{-1/2}[A][J]^{-1/2}$ where $\overline{A}^T = -[\overline{A}]$ is real.

4. The eigenvectors of $[\overline{A}]$ are then of the form

$$\{z\} = \{\phi\}e^{i\omega t} = \{\{u\} + i\{v\}\}\, e^{i\omega t}$$

   where the real part is

$$[\overline{A}]\{u\} - \omega\{v\} = \{0\}$$

   and the imaginary part is

$$[\overline{A}]\{v\} + \omega\{u\} = \{0\}$$

5. Combine these equations to obtain

$$[\overline{A}]^2\{u\} + \omega^2\{u\} = 0$$
$$[\overline{A}]^2\{v\} + \omega^2\{v\} = 0$$

6. Because $[\overline{A}]^2 = -[\overline{A}]^T[\overline{A}]$ one finds two equations for the real and imaginary parts of the eigenvectors:

$$\begin{aligned}\left[[\overline{A}]^T[\overline{A}] - \omega^2 I\right]\{u\} = 0 \\ \left[[\overline{A}]^T[\overline{A}] - \omega^2 I\right]\{v\} = 0\end{aligned} \tag{3.24}$$

We note that $[\overline{A}^T\overline{A}]$ is a symmetric real matrix, so we may use classical methods, and that a similar approach using the Cholesky decomposition is straightforward to develop. Also, the above form is appropriate when $[K]$ and $[M]$ are both positive definite. If we weaken this condition to say that $[K]$ and $[M]$ have no zero eigenvalues, then the problem instead turns out to be in terms of $[\overline{A}^H\overline{A}]$, which is Hermitian. The eigenvalues and eigenvectors of a Hermitian matrix have the same properties as those of a real, symmetric matrix, so classical methods apply here as well.

It can be shown that the eigenvalues of the real symmetric matrix $2n\times 2n$ matrix $[\overline{A}^T\overline{A}]$ (or of the Hermitian matrix $[\overline{A}^H\overline{A}]$) are real values of ω^2. There are only n distinct values, however, as each eigenvalue is a double root. Each double root has 2 eigenvectors $\{u\}$ and $\{v\}$. The eigenvectors of the problem are then $\{u \pm iv\}$; $\{v \mp iu\}$ are also eigenvectors.

c. When $[M]^{-1}$ exists but an otherwise general case, we can rewrite Eq. (3.21) as

$$\{\dot{y}\} + \begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix}\{y\} = \{0\}$$

so that $[J] = [I]$ and $[A]$ is general. Thus,

$$\{y\} = \{\phi\}e^{\eta t}$$

and

$$[A + I\eta]\{\phi\} = \{0\}$$

or

$$[\phi^{-1}][A][\phi] = -[\grave{}\,\eta\,\acute{}]$$

d. For low-order systems, one may apply the polynomial approach already discussed for special cases, viz.,

$$\left[M\eta^2 + C\eta + K\right]\{\phi\} = \{0\}$$

Rotor Dynamics Example

Consider a particle of mass m suspended by springs in a frame rotating about an axis fixed in inertial space with angular speed Ω, as shown in Fig. 3.18. The kinetic energy is easily found as

$$T = \frac{1}{2}m\left[(\dot{x}_1 - \Omega x_2)^2 + (\dot{x}_2 + \Omega x_1)^2\right]$$

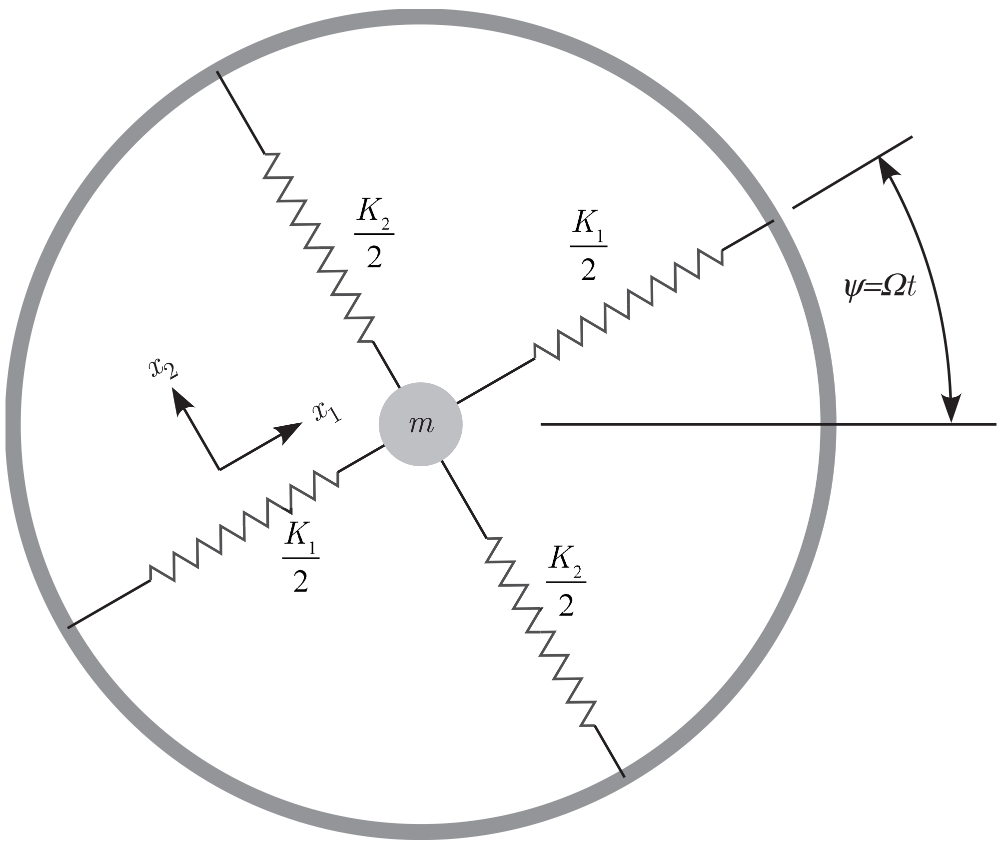

Figure 3.18: Spring-restrained particle in a rotating frame

where x_1 and x_2 are the displacement components of the particle relative to the axis of rotation in the directions indicated. Assuming stiffnesses K_1 and K_2 govern displacements of the particle in the x_1 and x_2 directions, and that the springs are relaxed when the particle is on the axis of rotation, then the potential energy is

$$V = \frac{1}{2}K_1 x_1^2 + \frac{1}{2}K_2 x_2^2$$

The equations of motion assume the form

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & -2\Omega \\ 2\Omega & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \lambda_1^2 - \Omega^2 & 0 \\ 0 & \lambda_2^2 - \Omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

where

$$\lambda_1^2 \equiv \frac{K_1}{m} \quad \text{and} \quad \lambda_2^2 \equiv \frac{K_2}{m}$$

This is a low-order problem, so the polynomial method is suitable. Let

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{\phi\} e^{i\omega t}$$

Therefore,

$$\begin{bmatrix} \lambda_1^2 - \Omega^2 - \omega^2 & -2\Omega i\omega \\ 2\Omega i\omega & \lambda_2^2 - \Omega^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The determinant of the coefficient matrix must vanish, so that

$$\omega^4 - \omega^2(\lambda_1^2 + \lambda_2^2 - 2\Omega^2 + 4\Omega^2) + (\lambda_1^2 - \Omega^2)(\lambda_2^2 - \Omega^2) = 0$$

The solution can be written as

$$\omega^2 = \frac{\lambda_1^2 + \lambda_2^2 + 2\Omega^2 \pm \sqrt{(\lambda_1^2 - \lambda_2^2)^2 + 8\Omega^2(\lambda_1^2 + \lambda_2^2)}}{2}$$

Special case In the special case that $\lambda_1^2 = \lambda_2^2 = \lambda^2$, the solution simplifies to

$$\omega^2 = \lambda^2 + \Omega^2 \pm 2\Omega\lambda = (\lambda \pm \Omega)^2$$

Thus,

$$\omega = \lambda + \Omega \quad \text{and} \quad |\lambda - \Omega|$$

The modes assume the form

$$\omega_1 = \lambda + \Omega \qquad \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ i \end{Bmatrix} \quad \text{regressing}$$

$$\omega_2 = \lambda - \Omega \quad \text{for } \Omega < \lambda \qquad \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -i \end{Bmatrix} \quad \text{progressing}$$

$$\omega_2 = \Omega - \lambda \quad \text{for } \Omega > \lambda \qquad \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ i \end{Bmatrix} \quad \text{regressing}$$

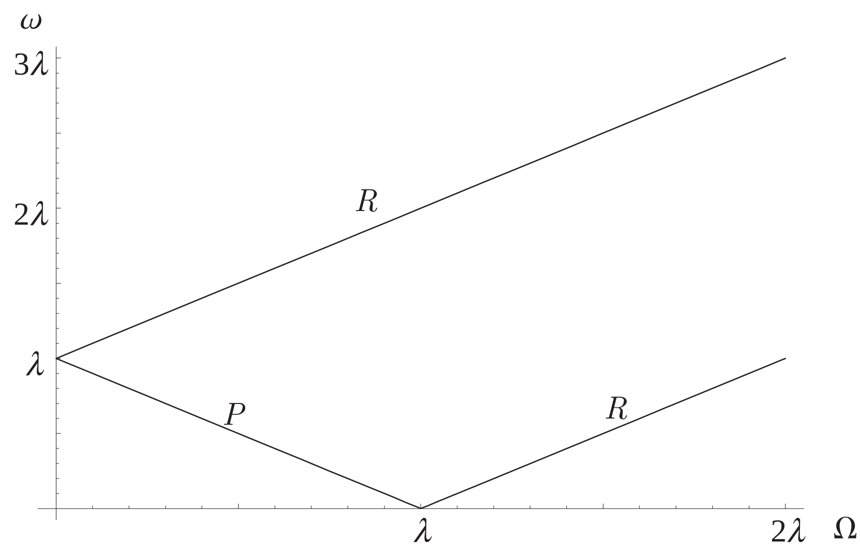

Figure 3.19: Frequencies of a rotating mass with P indicating a progressing mode and R a regressing mode

The names are descriptive of the way the motion appears to an observer in the nonrotating frame. A regressing mode appears as a clockwise, circular motion (opposite to the direction of rotation). A progressing mode appears as a counterclockwise, circular motion (in the direction of rotation). The regressing modes have x_2 lagging x_1 by 90°, whereas for the progressing mode x_2 leads x_1 by 90°. See Fig. 3.19.

State-Variable Method on Rotating Mass Here we illustrate the state-variable approach with the same example:

$$J = \begin{bmatrix} \lambda_1^2 - \Omega^2 & 0 & 0 & 0 \\ 0 & \lambda_2^2 - \Omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & -(\lambda_1^2 - \Omega^2) & 0 \\ 0 & 0 & 0 & -(\lambda_2^2 - \Omega^2) \\ (\lambda_1^2 - \Omega^2) & 0 & 0 & -2\Omega \\ 0 & (\lambda_2^2 - \Omega^2) & 2\Omega & 0 \end{bmatrix}$$

Assume $\lambda_1^2 > \Omega^2$. Then,

$$J^{-1/2}\, A\, J^{-1/2} = \begin{bmatrix} 0 & 0 & -\sqrt{\lambda_1^2 - \Omega^2} & 0 \\ 0 & 0 & 0 & -\sqrt{\lambda_2^2 - \Omega^2} \\ \sqrt{\lambda_1^2 - \Omega^2} & 0 & 0 & -2\Omega \\ 0 & \sqrt{\lambda_2^2 - \Omega^2} & 2\Omega & 0 \end{bmatrix} = \overline{A}$$

$$\overline{A}^2 = -\overline{A}^T\overline{A} = \begin{bmatrix} \lambda_1^2 - \Omega^2 & 0 & 0 & -2\Omega\sqrt{\lambda_1^2 - \Omega^2} \\ 0 & \lambda_2^2 - \Omega^2 & 2\Omega\sqrt{\lambda_2^2 - \Omega^2} & 0 \\ 0 & 2\Omega\sqrt{\lambda_2^2 - \Omega^2} & \lambda_1^2 + 3\Omega^2 & 0 \\ -2\Omega\sqrt{\lambda_1^2 - \Omega^2} & 0 & 0 & \lambda_2^2 + 3\Omega^2 \end{bmatrix}$$

Both the first and second of Eqs. (3.24) yield

$$\omega^4 - \omega^2(\lambda_1^2 + \lambda_2^2 - 2\Omega^2) + (\lambda_1^2 - \Omega^2)(\lambda_2^2 - \Omega^2) = 0$$

which is the same as the polynomial method.

View from the nonrotating system If we introduce displacement components in the nonrotating frame, denoting them as y_1 and y_2, the equations of motion become

$$\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} \lambda_1^2 \cos^2\psi + \lambda_2^2 \sin^2\psi & (\lambda_2^2 - \lambda_1^2)\sin\psi\cos\psi \\ (\lambda_2^2 - \lambda_1^2)\sin\psi\cos\psi & \lambda_1^2 \sin^2\psi + \lambda_2^2 \cos^2\psi \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

These equations are much more complex, possessing temporally periodic coefficients. The solution requires methodology beyond the scope of these notes and involves such techniques as Floquet-Lyapunov theory. To avoid this problem, we can consider only the case of isotropic stiffnesses, so that

$$\lambda_1 = \lambda_2 \qquad \omega_1^2 = \lambda^2 \qquad \omega_1 = \lambda$$

which may be progressing or regressing.

3.2.7 Stability Information

Looking directly at eigenvalues

1. $\{\ddot{x}\} + [D]\{x\} = \{0\}$, λ are eigenvalues of D .

