Rotorcraft Dynamics

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- By now most rotorcraft dynamicists are accustomed to writing lengthy, complicated equations for analysis of blade dynamics
- Many extant sets of such equations contain a host of oversimplifications
    - no initial curvature
    - uniaxial stress field (or alternatively cross-section rigid in its own plane)
    - only torsional warping
    - “moderate” deflections (or alternatively some sort of ordering scheme)
    - isotropic or transversely isotropic material construction
    - no shear deformation (invalidates theory for application to composite beams)

- Approximation concepts such as “ordering schemes” have been deeply ingrained in the thinking of most analysts
- However, some research has pointed out problems with this concept:
  - It is virtually impossible to apply an ordering scheme in a completely consistent manner (Stephens, Hodges, Avila, and Kung 1982)
  - Ordering schemes can lead to *more lengthy* equations due to expansion of transcendental functions (Crespo da Silva and Hodges 1986)
  - An ordering scheme that works for one set of configuration parameters may not be suitable for a different set (Hinnant and Hodges 1989)

In the present approach, there are several basic and exciting departures from the “old school”:

- No ordering scheme is needed or used
    - exact kinematics for beam reference line displacement and cross-sectional rotation
    - geometrically-exact equations of motion
- The beam constitutive law is
    - based on a separate finite element analysis
    - valid for anisotropic beams with inhomogeneous cross-sections

. . . departures from the "old school" (continued):

- A compact matrix notation is used
- The resulting mixed formulation can be put in the weakest form, so
  - the requirements for the shape functions are minimal, leading to the possibility of shape functions that are as simple as piecewise constant
  - approximate element quadrature is not required

- Kinematical Preliminaries
- Analysis of 3-D Beam Deformation
- Constitutive Equations from 2-D Finite Element Analysis
- 1-D Kinematics
- 1-D Equations of Motion
- 1-D Finite Element Solution
- Examples
- It should be noted that this material is found in the author's book *Nonlinear Composite Beam Theory*, published by AIAA in 2006.

- Consider a rigid body B moving in a frame A (frames and rigid bodies are kinematically equivalent)
- Introduce a dextral unit triad $\hat{\boldsymbol{A}}_i$ fixed in A (Roman subscripts vary from 1 to 3 unless otherwise specified, and unit vectors are denoted by a bold italic symbol with a "hat")
- Also, introduce a dextral triad $\hat{\boldsymbol{B}}_i$ fixed in B
- Now $\hat{\boldsymbol{B}}_i$ will vary in A as a function of time
- A vector is a *first*-order tensor
- A vector can always be expressed as a *linear* combination of dextral unit