   $[D]$ symmetric $\implies$ stable if $\lambda = \omega^2 > 0$
   unstable if $\lambda = \omega^2 < 0$
   displacement unstable and velocity stable if $\lambda = \omega^2 = 0$

   Reason:

   $$e^{\pm i\omega t} \implies e^{\pm at} \text{ if } \omega = ia$$

   $$\ddot{q} = 0 \implies q = q_0 + \dot{q}_0 t \ , \ \dot{q} = \dot{q}_0$$

2. $[D]$ not symmetric, distinct roots

   $$q = e^{\eta t} \qquad \eta^2 = -\lambda$$

$$\begin{aligned} \Longrightarrow \;& \lambda \text{ positive real, stable} \\ & \lambda \text{ negative real, unstable} \\ & \lambda \text{ complex, unstable} \\ & \lambda \text{ zero } \Longrightarrow \text{ displacement unstable, velocity stable} \end{aligned}$$

3. $[D]$ not symmetric, repeated root

$$\begin{aligned} \text{distinct eigenvectors} &\Longrightarrow \text{same as 2} \\ \text{only one eigenvector} &\Longrightarrow \text{unstable} \end{aligned}$$

Looking at eigenvalues of $[M]$, $[C]$, $[K]$

$$[M] = [M]^T \qquad [K] = [K]^T$$
$$[C] = [D] + [G] \qquad [D] = [D]^T \qquad [G] = -[G]^T$$
$$[D] \equiv \frac{1}{2}([C] + [C]^T) \qquad [G] \equiv \frac{1}{2}([C] - [C]^T)$$
$$[M]\{\ddot{x}\} + [D]\{\dot{x}\} + [G]\{\dot{x}\} + [K]\{x\} = \{0\}$$
$$M_{ij}\ddot{x}_j + D_{ij}\dot{x}_j + G_{ij}\dot{x}_j + K_{ij}x_j = 0$$

Multiply by $\dot{x}_i$

$$M_{ij}\dot{x}_i\ddot{x}_j + D_{ij}\dot{x}_i\dot{x}_j + G_{ij}\dot{x}_i\dot{x}_j + K_{ij}\dot{x}_i x_j = 0$$

Now, using symmetry properties and interchanging names of i and j, one finds that

$$M_{ij}\dot{x}_j\ddot{x}_i + D_{ij}\dot{x}_i\dot{x}_j - G_{ij}\dot{x}_i\dot{x}_j + K_{ij}x_i\dot{x}_j = 0$$

Adding these two equations and summing over i gives

$$M_{ij}\frac{d}{dt}(\dot{x}_i\dot{x}_j) + D_{ij}(2\dot{x}_i\dot{x}_j) + K_{ij}\frac{d}{dt}(x_i x_j) = 0$$

Finally, integrating and multiplying by 1/2, we get a work-energy balance of the form

$$\frac{1}{2}\{\dot{x}\}^T[M]\{\dot{x}\} + \frac{1}{2}\{x\}[K]\{x\} = -\int_0^t \{\dot{x}\}^T[D]\{\dot{x}\}dt$$

Let

$$\begin{aligned} [\phi_M]^T[M][\phi_M] &= [\grave{}\,\alpha\,\acute{}] \qquad & \{x\} &= [\phi_M]\{u\} \\ [\phi_K]^T[K][\phi_K] &= [\grave{}\,\beta\,\acute{}] \qquad & \{x\} &= [\phi_K]\{v\} \\ [\phi_D^T][D][\phi_D] &= [\grave{}\,\gamma\,\acute{}] \qquad & \{x\} &= [\phi_D]\{w\} \end{aligned}$$

$$\frac{1}{2}\sum_{i=1}^{n}\left(\underbrace{\alpha_i\dot{u}_i^2}_{\text{kinetic}} + \underbrace{\beta_i v_i^2}_{\text{potential}} + \underbrace{2\gamma_i\int_0^t \dot{w}_i^2 dt}_{\text{dissipated}}\right) = 0$$

Note: if $\alpha_i > 0$, $\beta_i > 0$, and $\gamma_i > 0$, the system must be stable. If $\gamma_i < 0$, the system *may* be unstable. Finally, if $\alpha_i\beta_i < 0$, the system may be divergent. ($\uparrow\ u \ \downarrow\ v$)

Kelvin-Tait-Chetaev Theorem

1. If $[M] = [M]^T$, $[K] = [K]^T$ and all eigenvalues of $[D]$ and $[M]$ are positive, then the system is stable if and only if all eigenvalues of $[K]$ are > 0. The number of unstable roots equals the number of negative eigenvalues of $[K]$.

2. If $[M]$ and $[K]$ have positive eigenvalues, then a necessary (but not sufficient) condition for instability is that $[D]$ have a negative eigenvalue.

Example: Rotor Dynamics with Damping If we add damping to the above example, we can put the theorem to the test. The equations of motion take on the form

$$\left\{\begin{matrix} \ddot{x}_1 \\ \ddot{x}_2 \end{matrix}\right\} + \left[\begin{matrix} 2\zeta_1\lambda_1 & -2\Omega \\ 2\Omega & 2\zeta_2\lambda_2 \end{matrix}\right] \left\{\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix}\right\} + \left[\begin{matrix} \lambda_1^2 - \Omega^2 & 0 \\ 0 & \lambda_2^2 - \Omega^2 \end{matrix}\right] \left\{\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\}$$

where ζ_1 and ζ_2 are the damping ratios for the two degrees of freedom.

The K-T-C theory implies that, for $\lambda_1 = \lambda_2 = \lambda$ and $\zeta_1 = \zeta_2 = \zeta$, there are two negative eigenvalues of the $[K]$ matrix when $\Omega > \lambda$; thus, there are two unstable roots when the eigenvalues of the $[D]$ matrix are positive. For positive definite $[K]$ (i.e. when $\Omega < \lambda$), one can have instability only when $\zeta < 0$.

3.2.8 Random excitation

$$\begin{aligned} \left[S_{y_i y_j}\right] &= [\phi] \left[\left\lceil \frac{1}{-i\omega - \eta} \right\rfloor\right] [\phi]^{-1} \left[S_{F_i F_j}\right] [\phi]^{-T} \left[\left\lceil \frac{1}{i\omega - \eta} \right\rfloor\right] [\phi]^T \\ &= \mathcal{H}^* \left[S_{F_i F_j}\right] \mathcal{H}^T \end{aligned}$$

Note if $\{F\} = [B]\{f\}$

$$\left[S_{F_i F_j}\right] = [B] \left[S_{f_i f_j}\right] [B]^T$$

$$R_{g_i g_j} = [\phi]^{-1} [B] \left[R_{f_i f_j}\right] [B]^T [\phi]^{-T}$$

$$R_{q_i q_j} = \int_0^\infty \int_0^\infty e^{\eta_i \lambda_i} e^{\eta_j \lambda_j} R_{g_i g_j}(\tau + \lambda_i - \lambda_j) d\lambda_i d\lambda_j$$

$$\left[R_{x_i x_j}\right] = [\phi] \left[R_{q_i q_j}\right] [\phi]^T$$

$$\begin{aligned} \mathcal{H} = [i\omega I + \overline{A}]^{-1} &= [\phi] \left[\left\lceil \frac{1}{i\omega - \eta} \right\rfloor\right] [\phi]^{-1} \\ \mathcal{H}^* = [-i\omega I + \overline{A}]^{-1} &= [\phi] \left[\left\lceil \frac{1}{-i\omega - \eta} \right\rfloor\right] [\phi]^{-1} \\ &= [\phi^*] \left[\left\lceil \frac{1}{-i\omega - \eta^*} \right\rfloor\right] [\phi^*]^{-1} \end{aligned}$$

Chapter 4

Continuous Systems

4.1 Classical approach: Newtonian or force method

4.1.1 Example: Vibrating String

$$F = ma \implies \left(T + \frac{1}{2}\frac{\partial T}{\partial x}dx\right)\left(\frac{\partial y}{\partial x} + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}dx\right) - \left(T - \frac{1}{2}\frac{\partial T}{\partial x}dx\right)\left(\frac{\partial y}{\partial x} - \frac{1}{2}\frac{\partial^2 y}{\partial x^2}dx\right)$$
$$+ \quad f dx = (\mu dx)\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial T}{\partial x}\frac{\partial y}{\partial x} + T\frac{\partial^2 y}{\partial x^2} + f = \left(\mu\frac{\partial^2 y}{\partial t^2}\right)$$
$$\mu\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = f$$
$$y(0,t) = y(l,t) = 0 \quad \implies \quad \text{fixed ends}$$
$$T\frac{\partial y}{\partial x}(0,t) = 0 \quad \implies \quad \text{no vertical force, } \textit{free left end}$$
$$T\frac{\partial y}{\partial x}(l,t) = 0 \quad \implies \quad \text{no vertical force, } \textit{free right end}$$

4.1.2 Example: Longitudinal vibrations of rods

$$\text{strain} \ \ \epsilon = \frac{\partial u}{\partial x}\ , \quad \sigma = E\frac{\partial u}{\partial x}\ , F = A\sigma = AE\frac{\partial u}{\partial x}$$
$$g dx + \left[AE\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right)dx\right] - \left[AE\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right)dx\right] = \mu\frac{\partial^2 u}{\partial t^2}$$
$$\mu\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = g \quad \begin{cases} u\ (0 \text{ or } l) = 0 \implies \text{fixed} \\ AE\ \partial u/\partial x\ (0 \text{ or } l) = 0 \implies \text{free} \end{cases}$$

Similarly, for torsion we have

$$\rho\mathcal{J}\frac{\partial^2\theta}{\partial t^2} - \frac{\partial}{\partial x}\left(GJ\frac{\partial\theta}{\partial x}\right) = M \qquad \begin{cases} \theta = 0 & \Longrightarrow \quad \text{fixed end} \\ GJ\theta' = 0 & \Longrightarrow \quad \text{free end} \end{cases}$$

For uniform mass and uniform tension (or AE) and no forcing function, above equations become

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \textbf{wave equation}$$

Same equation as for longitudinal sound waves in tube

$$\text{sound wave velocity} \quad v = \tfrac{\partial\phi}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\begin{cases} \frac{\partial\phi}{\partial x} = v = 0 \text{ closed end} \\ \phi = 0 \quad \text{open end} \end{cases}$$

4.1.3 Example: Beam

Vertical equilibrium:

$$\mu dx\ddot{v} = \left(V + \frac{\partial V}{\partial x}\frac{dx}{2}\right) - \left(V - \frac{\partial V}{\partial x}\frac{dx}{2}\right) + f dx$$

$$\mu\ddot{v} - \frac{\partial V}{\partial x} = f$$

Rotational equilibrium:

$$\left(M + \frac{\partial M}{\partial x}\frac{dx}{2}\right) - \left(M - \frac{\partial M}{\partial x}\frac{dx}{2}\right) + V dx = 0$$

$$-V = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 v}{\partial x^2}\right)$$

$$\Longrightarrow \mu\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) = f \qquad (\mu\ddot{v} + EIv'''' = f \;\; \text{if uniform})$$

Boundary Conditions:

a. End fixed in translation $\quad v = 0$

b. End fixed in rotation $\quad v' = 0$

c. End free in translation $\quad V = -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 v}{\partial x^2}\right) = 0$

d. End free in rotation $\quad M = EI\frac{\partial^2 v}{\partial x^2} = 0$

Both 4.1.3 and 4.1.3 are geometric boundary conditions; 4.1.3 and 4.1.3 are natural boundary conditions.

Fixed end $\Longrightarrow$ (a) & (b) Free end $\Longrightarrow$ (c) & (d)

Pinned end $\Longrightarrow$ (a) & (d) Roller end $\Longrightarrow$ (b) & (c)

16 possible cases, 10 of which are independent.

4.1.4 Flexibility Method

Let $G(x,\xi)$ be the displacement $w(x)$ due to a unit load at $x=\xi$. Thus,

$$w(x,t) = \int_0^1 G(x,\xi)[f(\xi) - \ddot{w}(\xi,t)\mu]d\xi$$

This is like a matrix equation of the form

$$\{w\} = [G]\{f\} - [G][\mu(x)]\{\ddot{w}\}$$

Note that $G(x,\xi)$ also called Green's function.

Example

String:

$$\begin{gathered}
T(\theta_1+\theta_2) = 1\ lb \\
wT\left(\frac{\xi}{L} + \frac{(L-\xi)}{L}\right) = 1\ lb \\
w = \frac{1}{T}\frac{\xi(L-\xi)}{L}\ ,\ wT\left(\frac{1}{\xi} + \frac{1}{L-\xi}\right) = 1 \\
G(x,\xi) = \begin{cases} \frac{x}{L}(L-\xi)\frac{1}{T} & x \le \xi \\ \frac{L-x}{L}\frac{\xi}{T} & x > \xi \end{cases} \\
w(x,t) = \int_0^x \frac{\xi}{T}\left(\frac{L-x}{L}\right)[f-\mu\ddot{w}]\ d\xi + \int_x^L \frac{L-\xi}{T}\left(\frac{x}{L}\right)[f-\mu\ddot{w}]d\xi
\end{gathered}$$

4.2 Energy method (Calculus of Variations)

4.2.1 Review Discrete Systems

$$\text{Lagrangian } = L = T - V \ , \delta W = f\delta q$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = f \quad \left(\frac{\partial V}{\partial \dot{q}} = 0\right)$$

$$J = \int_{t_1}^{t_2} L(t, q, \dot{q})dt \ , \quad \delta A \equiv \int_{t_1}^{t_2} \delta W \ dt - p\delta q|_{t_1}^{t_2}$$
$$\delta J = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right) dt \ , \quad \delta A = \int_{t_1}^{t_2} f\delta q \ dt - p\delta q|_{t_1}^{t_2}$$

Integrating by parts

$$\begin{aligned}\delta J + \delta A = \int_{t_1}^{t_2} \left[f + \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right] \delta q \ dt \quad &+ \quad \frac{\partial L}{\partial \dot{q}}\delta q|_{t_1}^{t_2} = 0 \\ &- \quad p\delta q|_{t_1}^{t_2}\end{aligned}$$

t_1, t_2 arbitrary

$$\Longrightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -f \ ; \quad p = \frac{\partial L}{\partial \dot{q}}$$

4.2.2 Lagrange's equations

Lagrange's equations form an extremum of $T - V$. Similarly, for spatial variable (potential energy only)

$$L = -\int_0^l \overline{V}(x, y, \frac{dy}{dx})dx$$

where $\overline{V}$ is potential energy per unit length.

$$\delta L + \delta W = \int_0^l \left[f - \frac{\partial \overline{V}}{\partial y} + \frac{d}{dx}\frac{\partial \overline{V}}{\partial y'}\right] \delta y dx - \frac{\partial \overline{V}}{\partial y'}\delta y|_0^l + F_i \delta y|_0^L = 0$$

Continuous system: space and time variable

$$L = T - V = \int_0^l F(t, x, u, \dot{u}, u')dx$$

$$\dot{u} = \frac{\partial u}{\partial t}\ ,\ u' = \frac{\partial u}{\partial x}\ ,\quad \delta W = \int_0^l f(x)\delta u\ dx$$

$$\int_{t_1}^{t_2}\int_0^l \left[f + \frac{\partial F}{\partial u} - \frac{\partial}{\partial \dot{u}} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u'}\right]\delta u\ dx\ dt + \int_{t_1}^{t_2}\frac{\partial F}{\partial u'}\delta u|_0^l\ dt = 0$$

$$+\int_{t_1}^{t_2} F_i \delta u|_0^\ell dt + \int_0^\ell \left(\frac{\partial F}{\partial \dot{u}} - p\right)\delta u\ dx|_{t_1}^{t_2}$$

$$\Longrightarrow \frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}} - \frac{d}{dx}\frac{\partial F}{\partial u'} = -f \qquad \text{Euler's equation of motion}$$

$$\left.\begin{array}{l}\frac{\partial F}{\partial u'}(0) = 0 \text{ or } u(0) = 0\\ \frac{\partial F}{\partial u'}(l) = 0 \text{ or } u(l) = 0\end{array}\right\}\quad \text{Boundary conditions}$$

Example: Vibrating String

$$T = \frac{1}{2}\int_0^l \mu\left(\frac{dy}{dt}\right)^2 dx\ ,\quad V = \int_0^l T\left[1 - \sqrt{1 - \left(\frac{dy}{dx}\right)^2}\right]dx$$

$$V \approx \int_0^l \frac{1}{2}T\left(\frac{dy}{dx}\right)^2 dx\ ,\quad L = \frac{1}{2}\int_0^l T\left[\mu\left(\frac{dy}{dx}\right)^2 - T\left(\frac{dy}{dx}\right)^2\right]dx$$

$$\frac{\partial F}{\partial y} = 0\ ,\ \frac{\partial F}{\partial \dot{y}} = \mu\frac{\partial y}{\partial t}\ ,\ \frac{\partial F}{\partial y'} = -T\frac{\partial y}{\partial x}$$

$$-\frac{\partial}{\partial t}\left(\mu\frac{\partial y}{\partial t}\right) - \frac{\partial}{\partial x}\left(-T\frac{\partial y}{\partial x}\right) = -f$$

$$\mu\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = f$$

It is noted that μ is not a function of time. If it is, Lagrange's equation is not applicable.

$$T\frac{\partial y}{\partial x} = 0 \quad \text{or} \quad y = 0 \text{ at ends}$$

Example: Longitudinal Vibration of Rods

$$T = \frac{1}{2}\int_0^l \mu\left(\frac{\partial u}{\partial t}\right)^2 \quad ,\quad V = \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$\Longrightarrow EA\frac{\partial u}{\partial x} = 0 \text{ or } u = 0 \text{ at boundaries}$$

Extensions

1. More dependent variables $F(t, x, u_i, \dot{u}_i, u_i')$

$$\delta W \int_0^l \sum f_i \delta u_i dx$$

$$\frac{\partial F}{\partial u_i} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{u}_i} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_i'} = -f_i$$

2. More derivatives $F(t, x, u, \dot{u}, \dot{u}', u', u'', u''')$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{u}} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u'} + \frac{\partial^2}{\partial x \partial t}\frac{\partial F}{\partial \dot{u}'} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial u''} - \frac{\partial^3}{\partial x^3}\frac{\partial F}{\partial u'''} = -f$$

Boundary conditions at $x = 0, l$

$$\left[\frac{\partial F}{\partial u'} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{u}'} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u''} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial u'''}\right] \delta u = 0$$

$$\left[\frac{\partial F}{\partial u''} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \dot{u}'''}\right] \delta u' = 0 \ , \quad \left[\frac{\partial F}{\partial u'''}\right] \delta u'' = 0$$

3. More independent variables $F(t, x, y, z, u, \dot{u}, u_x, u_y, u_z)$

$$f + \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{u}} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial F}{\partial u_z} = 0$$

Normal derivatives $\left(\frac{\partial F}{\partial u_n}\right) \delta u = 0$ on surface (edge).

4. Discrete potential and kinetic energy sources at boundaries.

$$\begin{aligned} L &= \int_0^l F(t, x, \dot{u}, u', u'') dx \\ &+ G[u(0,t), \dot{u}(0,t), u'(0,t), \dot{u}'(0,t), u(l,t), \dot{u}(l,t), u'(l,t), \dot{u}'(l,t)] \end{aligned}$$

$$J = \int_0^t L \ dt$$

$$\delta W = \int_0^l f(x) \delta u \ dx + f_l \ \delta u(l,t) + m_l \ \delta u'(l,t) + f_0 \ \delta u(0,t) + m_0 \ \delta u'(0,t)$$

$$
\begin{aligned}
\delta J + \delta A &= \int_{t_1}^{t_2} \left\{ \int_0^l \left[\delta u f + \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial \dot{u}}\delta \dot{u} + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u'' \right] dx \right. \\
&+ \frac{\partial G}{\partial u_0}\delta u_0 + \frac{\partial G}{\partial \dot{u}_0}\delta \dot{u}_0 + \frac{\partial G}{\partial \dot{u}'_0}\delta \dot{u}'_0 + \frac{\partial G}{\partial u'_0}\delta u'_0 \\
&+ \frac{\partial G}{\partial u_l}\delta u_l + \frac{\partial G}{\partial \dot{u}_l}\delta \dot{u}_l + \frac{\partial G}{\partial \dot{u}'_l}\delta \dot{u}'_l + \frac{\partial G}{\partial u'_l}\delta u'_l \\
&+ \left. f_l \delta u_l + m_l \delta u'_l + f_0 \delta u_0 + m_0\, \delta u'_0 \right\} dt
\end{aligned}
$$

Integration by parts yields (trailing time-terms dropped)

$$
\begin{aligned}
& \int_0^l \left[f + \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{u}} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u'} \right] \delta u \; dx \\
+ & \left[\frac{\partial F}{\partial u'_l} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{u}'_l} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u''_l} + \frac{\partial G}{\partial u_l} - \frac{d}{dt}\frac{\partial G}{\partial \dot{u}'_l} + f_l \right] \delta u_l \\
+ & \left[\frac{\partial F}{\partial u''_l} + \frac{\partial G}{\partial u'_l} - \frac{d}{dt}\frac{\partial G}{\partial \dot{u}'_l} + m_l \right] \delta u'_l \\
- & \left[\frac{\partial F}{\partial u'_0} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{u}'_0} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u''_0} - \frac{\partial G}{\partial u_0} + \frac{d}{dt}\frac{\partial G}{\partial \dot{u}'_0} - f_0 \right] \delta u_0 \\
- & \left[\frac{\partial F}{\partial u''_0} + \frac{\partial G}{\partial u'_0} + \frac{d}{dt}\frac{\partial G}{\partial \dot{u}'_0} - m_0 \right] \delta u'_0 = 0
\end{aligned}
$$

Boundary conditions become equations of motion.

Examples of Extensions

1. More independent variables

String in 2-dimensions

$$
\begin{aligned}
T_e &= \frac{1}{2}\int_0^l \rho\,[\dot{u}^2 + \dot{v}^2]\,dx \\
V &= \frac{1}{2}\int_0^l T\,[(u')^2 + (v')^2]\,dx \\
F &= \frac{1}{2}\rho(\dot{u}^2 + \dot{v}^2) - \frac{1}{2}T\,[(u')^2 + (v')^2]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u'} = 0 \quad &\Longrightarrow \quad \rho\ddot{u} = (Tu')' \\
\frac{\partial F}{\partial v} - \frac{d}{dt}\frac{\partial F}{\partial \dot{v}} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v'} = 0 \quad &\Longrightarrow \quad \rho\ddot{v} = (Tv')'
\end{aligned}
$$

2. More derivatives: vibrating beam

$$T_e = \frac{1}{2}\int_0^l \rho \left(\frac{\partial w}{\partial t}\right)^2 dx$$
$$V = \frac{1}{2}\int_0^l EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

$$F = \frac{1}{2}\left[\rho \left(\frac{\partial w}{\partial t}\right)^2 - EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] - F(t,x,w,w_t,w_{xx})$$
$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial t}\frac{\partial F}{\partial w_t} - \underbrace{\frac{\partial}{\partial x}\frac{\partial F}{\partial w_x}}_{=0} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{xx}} + f = 0$$
$$\rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = f$$

Boundary conditions:

$$f_\ell \delta w + \left(\frac{\partial F}{\partial w'} - \frac{\partial}{\partial x}\frac{\partial F}{\partial w''}\right)\delta w = 0 \quad \text{and} \quad \frac{\partial F}{\partial w''}\delta w' + m_\ell \delta w' = 0$$
$$f_\ell \delta w + \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w = 0 \ , -EI\frac{\partial^2 w}{\partial x^2}\delta w' + m_\ell \delta w' = 0$$
$$\implies \quad \text{Shear Force} \quad = 0 \quad (\text{free in } w) \quad \text{or} \quad w = 0 \quad (\text{fixed in } w)$$

and moment $= 0$ (free in slope) or $w' = 0$ (fixed slope)

$(EIw'')' = 0\ ,\quad EIw'' = 0$ free end

$(EIw'')' = 0\ ,\ w' = 0$ roller end

$w = 0\ ,\ EIw'' = 0$ pinned end

$w = 0\ ,\ w' = 0$ built-in or fixed end

3. More independent variables: vibrating square membrane

$$T_e = \frac{1}{2}\int_0^b\int_0^a \rho \left(\frac{\partial u}{\partial t}\right)^2 dx\ dy$$
$$V = \frac{1}{2}\int_0^b\int_0^a T \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\ dy$$
$$F = \frac{1}{2}\left[\rho \left(\frac{\partial u}{\partial t}\right)^2 - T \left(\frac{\partial u}{\partial x}\right)^2 - T \left(\frac{\partial u}{\partial y}\right)^2\right] = F(t,x,y,u,u_t,u_x,u_y)$$

For $\delta J + \delta A = 0$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\frac{\partial F}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_y} + f = 0$$
$$\rho\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(T\frac{\partial u}{\partial y}\right) = f$$

For $T = \text{constant}, \;\; f = 0$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \bigtriangledown^2 u \qquad \text{wave equation}$$

B.C.

$$T\frac{\partial u}{\partial n}\delta u = 0 \qquad \text{on boundary}$$
$$\Longrightarrow \qquad T\frac{\partial u}{\partial n} = 0\ , \quad \text{or} \quad u = 0 \quad \text{on boundary}$$

4. Lumped spring mass

Equation of motion:

$$\rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = 0 \qquad \text{(same as before)}$$
$$G = \frac{1}{2}M\dot{w}(l)^2 + \frac{1}{2}I\dot{w}'(l)^2 - \frac{1}{2}K_\theta w'(0)^2 - \frac{1}{2}K_w w(l)^2$$

$$\begin{aligned}\text{B.C.} \quad \Longrightarrow \quad & \frac{\partial F}{\partial w'}\delta w|_0^l + \frac{\partial F}{\partial w''}\delta w'|_0^l - \frac{\partial}{\partial x}\frac{\partial F}{\partial w''}\delta w|_0^l \\ + \quad & \frac{\partial G}{\partial w_l}\delta w_l + \frac{\partial G}{\partial w_0'}\delta w_0' - \frac{d}{dt}\frac{\partial G}{\partial \dot{w}_l}\delta w_l - \frac{d}{dt}\frac{\partial G}{\partial \dot{w}_l'}\delta w_l' = 0\end{aligned}$$

$$\text{at} \;\; x = 0 \begin{cases} \delta w = 0 \text{ (pinned)} & \Longrightarrow \quad w(0,t) = 0 \\ \delta w' \;\; \text{coeff.} = 0 & \Longrightarrow \quad EIw''(0,t) = K_\theta w'(0,t) \end{cases}$$
$$\text{at} \;\; x = l \begin{cases} \delta w \;\; \text{coeff.} \; = 0 \Longrightarrow & M\ddot{w}(l,t) + K_w w(l,t) = [EIw'']'(l,t) \\ \\ \delta w' \;\; \text{coeff.} \; = 0 \Longrightarrow & I\ddot{w}'(l,t) = -EIw''(l,t) \end{cases}$$

These boundary conditions can also be obtained from the force method.

4.2.3 Classical Solution of Continuous Systems

Example: Uniform string, transient vibrations

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ , \quad u(0) = u(l) = 0$$

Initial conditions: $u(x,0) = u_0(x)$, $\dot{u}(x,0) = \dot{u}_0(x)$

Mathematically, any function $f(x \pm ct)$ is a solution; but when boundary conditions are present, the use of this fact (called method of characteristics) is not always convenient. A more viable approach is "separation of variables."

$$u(x,t) = \phi(x)q(t) \qquad \frac{1}{c^2}\phi(x)\ddot{q}(t) - \phi''(x)q(t) = 0$$

$$\phi(0) = \phi(l) = 0$$

Before solving for transient response, we look for natural frequencies. So we look for $q(t) = \overline{q}e^{i\omega t}$

$$\Longrightarrow \overline{q}[\phi''(x) + \frac{\omega^2}{c^2}\phi(x)] = 0 \quad \text{eigenvalue problem}$$

$$\text{for } \ \overline{q} \neq 0 \ , \quad \phi'' + \frac{\omega^2}{c^2}\phi(x) = 0$$

$$\phi = a\cos\left(\frac{\omega x}{c}\right) + b\sin\left(\frac{\omega x}{c}\right)$$

$$\phi(0) = 0 \quad \Longrightarrow \quad a = 0$$

$$\phi(l) = 0 \quad \Longrightarrow \quad b\sin\left(\frac{\omega l}{c}\right) = 0 \quad \Longrightarrow \quad \frac{\omega l}{c} = n\pi \ , \ n = 1,2,3\ldots$$

$$\omega_n = \frac{n\pi c}{l} \ , \ n = 1,2,3\ldots$$

Comments:

1. This eigenvalue problem has infinite number of frequencies ω_n and mode shapes $b\sin(n\pi x/l) = \phi_n$.

2. Note arbitrary multiplier on modes.

3. Modes often normalized so that $\int_0^l \phi_n^2 dx = 1 \quad \Longrightarrow \quad b = \sqrt{2/l}$

4. When modes normalized $\int_0^l \phi_i\phi_j dx = \begin{cases} 0 & i \neq j \text{ orthogonal} \\ 1 & i = j \end{cases}$

   just like big identity matrix.

5. Modes complete: only $f(x) = \sum \alpha_i\phi_i(x)$ on interval $0 \to l$.

   Other boundary conditions $\begin{pmatrix} \text{longitudinal vibrations} \\ \text{torsional vibrations} \\ \text{sound waves} \end{pmatrix}$

Free-Free

$$\phi'(0) = \phi'(l) = 0 \ , \ \phi' = \frac{-\omega}{c} a \sin\left(\frac{\omega x}{c}\right) + \frac{\omega}{c} b \cos\left(\frac{\omega x}{c}\right)$$
$$\implies \quad b = 0 \ , \ \frac{\omega l}{c} = n\pi \ , \quad n = 0, 1, 2, 3 \ldots$$
$$\omega_n = \frac{n\pi c}{l} \ , \quad \phi_0 = \frac{1}{\sqrt{l}} \ , \quad \phi_n = \sqrt{\frac{2}{l}} \cos\left(\frac{n\pi x}{l}\right) \ , \ n \geq 1$$

Fixed-Free or Free-Fixed

$$\phi(0) = 0 \ , \ \phi'(l) = 0 \quad \implies \quad \frac{\omega l}{c} = \frac{n\pi}{2} \ , \quad n = 1, 3, 5, 7 \ldots$$
$$\omega_n = \frac{n\pi c}{2l} \ , \ n = 1, 3, 5, 7 \ldots \qquad \phi_n = \sqrt{2/l} \sin\left(\frac{n\pi x}{2l}\right)$$

Uniform Beam

$$\rho \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = 0$$

Separation of variables: $u(x,t) = w(x) q(t) \ , \ q = e^{i\omega t}$

$$\implies \quad EI \frac{d^4 w}{dx^4} - \rho\omega^2 w = 0 \implies \frac{d^4 w}{dx^4} - \beta^4 w = 0 \ , \ \beta = \cdot \sqrt[4]{\frac{\rho\omega^2}{EI}}$$
$$w = e^{\alpha x} \implies \alpha^4 - \beta^4 = 0 \ , \ \alpha = \pm\beta \ , \ \pm i\beta$$
$$w = \gamma_1 e^{\beta x} + \gamma_2 e^{-\beta x} + \gamma_3 e^{i\beta x} + \gamma_4 e^{-\beta x}$$

or

$$w = a \cosh \beta x + b \cos \beta x + c \sinh \beta x + d \sin \beta x$$

Ten possible boundary condition pairs:

1. fixed-fixed	6. pinned-roller
2. fixed-pinned	7. pinned-free
3. fixed-roller	8. roller-roller
4. fixed-free	9. roller-free
5. pinned-pinned	10. free-free

$$w' = \beta(a \sinh \beta x - b \sin \beta x + c \cosh \beta x + d \cos \beta x)$$
$$w'' = \beta^2(a \cosh \beta x - b \cos \beta x + c \sinh \beta x - d \sin \beta x)$$
$$w''' = \beta^3(a \sinh \beta x + b \sin \beta x + c \cosh \beta x - d \cos \beta x)$$
$$w'''' = \beta^4(a \cosh \beta x + b \cos \beta x + c \sinh \beta x + d \sin \beta x) = \beta^4 w$$

Example: Fixed-free beam (cantilever)

$$
\begin{gathered}
w(0) = w'(0) = 0 \qquad \text{geometric} \\
w''(l) = w'''(l) = 0 \qquad \text{natural} \\
w(0) = 0 \quad \Longrightarrow \quad a + b = 0 \;, \quad a = -b \\
w'(0) = 0 \quad \Longrightarrow \quad c + d = 0 \;, \quad c = -d \\
w = a(\cosh \beta x - \cos \beta x) + c(\sinh \beta x - \sin \beta x) \\
w''(l) = \beta^2 a(\cosh \beta l + \cos \beta l) + \beta^2 c(\sinh \beta l + \sin \beta l) = 0 \\
w'''(l) = \beta^3 a(\sinh \beta l - \sin \beta l) + \beta^3 c(\cosh \beta l + \cos \beta l) = 0 \\
\text{Det } = 0 \quad \Longrightarrow \quad \cosh^2 \beta l + \cos^2 \beta l + 2 \cosh \beta l \cos \beta l - \sinh^2 \beta l + \sin^2 \beta l = 0 \\
2(1 + \cosh \beta l \cos \beta l) = 0 \qquad , \qquad \cos \beta l = -1/ \cosh \beta l
\end{gathered}
$$

$$
\begin{aligned}
l\beta_1 &= 1.875 \\
l\beta_2 &= 4.694 \\
l\beta_n &\approx \frac{2n-1}{2}\pi \qquad n > 2
\end{aligned}
$$

4.3 Timoshenko Beam Theory

Historically, it was noted that Euler-Bernoulli beam theory predicts an infinite wave speed. Any disturbance is "felt" throughout the beam the instant it occurs. Moreover, studies of wave propagation through an elastic body showed that there was a branch of high-frequency response which Euler-Bernoulli theory completely misses. This led to the development of an improved beam theory by Timoshenko. In this handout we will derive both theories, specialized for homogeneous, isotropic, prismatic beams.

4.3.1 3-D Strain Energy

The 3-D strain and stress can be written as

$$\epsilon = \lfloor \epsilon_{11} \;\; \epsilon_{22} \;\; \epsilon_{33} \;\; 2\epsilon_{23} \;\; 2\epsilon_{31} \;\; 2\epsilon_{12} \rfloor^T \tag{4.1}$$

and

$$\sigma = \lfloor \sigma_{11} \;\; \sigma_{22} \;\; \sigma_{33} \;\; \sigma_{23} \;\; \sigma_{31} \;\; \sigma_{12} \rfloor^T \tag{4.2}$$

respectively. The 3-D strain energy per unit volume is thus

$$\Psi = \frac{1}{2}\epsilon^T D \epsilon \tag{4.3}$$

where D for isotropic materials is

$$D = \begin{bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{4.4}$$

with constitutive law as

$$\sigma = D\epsilon \tag{4.5}$$

The Bernoulli hypothesis assumes that stress components in the plane of the cross section are very small relative to others, because of the slenderness of the beam geometry. Thus, one can set $\sigma_{22} = \sigma_{23} = \sigma_{33} = 0$, which allows these equations to be simplified significantly to

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \end{Bmatrix} \tag{4.6}$$

and the strain energy is simply

$$\Psi = \frac{1}{2} \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \end{Bmatrix}^T \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \end{Bmatrix} \tag{4.7}$$

The cross-sectional strains ϵ_{22}, ϵ_{23}, and ϵ_{33} do not appear in the strain energy. Thus, the nonzero in-plane distortion caused by the Poisson effect does not appear explicitly in the strain energy, but it can be recovered. It is important to not that we are *not* assuming the cross section to be rigid in its own plane! In fact

$$\epsilon_{23} = 0 \quad \epsilon_{22} = \epsilon_{33} = -\nu\epsilon_{11} \tag{4.8}$$

The nonzero values of strains ϵ_{22} and ϵ_{33} show that the cross section is distorting.

4.3.2 Kinematics

The position vector to any material point in a prismatic, undeformed beam can be written as

$$\mathbf{r} = x_i\hat{\mathbf{a}}_i \tag{4.9}$$

where the unit vectors $\hat{\mathbf{a}}_i$ are along corresponding Cartesian axes x_i for i=1, 2, and 3; x_1 is along the beam axis, and x_2 and x_3 are cross-sectional coordinate axes. Assuming that the cross-sectional plane displaces by an amount $\mathbf{u}$ such that

$$\mathbf{u} = u_1(x_1,t)\hat{\mathbf{a}}_1 + u_2(x_1,t)\hat{\mathbf{a}}_2 + u_3(x_1,t)\hat{\mathbf{a}}_3 \tag{4.10}$$

and rotates by the small angles $\theta_i(x_1,t)$ about $\hat{\mathbf{a}}_i$ during deformation such that the unit vectors fixed in the cross-sectional frame $\hat{\mathbf{B}}_i$ become

$$\begin{aligned}\hat{\mathbf{B}}_1 &= \hat{\mathbf{a}}_1 + \theta_3\hat{\mathbf{a}}_2 - \theta_2\hat{\mathbf{a}}_3\\ \hat{\mathbf{B}}_2 &= -\theta_3\hat{\mathbf{a}}_1 + \hat{\mathbf{a}}_2 + \theta_1\hat{\mathbf{a}}_3\\ \hat{\mathbf{B}}_3 &= \theta_2\hat{\mathbf{a}}_1 - \theta_1\hat{\mathbf{a}}_2 + \hat{\mathbf{a}}_3\end{aligned} \tag{4.11}$$

one can then write the position vector to the same material point as

$$\mathbf{R} = (x_1+u_1)\hat{\mathbf{a}}_1 + u_2\hat{\mathbf{a}}_2 + u_3\hat{\mathbf{a}}_3 + x_2\hat{\mathbf{B}}_2 + x_3\hat{\mathbf{B}}_3 + \Lambda(x_2,x_3)\theta_1'\hat{\mathbf{B}}_1 \tag{4.12}$$

The last term is the out-of-plane Saint-Venant warping caused by torsion. Note that we are here ignoring the out-of-plane Saint-Venant warping caused by transverse shearing for the sake of simplicity. We will take it into account approximately below by introduction of shear-correction factors.

Defining $\mathbf{G}_i = \partial\mathbf{R}/\partial x_i$ and using the engineering strain definitions

$$\epsilon_{ij} = \frac{1}{2}\left(\hat{\mathbf{B}}_i\cdot\mathbf{G}_j + \hat{\mathbf{B}}_j\cdot\mathbf{G}_i\right) - \delta_{ij} \tag{4.13}$$

we can find

$$\begin{aligned}\epsilon_{11} &= \hat{\mathbf{B}}_1\cdot\mathbf{G}_1 - 1 = u_1' - x_2\theta_3' + x_3\theta_2' + \Lambda\theta_1''\\ 2\epsilon_{12} &= \hat{\mathbf{B}}_1\cdot\mathbf{G}_2 + \hat{\mathbf{B}}_2\cdot\mathbf{G}_1 = u_2' - \theta_3 + (\Lambda_2 - x_3)\,\theta_1'\\ 2\epsilon_{13} &= \hat{\mathbf{B}}_1\cdot\mathbf{G}_3 + \hat{\mathbf{B}}_3\cdot\mathbf{G}_1 = u_3' + \theta_2 + (\Lambda_3 + x_2)\,\theta_1'\end{aligned} \tag{4.14}$$

where Λ_2 and Λ_3 are the partial derivatives of Λ with respect to x_2 and x_3, respectively. The term involving θ_1'' can be ignored for cases other than thin-walled beams with open cross sections.

Making use of the angular velocity of the cross-sectional frame, given by

$$\boldsymbol{\omega} = \dot\theta_1\hat{\mathbf{a}}_1 + \dot\theta_2\hat{\mathbf{a}}_2 + \dot\theta_3\hat{\mathbf{a}}_3 \tag{4.15}$$

the above displacement field (excluding the torsional warping) can be used to find the velocity field, yielding the velocity of an arbitrary material point as $\mathbf{v} = v_1\hat{\mathbf{a}}_1 + v_2\hat{\mathbf{a}}_2 + v_3\hat{\mathbf{a}}_3$ where

$$\begin{aligned}v_1 &= \dot u_1 - x_2\dot\theta_3 + x_3\dot\theta_2\\ v_2 &= \dot u_2 - x_3\dot\theta_1\\ v_3 &= \dot u_3 + x_2\dot\theta_1\end{aligned} \tag{4.16}$$

4.3.3 Strain and Kinetic Energy

The kinetic energy is most easily developed, so we'll write that one first. The kinetic energy per unit volume is just $\rho \mathbf{v} \cdot \mathbf{v}/2$, or

$$\tau = \frac{1}{2}\rho \left(v_1^2 + v_2^2 + v_3^2 \right) \tag{4.17}$$

where ρ is the material density. The integral over the cross-sectional plane yields a total kinetic energy per unit length as

$$T = \frac{1}{2} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 & 0 & m\overline{x}_3 & -m\overline{x}_2 \\ 0 & m & 0 & -m\overline{x}_3 & 0 & 0 \\ 0 & 0 & m & m\overline{x}_2 & 0 & 0 \\ 0 & -m\overline{x}_3 & m\overline{x}_2 & \rho(I_2+I_3) & 0 & 0 \\ m\overline{x}_3 & 0 & 0 & 0 & \rho I_2 & \rho I_{23} \\ -m\overline{x}_2 & 0 & 0 & 0 & \rho I_{23} & \rho I_3 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \tag{4.18}$$

Here m is the mass per unit length, I_2 and I_3 are area moments of inertia for the cross section, I_{23} is the cross-sectional product of inertia, and $\overline{x}_2$ and $\overline{x}_3$ are the coordinates of the cross-sectional centroid. Note that $\rho(I_2 + I_3)$ is the torsional inertia.

The strain energy per unit length also involves integration over the cross-sectional plane of the 3-D strain energy. If the Saint-Venant warping function is known, we can evaluate the resulting cross-sectional integrals. This function is determined by solution of an appropriate cross-sectional boundary-value problem. The resulting strain energy per unit length is

$$U = \frac{1}{2} \begin{Bmatrix} u_1' \\ u_2' - \theta_3 \\ u_3' + \theta_2 \\ \theta_1' \\ \theta_2' \\ \theta_3' \end{Bmatrix}^T \begin{bmatrix} EA & 0 & 0 & 0 & EA\overline{x}_3 & -EA\overline{x}_2 \\ 0 & GA_2 & 0 & -GA_2 s_3 & 0 & 0 \\ 0 & 0 & GA_3 & GA_3 s_2 & 0 & 0 \\ 0 & -GA_2 s_3 & GA_3 s_2 & GJ & 0 & 0 \\ EA\overline{x}_3 & 0 & 0 & 0 & EI_2 & EI_{23} \\ -EA\overline{x}_2 & 0 & 0 & 0 & EI_{23} & EI_3 \end{bmatrix} \begin{Bmatrix} u_1' \\ u_2' - \theta_3 \\ u_3' + \theta_2 \\ \theta_1' \\ \theta_2' \\ \theta_3' \end{Bmatrix} \tag{4.19}$$

where s_2 and s_3 are the coordinates of the shear center, defined here as

$$\begin{aligned} GA_3 s_2 =& G \int_A (\Lambda_3 + x_2) dA \\ -GA_2 s_3 =& G \int_A (\Lambda_2 - x_3) dA \end{aligned} \tag{4.20}$$

It is important to note that without the warping caused by torsion, the shear center is at the centroid. Also, without the warping caused by transverse shearing, $GA_2 = GA_3 = GA$. However, when transverse shearing is taken into account, A_2 and A_3 are less than A, often written as kA where k is a nondimensional constant known as the shear correction factor. The value of k is a function of the cross-sectional geometry and Poisson's ratio for homogeneous isotropic beams. For homogeneous, isotropic beams with rectangular cross sections, $k \approx 5/6$ and is a function of the aspect ratio and Poisson's ratio. For other sections it may assume values much smaller but hardly ever any larger.

4.3.4 Euler-Bernoulli Theory

To obtain Euler-Bernoulli theory, one need only take the locus of shear centers as the x_1 axis, thus setting s_2 and s_3 equal to zero, and minimize the strain energy density from Timoshenko theory with respect to shear strain measures $u_2'-\theta_3$ and $u_3'+\theta_2$. Because there is no coupling between bending and shear for isotropic beams, this in effect sets $\theta_2 = -u_3'$ and $\theta_3 = u_2'$. Thus, the strain energy simplifies to

$$U = \frac{1}{2}\begin{Bmatrix} u_1' \\ \theta_1' \\ u_2'' \\ u_3'' \end{Bmatrix}^T \begin{bmatrix} EA & 0 & -EA\overline{x}_2 & -EA\overline{x}_3 \\ 0 & GJ & 0 & 0 \\ -EA\overline{x}_2 & 0 & EI_3 & -EI_{23} \\ -EA\overline{x}_3 & 0 & -EI_{23} & EI_2 \end{bmatrix} \begin{Bmatrix} u_1' \\ \theta_1' \\ u_2'' \\ u_3'' \end{Bmatrix} \tag{4.21}$$

and the kinetic energy is simplified by ignoring the rotary inertia terms connected with bending, so that

$$T = \frac{1}{2}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{\theta}_1 \\ \dot{u}_2' \\ \dot{u}_3' \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 & 0 & -m\overline{x}_2 & -m\overline{x}_3 \\ 0 & m & 0 & -m\overline{x}_3 & 0 & 0 \\ 0 & 0 & m & m\overline{x}_2 & 0 & 0 \\ 0 & -m\overline{x}_3 & m\overline{x}_2 & \rho(I_2+I_3) & 0 & 0 \\ -m\overline{x}_2 & 0 & 0 & 0 & 0 & 0 \\ -m\overline{x}_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{\theta}_1 \\ \dot{u}_2' \\ \dot{u}_3' \end{Bmatrix} \tag{4.22}$$

It should be noted that for the Euler-Bernoulli theory the cross-sectional integrals are with respect to the shear center, whereas for the Timoshenko theory they are for the point at which the x_1 axis passes through the section, which is arbitrary.

4.3.5 Simplifications for Planar Deformation

For bending and shearing deformation in the 1-2 plane we can simplify the theory considerably to obtain

$$U = \frac{1}{2}\int_0^\ell \left[EI{\theta_3'}^2 + GAk(u_2' - \theta_3)^2 \right] dx_1 \tag{4.23}$$

where $EI = EI_3$ and $GAk = GA_2$. The first term is the strain energy due to bending. The physical phenomenon of transverse shearing deformation is seen in the second term, and its effect compared to the first term is of the order of h^2/L^2 compared to unity, where L is the wavelength of the deformation (i.e., $L \approx \ell/n$ where n is the number of waves over the span of the beam).

The kinetic energy from the Timoshenko theory reduces to

$$T = \frac{1}{2}\int_0^\ell (m\dot{u}_2^2 + \rho I\dot{\theta}_3^2)dx_1 \tag{4.24}$$

where $m = \rho A$ and $I = I_3$. The first term is the same as from Euler-Bernoulli theory. The second is the rotary inertia term, and its effect compared to the first term is also of the order of h^2/L^2 compared to unity.

We may invoke Hamilton's principle to derive the governing equations and boundary conditions. The statement of Hamilton's principle for this problem is

$$\delta\int_{t_1}^{t_2}\frac{1}{2}\int_0^\ell \left[m\dot{u}_2^2 + \rho I\dot{\theta}_3^2 - EI{\theta_3'}^2 - GkA(u_2' - \theta_3)^2\right]dx_1 dt = 0 \tag{4.25}$$

Carrying out the variation, integrating by parts in time, and assuming that the variations of u_2 and θ_3 vanish at the times t_1 and t_2, one obtains the weak form

$$\int_0^\ell \left[EI\theta_3'\delta\theta_3' + GAk(u_2' - \theta_3)(\delta u_2' - \delta\theta_3) + m\ddot{u}_2\delta u_2 + \rho I\ddot{\theta}_3\delta\theta_3\right]dx_1 = 0 \tag{4.26}$$

Since the highest spatial derivative is 1, the essential boundary conditions affect only the functions u_2 and θ_3. Spatial integration by parts shows that at both ends either u_2 or the shear force $V = GAk(u_2' - \theta_3)$ must vanish; similarly, either θ_3 or the bending moment $M = EI\theta_3'$ must vanish. The resulting Euler-Lagrange equations are

$$\begin{aligned}\rho I\ddot{\theta}_3 - (EI\theta_3')' - GAk(u_2' - \theta_3) &= 0\\ m\ddot{u}_2 - [GAk(u_2' - \theta_3)]' &= 0\end{aligned} \tag{4.27}$$

From the equations of motion, the reduction to Euler-Bernoulli theory is straightforward. First, we note from the weak form that if GAk tends to infinity, then $\theta_3 = u_2'$. The simplified weak form is then

$$\int_0^\ell (EIu_2''\delta u_2'' + m\ddot{u}_2\delta u_2 + \rho I\ddot{u}_2'\delta u_2')\,dx_1 = 0 \tag{4.28}$$

The last term can be discarded, because Euler-Bernoulli theory assumes that $h^2 << L^2$. It is noted that one cannot get the moment equation of motion, i.e., the first of the Timoshenko Euler-Lagrange equations, from the Euler-Bernoulli energy.

To undertake this reduction from the Euler-Lagrange equations, first, we note that for finite V and infinite GAk, $\theta_3 = u_2'$. Next, we ignore the rotary inertia term in the first of Eqs. (4.27). Finally, solving that equation for V, one finds that $V = -M'$ which, when substituted into the second of Eqs. (4.27), yields

$$m\ddot{u}_2 + (EIu_2'')'' = 0 \tag{4.29}$$

The boundary conditions transform in a similar way.

4.4 Solution for Uniform, Simply-Supported Beam

The governing equations for a uniform beam are

$$\begin{aligned} \rho I\ddot{\theta}_3 - EI\theta_3'' - GAk(u_2' - \theta_3) &= 0 \\ m\ddot{u}_2 - GAk(u_2' - \theta_3)' &= 0 \end{aligned} \tag{4.30}$$

and the boundary conditions for the simply-supported case are simply that $u_2(0,t) = u_2(\ell,t) = 0$ and $M(0,t) = M(\ell,t) = 0$.

4.4.1 General Solution

For convenience, let us nondimensionalize the governing equations and boundary conditions. Let

$$u_2 = \ell v \exp(i\omega t) \qquad \theta_3 = \theta \exp(i\omega t) \qquad x_1 = \ell x \tag{4.31}$$

and introduce the nondimensional parameters

$$e = \frac{E}{Gk} \qquad \sigma^2 = \frac{I}{A\ell^2} \qquad \Omega^2 = \frac{m\omega^2\ell^4}{EI} \tag{4.32}$$

Note here that σ plays the role of a slenderness parameter of the order of h/ℓ. With these substitutions, one can show that Eqs. (4.30) become

$$\begin{aligned} \theta'' + \frac{1}{\sigma^2 e}(v' - \theta) + \sigma^2\Omega^2\theta &= 0 \\ \sigma^2\Omega^2 e v + v'' - \theta' &= 0 \end{aligned} \tag{4.33}$$

To solve these equations, we let $v = \overline{v}\exp(\beta x)$ and $\theta = \overline{\theta}\exp(\beta x)$. In order for a non-trivial solution of this form to exist, one must have

$$\begin{vmatrix} \beta^2 - \frac{1}{\sigma^2 e} + \sigma^2\Omega^2 & \frac{1}{\sigma^2 e}\beta \\ -\beta & \sigma^2\Omega^2 e + \beta^2 \end{vmatrix} = 0 \tag{4.34}$$

so that

$$\beta^4 + \sigma^2\Omega^2(1+e)\beta^2 + \sigma^4\Omega^4 e - \Omega^2 = 0 \tag{4.35}$$

This biquadratic has the solution

$$\beta^2 = -\frac{\sigma^2\Omega^2(1+e)}{2} \pm \Omega\sqrt{1 + \frac{\sigma^4\Omega^2}{4}(1-e)^2} \tag{4.36}$$

Since the radicand is > 0, we will always have at least two imaginary roots. These are obained by taking the minus and letting $\mu^2 = -\beta_-^2$ to obtain

$$\mu^2 = \frac{\sigma^2\Omega^2(1+e)}{2} + \Omega\sqrt{1 + \frac{\sigma^4\Omega^2}{4}(1-e)^2} \tag{4.37}$$

Taking the plus, one finds that two situations can arise concerning the nature of the other two roots. One can either have $\beta_+^2 > 0$ or $\beta_+^2 < 0$. The determining factor for which of these cases is true can be found by setting $\beta_+^2 = 0$ so that

$$\frac{\sigma^2\Omega^2(1+e)}{2} = \Omega\sqrt{1+\frac{\sigma^4\Omega^2}{4}(1-e)^2} \tag{4.38}$$

Simplifying the above expression, one finds that $\beta_+^2 > 0$ if

$$\Omega^2 < \frac{1}{e\sigma^4} \tag{4.39}$$

We call this the "low-frequency branch" because it corresponds to the Euler-Bernoulli solution, albeit with slightly different results. The solution for v becomes

$$v = a\sin(\mu x) + b\cos(\mu x) + c\sinh(\beta_+ x) + d\cosh(\beta_+ x) \tag{4.40}$$

The other root is for the case when $\beta_+^2 < 0$ for which

$$\Omega^2 > \frac{1}{e\sigma^4} \tag{4.41}$$

This is the so-called "high-frequency" branch, which has no counterpart in the Euler-Bernoulli theory. For this case, we define $\gamma^2 = -\beta_+^2$ so that

$$\gamma^2 = \frac{\sigma^2\Omega^2(1+e)}{2} - \Omega\sqrt{1+\frac{\sigma^4\Omega^2}{4}(1-e)^2} \tag{4.42}$$

and the solution for v is

$$v = a\sin(\mu x) + b\cos(\mu x) + c\sin(\gamma x) + d\cos(\gamma x) \tag{4.43}$$

Notice the completely different form of the solution. The borderline case, for which $\Omega^2 = \frac{1}{e\sigma^4}$ gives $v = 0$ and $\theta = \theta(t)$. This is the so-called "pure shear" mode, which does not exist for all boundary conditions.

4.4.2 Application of Boundary Conditions

The boundary conditions on v and θ are simply that $v(0) = v(1) = 0$ and $\theta'(0) = \theta'(1) = 0$. We note that once v is known, one can find θ by using the second of Eqs. (4.33) which yields $\theta' = v'' + \sigma^2\Omega^2 ev$. Thus, $\theta'(0) = v''(0)$ and $\theta'(1) = v''(1)$.

Low-Frequency Branch

Considering the low-frequency branch, Eq. (4.40), the boundary conditions at $x = 0$ give $b + d = 0$ and $d\beta^2 - b\mu^2 = 0$. Since $\mu^2 + \beta^2 > 0$, this leads to $b = d = 0$. Thus, the boundary conditions at $x = 1$ give

$$\begin{aligned} v(1) =& a \sin\mu + c \sinh\beta = 0 \\ v''(1) =& -a\mu^2 \sin\mu + c\beta^2 \sinh\beta = 0 \end{aligned} \tag{4.44}$$

The determinant of the coefficients must vanish, yielding

$$(\beta^2 + \mu^2) \sin\mu \sinh\beta = 0 \tag{4.45}$$

Since $\mu^2 + \beta^2 > 0$ and β does not vanish, we must have $\sin\mu = 0$ so that $\mu = n\pi$ and $c = 0$. Therefore,

$$v = a \sin(n\pi x) \tag{4.46}$$

Putting the solution $\mu = n\pi$ back into its definition and simplifying, one obtains

$$\sigma^4 \Omega_n^4 e - \Omega_n^2 [1 + (n\pi)^2 \sigma^2 \Omega_n^2 (1 + e)] + (n\pi)^4 = 0 \tag{4.47}$$

The smaller root is given by

$$\Omega_n^2 = \frac{1 + (n\pi)^2\sigma^2(1+e) - \sqrt{1 + 2(n\pi)^2\sigma^2(1+e) + (n\pi)^4\sigma^4(1-e)^2}}{2e\sigma^4} \qquad n = 1, 2, \ldots \tag{4.48}$$

Recognizing that the Euler-Bernoulli solution is $\Omega_n^2 = (n\pi)^4$, we can normalize the above to get a better idea of the effect of the correction on the low-frequency branch. Letting $\lambda_n = \Omega_n/(n\pi)^2$, one finds that

$$\begin{aligned} \lambda_n^2 =& \frac{1 + (n\pi)^2\sigma^2(1+e) - \sqrt{1 + 2(n\pi)^2\sigma^2(1+e) + (n\pi)^4\sigma^4(1-e)^2}}{2e(n\pi)^4\sigma^4} \qquad n = 1, 2, \ldots \\ =& 1 - (1+e)(n\pi)^2\sigma^2 + O(\sigma^4) \qquad n = 1, 2, \ldots \end{aligned} \tag{4.49}$$

The series approximation shows that as long as σ is a small parameter, the correction to the frequency for the lowest mode (n=1) is not significant. However, as shown in Fig. 4.1 the correction becomes more significant as any one of these parameters become larger: the mode number n, the slenderness parameter σ, or the stiffness ratio e. Moreover, the correction to the frequency is clearly $O(h^2/L^2)$ as expected.

High-Frequency Branch

Considering the high-frequency branch, Eq. (4.43), the boundary conditions show that $b+d = 0$ and $b\mu^2 + d\gamma^2 = 0$. Thus, $b = d = 0$ since $\mu^2 - \gamma^2 \neq 0$. The frequency determinant shows that $\gamma = n\pi$ and $a = 0$ so that

$$v = c \sin(n\pi x) \tag{4.50}$$

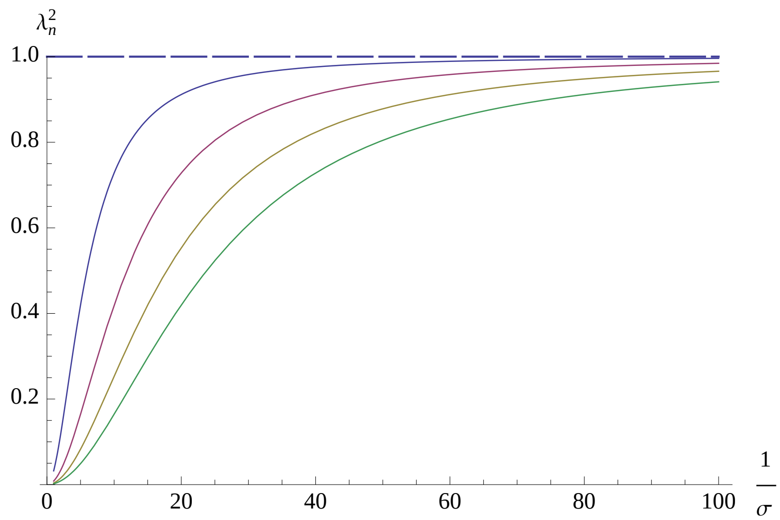

Figure 4.1: Plot of the ratios of the first four frequencies of the classical branch relative to their Euler-Bernoulli counterparts for $e = 3$ (lower modes at top); note that the slenderer the beam or the lower the mode number, the closer the result is to unity

and, using the same arguments as above,

$$\Omega_n^2 = \frac{1 + (n\pi)^2\sigma^2(1+e) + \sqrt{1 + 2(n\pi)^2\sigma^2(1+e) + (n\pi)^4\sigma^4(1-e)^2}}{2e\sigma^4} \geq \frac{1}{e\sigma^4} \quad n = 0, 1, 2, \ldots \tag{4.51}$$

Thus, there is an infinite number of high-frequency modes approaching $1/(e\sigma^4)$ as n becomes small. The $n = 0$ mode is the "pure shear" mode, which does not exist for all boundary conditions but does for the simply-supported case. (The reader should verify this by substitution into the governing equations and boundary conditions.) The frequencies of the other modes of the high-frequency branch increase slightly above the pure shear mode as the mode number increases. Introducing $\nu_n^2 = e\sigma^4\Omega_n^2$, one can plot the various values of ν_n^2 versus $1/\sigma$, as in Fig. 4.2. Notice that as the beam becomes slenderer, all the frequencies tend to "pile up" on that of the pure shear mode.

For slender beams, frequencies of these modes may be much larger than those of the classical branch. Note, however, that when the formulae for the frequencies are evaluated in a straightforward manner, there may be overlap between the frequencies of the two types of modes for beams that are not so slender. Indeed, the overlapping area shows the two sets of frequencies mingling; see Fig. 4.3. This happens when frequencies from the classical branch exceed the lowest frequency of the high-frequency branch, which is $1/(e\sigma^4)$ for the pinned-pinned beam. This frequency is sometimes called the cut-off frequency. Table 4.4.2 shows the first 16 normalized natural frequencies (Ω_n) from Timoshenko theory along with Euler-Bernoulli frequencies for modes that correspond to the low-frequency branch.

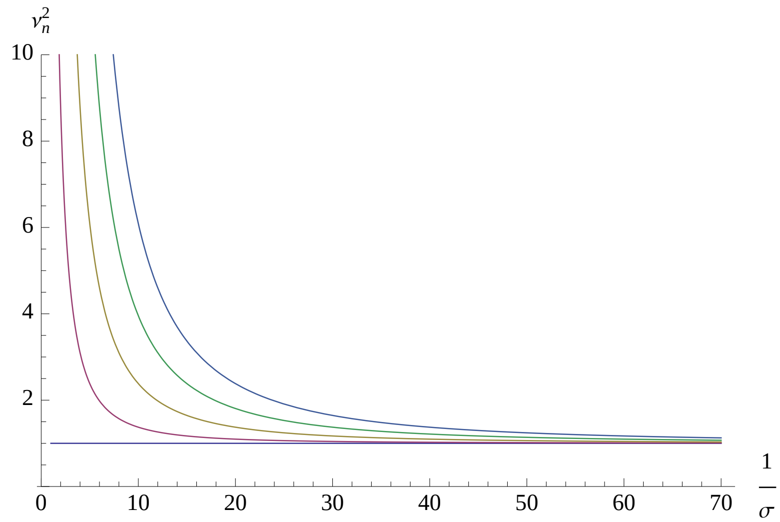

Figure 4.2: Plot of the ratios of the first four frequencies of the high-frequency branch multiplied by $e\sigma^4$, $e = 3$ (lower modes at bottom); note that the slenderer the beam or the lower the mode number, the closer the result is to unity

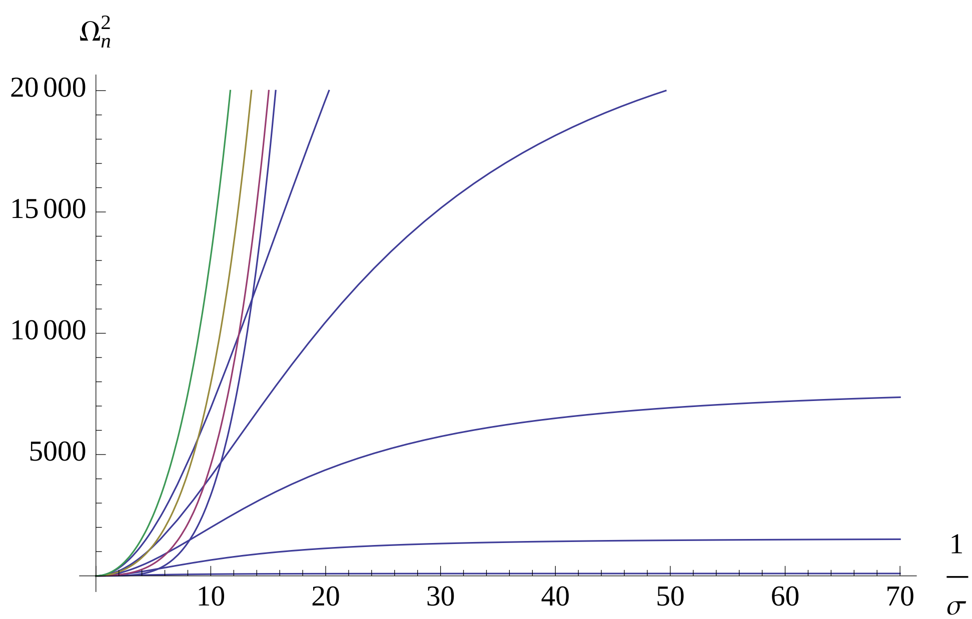

Figure 4.3: Plot of the first five frequencies of the classical and high-frequency branches for $e = 3$

Euler-Bernoulli	Timoshenko
9.86960	9.42302
39.4784	33.6856
88.8264	66.0906
157.914	102.297
246.740	140.160
355.306	178.672
483.611	217.361
–	230.940
–	241.885
631.655	256.008
–	270.654
799.438	294.517
–	310.386
986.960	332.850
–	356.496
1194.22	371.001

Table 4.1: Exact normalized frequencies (Ω_n) from Euler-Bernoulli and Timoshenko beam theories for $e = 3$ and $\sigma = 0.05$

Analysis of Mode Shapes

For both regimes, the solution has the form

$$\begin{aligned} v_n &= a_n \sin(n\pi x) \\ \theta_n &= a_n \cos(n\pi x) \left(n\pi - \frac{\sigma^2 \Omega_n^2 e}{n\pi} \right) \end{aligned} \tag{4.52}$$

In the "classical" branch, $\Omega_n^2 = O(n^4\pi^4)$ so that

$$\theta_n \approx a_n n\pi \cos(n\pi x) \left[1 - e\sigma^2 (n\pi)^2 \right] \tag{4.53}$$

which means that the transverse shear strain, proportional to $\Gamma = v' - \theta$, is of the form

$$\Gamma = v' - \theta \approx a_n e\sigma^2 (n\pi)^3 \cos(n\pi x) \tag{4.54}$$

Thus, a direct way to compare the mode shape from Timoshenko theory with that from the Euler-Bernoulli theory is to normalize all results for maximum bending θ' of unity and look at Γ. The larger n becomes, the more shear there is in the mode shape and thus the more inaccuracy is present in the Euler-Bernoulli analysis. The error is also proportional to $e\sigma^2$. For the isotropic case, e is of the order of 3, but σ can vary significantly depending on the

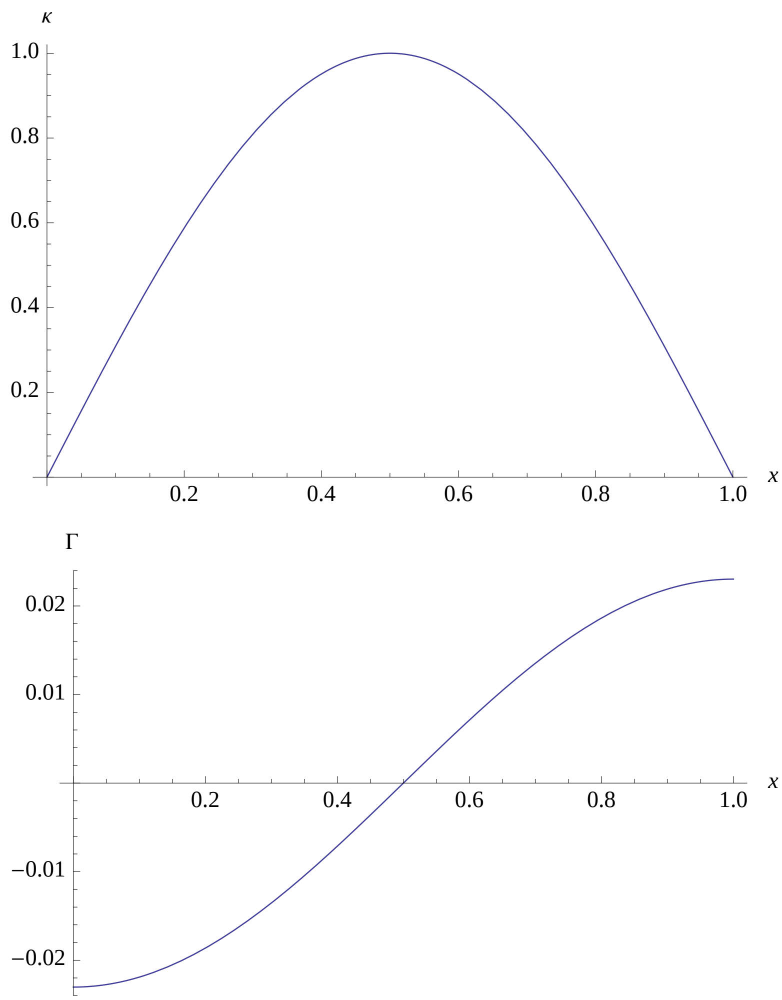

Figure 4.4: Bending and transverse shear (normalized to unit bending amplitude) in classical branch for $a_n = 1$, $n = 1$, $e = 3$, and $\sigma = 0.05$

slenderness of the beam. The slenderer the beam, the less transverse shear is present in the modes of the classical branch. Figs. 4.4 – 4.6 show this progression. Note that increasing σ from 0.05 to 0.2 increases the magnitude of shear by a factor of about 15, while increasing n from 1 to 2 increases the magnitude of shear by a factor of about 2.

For the high-frequency branch, $\Omega_n^2 = O(e^{-1}\sigma^{-4})$. Thus, the transverse shear is proportional to

$$\Gamma = v' - \theta \approx \frac{a_n \cos(n\pi x)}{n\pi\sigma^2} \tag{4.55}$$

Thus, the amount of transverse shear present in the modes of the high-frequency branch increases for slenderer beams and decreases as mode number increases, just the opposite as in the classical branch. Figs. 4.7 – 4.9 show this trend. First note that the magnitude of the

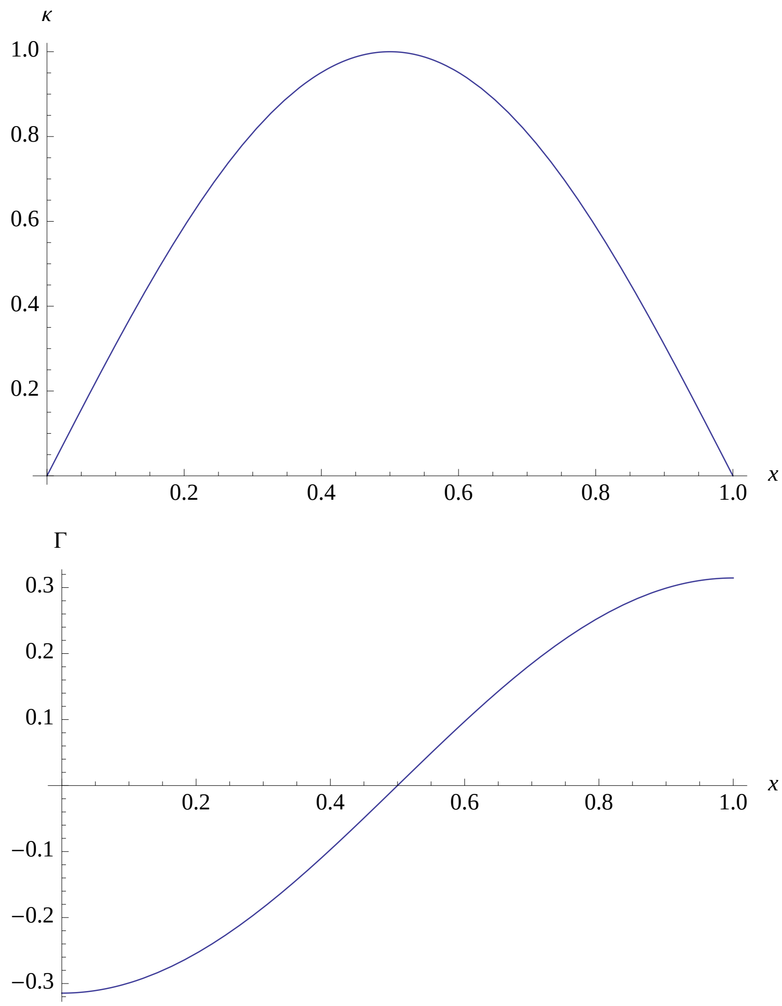

Figure 4.5: Bending and transverse shear (normalized to unit bending amplitude) in classical branch for $a_n = 1$, $n = 1$, $e = 3$, and $\sigma = 0.2$

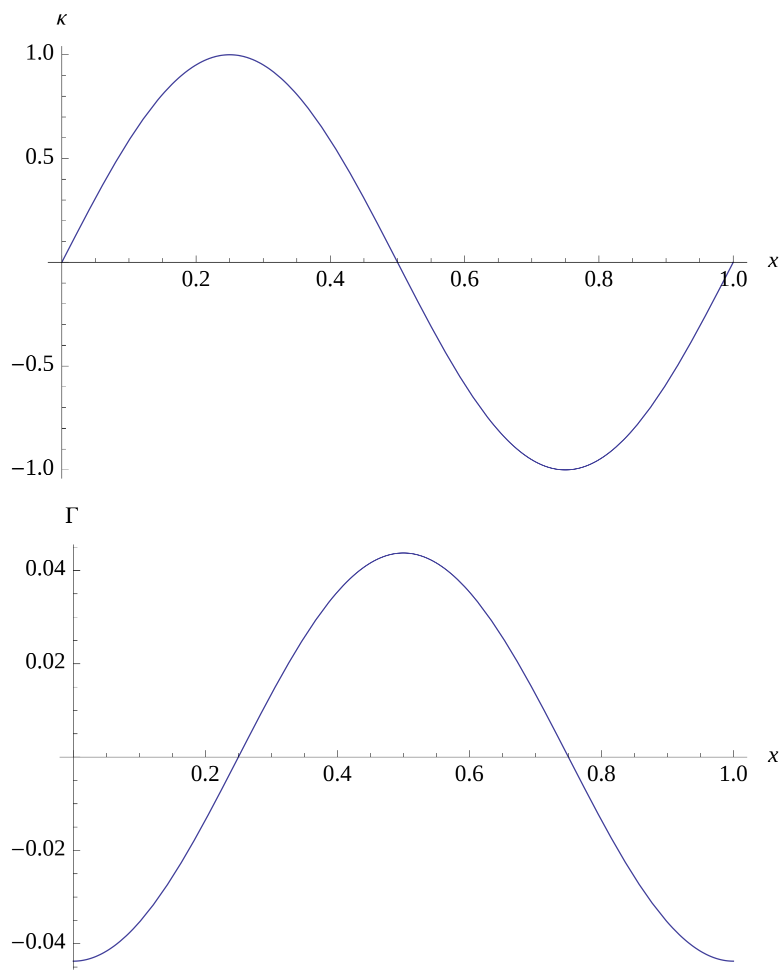

Figure 4.6: Bending and transverse shear (normalized to unit bending amplitude) in classical branch for $a_n = 1$, $n = 2$, $e = 3$, and $\sigma = 0.05$

shear is much larger in these modes than in those of the classical branch. Second, note that making the beam slenderer or raising the mode number both tend to reduce the magnitude of the transverse shear in the vibration modes. Consistent with this trend is the fact that the mode for $n = 0$ has no bending in it whatsoever; it is a pure shear mode with frequency equal to $\Omega_0 = 1/(\sqrt{e}\sigma^2)$.

4.5 Membranes and Plates

The resulting theory for membranes and plates is pretty standard in textbooks, but the underlying foundations are not so well known. Here is one point of view for how to get there.

4.5.1 Derivation of Strain Energy

3-D Strain Energy/Constitutive Relations

As with the earlier discussion of the beam, we start with the 3-D strain energy per unit volume for an isotropic material, in a form suitable for use with Cartesian coordinates

$$2U_3 = \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ \Gamma_{33} \\ 2\Gamma_{23} \\ 2\Gamma_{31} \\ 2\Gamma_{12} \end{Bmatrix}^T \begin{bmatrix} \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ \Gamma_{33} \\ 2\Gamma_{23} \\ 2\Gamma_{31} \\ 2\Gamma_{12} \end{Bmatrix} \tag{4.56}$$

where the Γ_{ij} terms are the strain components, E is the Young's modulus, $G = \frac{E}{2(1+\nu)}$ is the shear modulus, and ν is Poisson's ratio. It is well known that the stress components, σ_{ij}, in the same Cartesian frame are

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ \Gamma_{33} \\ 2\Gamma_{23} \\ 2\Gamma_{31} \\ 2\Gamma_{12} \end{Bmatrix} \tag{4.57}$$

For membranes and plates, one may safely assume that stresses through the thickness are much smaller than others. For a coordinate system with x_3 through the thickness, this means that

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \tag{4.58}$$

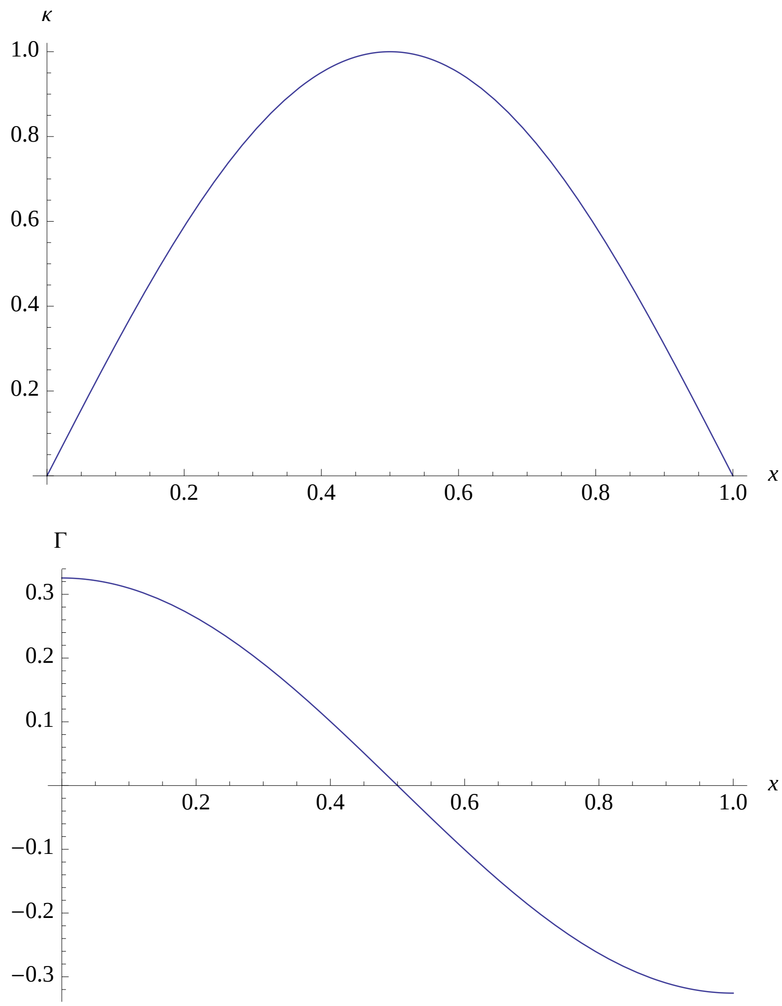

Figure 4.7: Bending and transverse shear (normalized to unit bending amplitude) in high-frequency branch for $a_n = 1$, $n = 1$, $e = 3$, and $\sigma = 0.05$

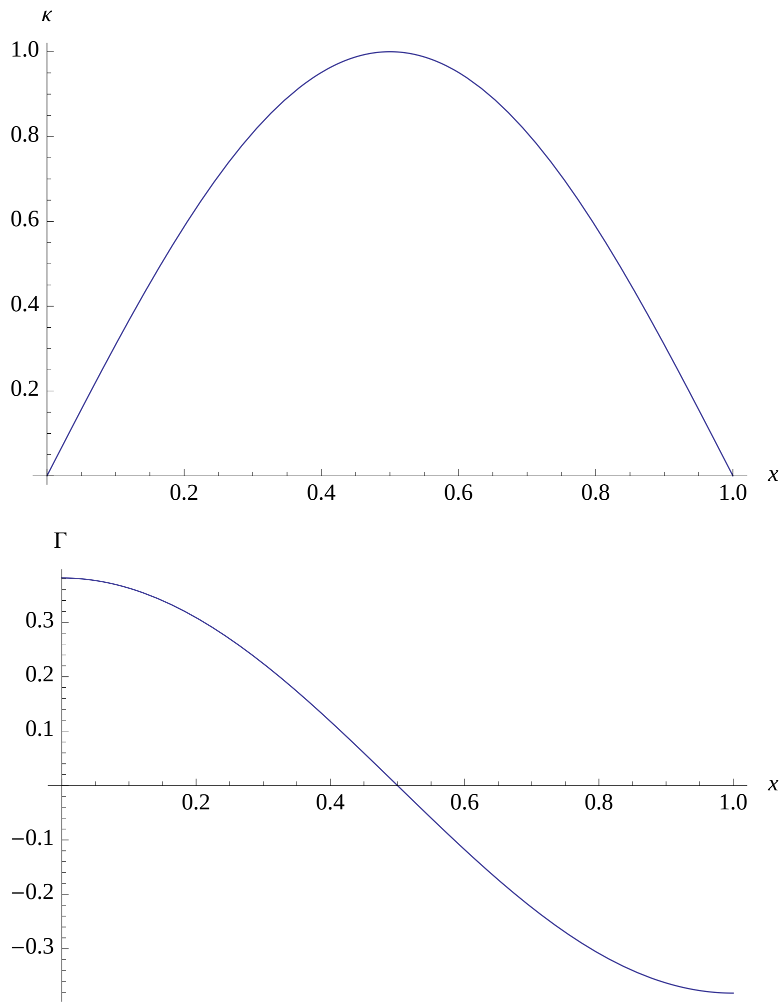

Figure 4.8: Bending and transverse shear (normalized to unit bending amplitude) in high-frequency branch for $a_n = 1$, $n = 1$, $e = 3$, and $\sigma = 0.2$

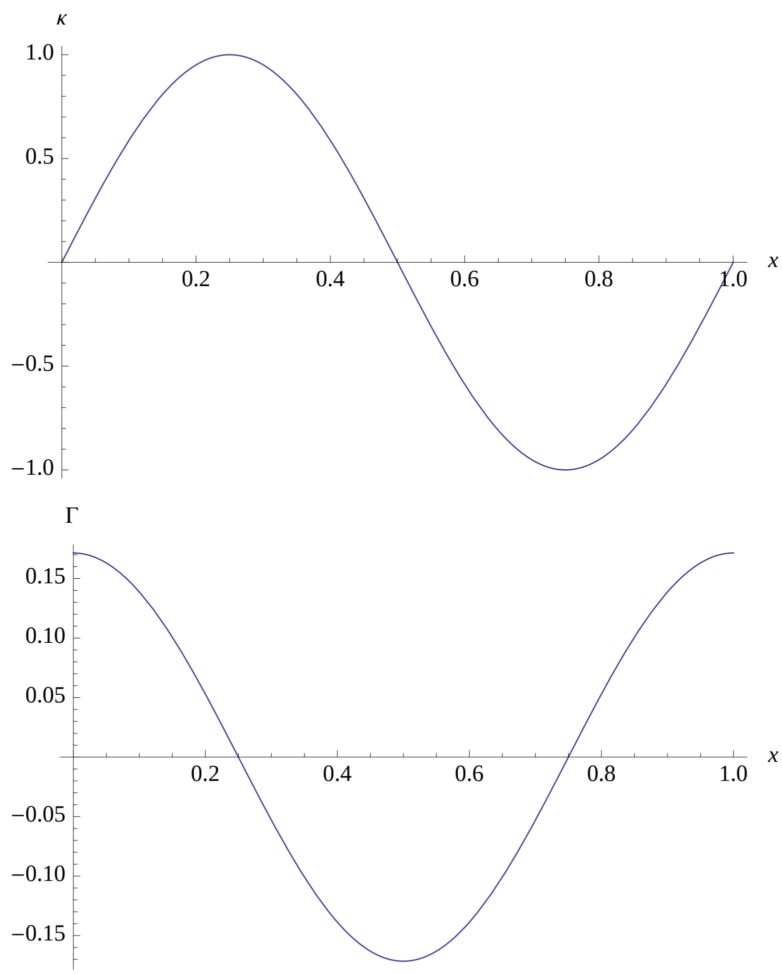

Figure 4.9: Bending and transverse shear (normalized to unit bending amplitude) in high-frequency branch for $a_n = 1$, $n = 2$, $e = 3$, and $\sigma = 0.05$

After substitution of Eq. (4.58) into Eq. (4.57), one obtains

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 \\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ 2\Gamma_{12} \end{Bmatrix} \tag{4.59}$$

Both membrane and thin-plate theories may be derived from this reduced form of Hooke's law, sometimes referred to as the plane-stress-reduced form. The corresponding strain energy per unit volume can then be written as

$$2U_3^* = \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ 2\Gamma_{12} \end{Bmatrix}^T \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 \\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ 2\Gamma_{12} \end{Bmatrix} \tag{4.60}$$

2-D Strain Energy/Constitutive Relations

The position vector to any point in a flat plate or membrane can be written as

$$\bar{\mathbf{r}} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = x_i\mathbf{a}_i \tag{4.61}$$

where x_i are the Cartesian coordinates, and $\mathbf{a}_i$ are unit vectors in those respective directions.

The position vector to the same point in the deformed plate or membrane may be written as

$$\overline{\mathbf{R}} = x_\alpha\mathbf{a}_\alpha + u_i(x_1, x_2)\mathbf{a}_i + x_3\mathbf{B}_3(x_1, x_2) + w_i(x_1, x_2, x_3)\mathbf{B}_i(x_1, x_2) \tag{4.62}$$

where $\mathbf{B}_i$ is a triad associated with the deformed plate such that the normal to the deformed plate middle surface is $\mathbf{B}_3$, and w_i is the warping of a line element that prior to deformation is perpendicular to the undeformed plate. As before, there is an implied summation of repeated indices over their range. This expression exhibits that most of the deformation in the plate is captured by displacements of the middle surface (u_i) and rotation of the normal $\mathbf{B}_3$. The remainder is found in w_i, which can be shown to be $O(h\varepsilon)$ where h is the thickness of the plate and ε is the order of the maximum strain in the plate.

The covariant basis vectors for the deformed state are

$$\begin{aligned} \mathbf{G}_\alpha &= \overline{\mathbf{R}}_{,\alpha} = (\delta_{\alpha\beta} + u_{\beta,\alpha})\mathbf{a}_\beta + u_{3,\alpha}\mathbf{a}_3 + x_3\mathbf{B}_{3,\alpha} + O\left(\frac{h}{\ell}\varepsilon, \varepsilon^2\right) \\ \mathbf{G}_3 &= \overline{\mathbf{R}}_{,3} = \mathbf{B}_3 + w_{i,3}\mathbf{B}_i \end{aligned} \tag{4.63}$$

where $(\bullet)_{,i} = \partial(\bullet)/\partial x_i$ and where Greek indices only vary from 1 to 2. The neglected terms in $\mathbf{G}_\alpha$ stem from the order of the warping and the introduction of the fact that $w_{i,\alpha} = O(\frac{w_i}{\ell})$ with ℓ being the wavelength of the deformation in the plate or membrane. To calculate the strain energy for a plate or membrane, then, we can introduce the membrane strains as $\epsilon_{\alpha\beta}$ and the bending/twisting strains as $\kappa_{\alpha\beta}$ where

$$(\delta_{\alpha i} + u_{i,\alpha})\mathbf{a}_i = (\delta_{\alpha\beta} + \epsilon_{\alpha\beta})\mathbf{B}_\beta \tag{4.64}$$

and

$$\mathbf{B}_{3,\alpha} = \kappa_{\alpha\beta}\mathbf{B}_\beta \tag{4.65}$$

subject to

$$\begin{aligned} \epsilon_{\alpha\beta} &= \epsilon_{\beta\alpha} \\ \kappa_{\alpha\beta} &= \kappa_{\beta\alpha} \end{aligned} \tag{4.66}$$

The 3-D strains can be found from the definition of Green-Lagrange strain components, given by

$$2\Gamma_{\alpha\beta} = \mathbf{G}_\alpha \cdot \mathbf{G}_\beta - \delta_{\alpha\beta} \tag{4.67}$$

which, in light of the above, become

$$\Gamma_{\alpha\beta} = \epsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta} \tag{4.68}$$

Substituting Eq. (4.68) into Eq. (4.60) and integrating through the thickness yields an expression for the strain energy per unit area of the plate or membrane of the form

$$2U_2 = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \\ \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{Bmatrix}^T \begin{bmatrix} \frac{Eh}{1-\nu^2} & \frac{Eh\nu}{1-\nu^2} & 0 & 0 & 0 & 0 \\ \frac{Eh\nu}{1-\nu^2} & \frac{Eh}{1-\nu^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & Gh & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{Eh^3}{12(1-\nu^2)} & \frac{Eh^3\nu}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & \frac{Eh^3\nu}{12(1-\nu^2)} & \frac{Eh^3}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Gh^3}{12} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \\ \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{Bmatrix} \tag{4.69}$$

It is clear from Eq. (4.69) that the membrane energy and the bending/twist energy are decoupled for this problem. There are three things that will introduce coupling terms: (1) accounting for anisotropy of materials, such as in laminated plate theory; (2) initial curvature, as in shell theory; and (3) large in-plane forces, such as in Von Kármán theory.

4.5.2 2-D Strain-Displacement Relations

For the linear theory,

$$\mathbf{B}_3 = -u_{3,1}\mathbf{a}_1 - u_{3,2}\mathbf{a}_2 + \mathbf{a}_3 \tag{4.70}$$

(This effectively locks out transverse shear deformation, but one can derive a theory that contains this effect by introducing independent rotation variables just as was done for the beam.) With Eq. (4.70) one finds the membrane and bending/twist strain measures to be

$$\begin{aligned} \epsilon_{11} &= u_{1,1} \\ \epsilon_{22} &= u_{2,2} \\ 2\epsilon_{12} &= u_{1,2} + u_{2,1} \\ \kappa_{11} &= -u_{3,11} \\ \kappa_{22} &= -u_{3,22} \\ 2\kappa_{12} &= -2u_{3,12} \end{aligned} \tag{4.71}$$

For dealing with membranes, which are under large in-plane forces but without bending (so that only the terms in the strain energy, Eq. 4.69, which involve ϵ are kept), we must use nonlinear strain-displacement relations for the membrane terms. This can also be a way of deriving the Von Kármán theory. These relations can be found as

$$\begin{aligned}\epsilon_{11} =& u_{1,1} + \frac{u_{3,1}^2}{2}\\ \epsilon_{22} =& u_{2,2} + \frac{u_{3,2}^2}{2}\\ 2\epsilon_{12} =& u_{1,2} + u_{2,1} + u_{3,1}u_{3,2}\end{aligned} \tag{4.72}$$

and follow directly from the direction cosine matrix which relates the triad $\mathbf{B}_i$ to $\mathbf{a}_i$, viz.

$$\begin{Bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\\ \mathbf{B}_3\end{Bmatrix} = \begin{bmatrix} 1-\frac{u_{3,1}^2}{2} & 0 & u_{3,1}\\ 0 & 1-\frac{u_{3,2}^2}{2} & u_{3,2}\\ -u_{3,1} & -u_{3,2} & 1-\frac{u_{3,1}^2}{2}-\frac{u_{3,2}^2}{2}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{Bmatrix} \tag{4.73}$$

and Eq. (4.64).

4.5.3 Membranes

Recalling our strategy for the string, let us consider the static behavior first, so as to derive a simple potential energy function suitable for small deflections of a taut membrane. Consider a rectangular membrane under tension forces applied uniformly around the edges. Let the Cartesian system be $x = x_1$ and $y = x_2$ with displacements $u = u_1$, $v = u_2$, and $w = u_3$.

Strain Energy

The strain energy can be written as

$$\begin{aligned}U =& \frac{1}{2}\iint_{\mathcal{A}}\left\{\frac{Eh}{1-\nu^2}\left[\left(u_x + \frac{w_x^2}{2}\right)^2 + \nu\left(u_x + \frac{w_x^2}{2}\right)\left(v_y + \frac{w_y^2}{2}\right) + \left(v_y + \frac{w_y^2}{2}\right)^2\right]\right.\\ &\left. + G(u_y + v_x + w_x w_y)^2\right\}dxdy\end{aligned} \tag{4.74}$$

Here $(\bullet)_x = \partial(\bullet)/\partial x$ and similarly for y.

Inplane Forces

The virtual work of the edge forces is

$$\overline{\delta W} = T\iint_{\mathcal{A}}(\delta u_x + \delta v_y)\,dxdy \tag{4.75}$$

for constant force for unit length T. Thus, the potential of the applied forces is

$$V = -T \iint_{\mathcal{A}} (u_x + v_y)\, dxdy \tag{4.76}$$

Taking the variation of the total potential $U + V$, one finds the equilibrium equations and boundary conditions, which yield simply that

$$\begin{aligned} N_{xx} =& \frac{\partial U}{\partial \epsilon_{xx}} = \frac{Eh}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy}) = T \\ N_{xy} =& \frac{\partial U}{\partial \epsilon_{xy}} = 2G\epsilon_{xy} = 0 \\ N_{yy} =& \frac{\partial U}{\partial \epsilon_{yy}} = \frac{Eh}{1-\nu^2}(\epsilon_{yy} + \nu\epsilon_{xx}) = T \end{aligned} \tag{4.77}$$

Therefore membrane strains can be seen to be $O(\frac{T}{Eh}) << 1$. Thus, we can set $\epsilon_{xx} \approx 0$ and $\epsilon_{yy} \approx 0$, so that

$$\begin{aligned} u_x =& -\frac{w_x^2}{2} \\ v_y =& -\frac{w_y^2}{2} \end{aligned} \tag{4.78}$$

Substitution of Eq. (4.78) into the strain energy, Eq. (4.74) yields only higher-order terms. On the other hand, when Eq. (4.78) is substituted into the potential energy, Eq. (4.76), the resulting total potential energy is then

$$U + V = V = \frac{T}{2} \iint_{\mathcal{A}} \left(w_x^2 + w_y^2\right) dxdy \tag{4.79}$$

Thus, as with the tensile forces in a string, the inplane forces in a flat, taut membrane serve to contribute to the potential energy in a simple way.

Kinetic Energy

To complete the theory of vibration for taut membranes, we need the kinetic energy. As with the string, the in-plane displacements u and v are very small relative to the transverse displacement w so that for a membrane with constant mass per unit area $\mu = \rho h$, the kinetic energy is

$$K = \frac{\mu}{2} \iint_{\mathcal{A}} \dot{w}^2 dxdy \tag{4.80}$$

Equations of Motion and Boundary Conditions

Substitution of these energy expressions into Hamilton's principle for a membrane of arbitrary shape shows that the equation of motion is a 2-D wave equation

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = \nabla^2 w \tag{4.81}$$

where $c = \sqrt{\frac{T}{\mu}}$. The boundary conditions show that on the boundary either $w = 0$ or $T\partial w/\partial n = 0$, where a partial with respect to n means the partial with respect to a direction in the plane of the undeformed membrane normal to the edge.

Solutions for Rectangular Case

For a rectangular membrane, with $0 < x < a$ and $0 < y < b$, we can solve the equation of motion by separation of variables

$$w(x, y, t) = \phi(x)\psi(y)q(t) \tag{4.82}$$

where $q = \overline{q}e^{i\omega t}$. This yields

$$\frac{\phi_{xx}}{\phi} + \frac{\psi_{yy}}{\psi} + \frac{\omega^2}{c^2} = 0 \tag{4.83}$$

The first term is a function only of x, and the second term of y only. These can all be satisfied only by introducing two separation constants α^2 and β^2 so that

$$\phi_{xx} = -\alpha^2\phi \qquad \psi_{yy} = -\beta^2\psi \qquad \alpha^2 + \beta^2 = \frac{\omega^2}{c^2} \tag{4.84}$$

The solutions are

$$\begin{aligned}\phi =& a_1 \cos(\alpha x) + b_1 \sin(\alpha x)\\ \psi =& a_2 \cos(\beta y) + b_2 \sin(\beta y)\end{aligned} \tag{4.85}$$

For a membrane fixed on all boundaries

$$\begin{aligned}\phi(0) =& \phi(a) = 0\\ \psi(0) =& \psi(b) = 0\end{aligned} \tag{4.86}$$

so that

$$\begin{aligned}\phi = \quad & \sin(\alpha x) \quad \alpha a = n\pi \quad n = 1, 2, \ldots\\ \psi = \quad & \sin(\beta y) \quad \beta b = m\pi \quad m = 1, 2, \ldots\end{aligned} \tag{4.87}$$

Thus,

$$\frac{\omega^2}{c^2} = \alpha^2 + \beta^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \tag{4.88}$$

so that

$$\omega = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} = \omega_{mn} \tag{4.89}$$

n	m	$\frac{\omega_{nm}a}{c\pi}$	description of mode
1	1	$\sqrt{2}$	single bump, no nodal lines
1	2	$\sqrt{5}$	2 bumps, one nodal line
2	1	$\sqrt{5}$	2 bumps, one nodal line
2	2	$2\sqrt{2}$	4 bumps, two nodal lines

Table 4.2: Lowest frequencies and modes for square membrane with $a = b$

The solution is then

$$w(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{i\omega_{mn}t} \tag{4.90}$$

Consider a square membrane with $a = b$. The lowest natural frequencies and their corresponding modes are described in Table 4.2. See Meirovitch (1997) for some other solved problems, including circular membranes.

4.5.4 Plates

First, we introduce the symbols

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad \mu = \rho h \tag{4.91}$$

Strain and Kinetic Energies

Now, for the problem of a flat, rectangular, isotropic plate, with membrane forces equal to zero, the strain energy is given by

$$U = \frac{1}{2}\int_0^a \int_0^b D\left[w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx}w_{yy} + 2(1-\nu)w_{xy}^2\right] dydx \tag{4.92}$$

The kinetic energy is

$$K = \frac{1}{2}\int_0^a \int_0^b \mu\dot{w}^2 dydx \tag{4.93}$$

Equation of Motion and Boundary Conditions

Hamilton's principle requires that

$$\begin{aligned} 0 = \int_{t_1}^{t_2} \int_0^a \int_0^b &[\mu\dot{w}\delta\dot{w} - Dw_{xx}\delta w_{xx} - Dw_{yy}\delta w_{yy} - D\nu w_{xx}\delta w_{yy} \\ &- D\nu w_{yy}\delta w_{xx} - 2D(1-\nu)w_{xy}\delta w_{xy}]dydxdt \end{aligned} \tag{4.94}$$

which, upon integration by parts over t, x, and y, yields

$$
\begin{aligned}
0 = & \int_{t_1}^{t_2} \int_0^a \int_0^b \{-\mu \ddot{w} \delta w + [D(w_{xx} + \nu w_{yy})]_x \delta w_x + [D(w_{yy} + \nu w_{xx})]_y \delta w_y \\
& + 2[D(1-\nu) w_{xy}]_x \delta w_y\} dydxdt \\
& + \int_0^a \int_0^b \mu \dot{w} \delta w dydx|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^b D(w_{xx} + \nu w_{yy}) \delta w_x dydt|_0^a \\
& - \int_{t_1}^{t_2} \int_0^a D(w_{yy} + \nu w_{xx}) \delta w_y dxdt|_0^b \\
& - 2 \int_{t_1}^{t_2} \int_0^b D(1-\nu) w_{xy} \delta w_y dydt|_0^a
\end{aligned}
\tag{4.95}
$$

Setting to zero the variations of all quantities at the ends of the time interval and integrating by parts a second time in space we get

$$
\begin{aligned}
0 = & \int_{t_1}^{t_2} \int_0^a \int_0^b \{-\mu \ddot{w} - [D(w_{xx} + \nu w_{yy})]_{xx} - [D(w_{yy} + \nu w_{xx})]_{yy} \\
& - 2[D(1-\nu) w_{xy}]_{xy}\} \delta w dydxdt \\
& - \int_{t_1}^{t_2} \int_0^b D(w_{xx} + \nu w_{yy}) \delta w_x dydt|_0^a - \int_{t_1}^{t_2} \int_0^a D(w_{yy} + \nu w_{xx}) \delta w_y dxdt|_0^b \\
& + \int_{t_1}^{t_2} \int_0^b [D(w_{xx} + \nu w_{yy})]_x \delta w dydt|_0^a + \int_{t_1}^{t_2} \int_0^a [D(w_{yy} + \nu w_{xx})]_y \delta w dxdt|_0^b \\
& + 2 \int_{t_1}^{t_2} \int_0^b [D(1-\nu) w_{xy}]_y \delta w dydt|_0^a + 2 \int_{t_1}^{t_2} \int_0^a [D(1-\nu) w_{xy}]_x \delta w dxdt|_0^b \\
& - 2 \int_{t_1}^{t_2} D(1-\nu) w_{xy} \delta w dt|_0^a|_0^b
\end{aligned}
\tag{4.96}
$$

Finally, collecting terms, removing the time integration, and changing the sign, we obtain

$$
\begin{aligned}
0 = & \int_0^a \int_0^b \{\mu \ddot{w} + [D(w_{xx} + \nu w_{yy})]_{xx} + [D(w_{yy} + \nu w_{xx})]_{yy} \\
& + 2[D(1-\nu) w_{xy}]_{xy}\} \delta w dydx \\
& + \int_0^b D(w_{xx} + \nu w_{yy}) \delta w_x dy|_0^a + \int_0^a D(w_{yy} + \nu w_{xx}) \delta w_y dx|_0^b \\
& - \int_0^b \{[D(w_{xx} + \nu w_{yy})]_x + 2[D(1-\nu) w_{xy}]_y\} \delta w dy|_0^a \\
& - \int_0^a \{[D(w_{yy} + \nu w_{xx})]_y + 2[D(1-\nu) w_{xy}]_x\} \delta w dx|_0^b \\
& + 2D(1-\nu) w_{xy} \delta w|_0^a|_0^b
\end{aligned}
\tag{4.97}
$$

For uniform plates, then, the Euler-Lagrange equation becomes

$$\mu\ddot{w} + D(w_{xxxx} + 2w_{xxyy} + w_{yyyy}) = 0 \tag{4.98}$$

with boundary conditions involving the following quantities

$$\begin{aligned}
w_{xx} + \nu w_{yy} &= 0 \quad \text{zero moment if } x \text{ boundary is free or simply supported} \\
& \text{(reverse } x \text{ and } y \text{ for } y \text{ boundary)} \\
w_{xxx} + (2-\nu)w_{yyx} &= 0 \quad \text{zero shear force if } x \text{ boundary is free or roller} \\
& \text{(reverse } x \text{ and } y \text{ for } y \text{ boundary)} \\
w_{xy} &= 0 \quad \text{zero twisting moment at free corner} \\
w_y &= 0 \quad \text{zero slope on } x \text{ boundary} \\
& \text{(reverse } x \text{ and } y \text{ for } y \text{ boundary)} \\
w &= 0 \quad \text{zero deflection}
\end{aligned} \tag{4.99}$$

For the free corner case, δw is not zero and w_{xy} must be zero at that point; there the boundary condition reduces to $\phi_x \psi_y|_{\text{corner}} = 0$.

Various Solution Cases

We again consider separation of variables so that

$$w = \overline{q} e^{i\omega t} \phi(x)\psi(y) \tag{4.100}$$

The equation of motion becomes

$$D(\phi''''\psi + 2\phi''\psi'' + \phi\psi'''') - \omega^2 \mu \phi\psi = 0 \tag{4.101}$$

or

$$\frac{\phi''''}{\phi} + 2\frac{\phi''}{\phi}\frac{\psi''}{\psi} + \frac{\psi''''}{\psi} = \frac{\omega^2 \mu}{D} \tag{4.102}$$

Now consider four types of edges: free, simply-supported, roller, and clamped.

Free Edge:

$$\begin{aligned}
\phi_{xx}\psi + \nu\phi\psi_{yy} = 0 &\qquad \text{for all } y \quad (\phi_{xx} = 0 \text{ and } \phi = 0 \text{ possible}) \\
\phi_{xxx}\psi + (2-\nu)\phi_x\psi_{yy} = 0 &\qquad \text{for all } y \quad (\phi_{xxx} = 0 \text{ and } \phi_x = 0 \text{ possible})
\end{aligned} \tag{4.103}$$

Pinned Edge:

$$\phi = 0 \quad \phi_{xx}\psi + \nu\phi\psi_{yy} = 0 \quad \text{for all } y \quad (\phi_{xx} = 0 \text{ and } \phi = 0 \text{ possible}) \tag{4.104}$$

Roller Edge:

$$\phi_x = 0 \quad \phi_{xxx} + (2-\nu)\phi_x\psi_{yy} = 0 \quad \text{for all } y \quad (\phi_{xxx} = 0 \text{ and } \phi_x = 0 \text{ possible}) \tag{4.105}$$

Clamped Edge:

$$\phi = 0 \quad \phi_x = 0 \tag{4.106}$$

Table of Solutions for Rectangular Case

Below are some known exact solutions. The convention for the four sequential letters is that the first letter is for the condition at $x = a$, the second for $y = b$, the third for $x = 0$, and the fourth for $y = 0$; "p" is for pinned, and "r" is for roller. For many other combinations of boundary conditions, especially those which involve clamped and free edges, exact solutions are not known, because separation of variables is not possible. See the text for some other solved problems, including circular plates.

p-p-p-p

$$\begin{aligned} \phi &= \sin\left(\frac{n\pi x}{a}\right) && n = 1, 2, 3, 4, \ldots \\ \psi &= \sin\left(\frac{m\pi y}{b}\right) && m = 1, 2, 3, 4, \ldots \\ \omega_{mn} &= \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2\sqrt{\frac{D}{\mu}} \end{aligned} \tag{4.107}$$

r-p-p-p

$$\begin{aligned} \phi &= \sin\left(\frac{p\pi x}{2a}\right) && p = 1, 3, 5, 7, \ldots \\ \psi &= \sin\left(\frac{m\pi y}{b}\right) && m = 1, 2, 3, 4, \ldots \\ \omega_{mp} &= \left(\frac{p^2}{4a^2} + \frac{m^2}{b^2}\right)\pi^2\sqrt{\frac{D}{\mu}} \end{aligned} \tag{4.108}$$

r-p-r-p

$$\begin{aligned} \phi &= \cos\left(\frac{n\pi x}{a}\right) && n = 0, 1, 2, 3, \ldots \\ \psi &= \sin\left(\frac{m\pi y}{b}\right) && m = 1, 2, 3, 4, \ldots \\ \omega_{mn} &= \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2\sqrt{\frac{D}{\mu}} \end{aligned} \tag{4.109}$$

r-r-p-p

$$\begin{aligned}
\phi &= \sin\left(\frac{p\pi x}{2a}\right) && p = 1,3,5,7,\ldots \\
\psi &= \sin\left(\frac{q\pi y}{2b}\right) && q = 1,3,5,7,\ldots \\
\phi_x(a)\psi_y(b) &= 0 \\
\omega_{pq} &= \left(\frac{p^2}{4a^2} + \frac{q^2}{4b^2}\right)\pi^2\sqrt{\frac{D}{\mu}}
\end{aligned} \tag{4.110}$$

r-r-p-r

$$\begin{aligned}
\phi &= \sin\left(\frac{p\pi x}{2a}\right) && p = 1,3,5,7,\ldots \\
\psi &= \cos\left(\frac{m\pi y}{b}\right) && m = 0,1,2,3,\ldots \\
\phi_x\psi_y &= 0 && \text{at free corner} \\
\omega_{mp} &= \left(\frac{p^2}{4a^2} + \frac{m^2}{b^2}\right)\pi^2\sqrt{\frac{D}{\mu}}
\end{aligned} \tag{4.111}$$

r-r-r-r

$$\begin{aligned}
\phi &= \cos\left(\frac{n\pi x}{a}\right) && n = 0,1,2,3,\ldots \\
\psi &= \cos\left(\frac{m\pi y}{b}\right) && m = 0,1,2,3,\ldots \\
\omega_{mn} &= \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2\sqrt{\frac{D}{\mu}}
\end{aligned} \tag{4.112}$$

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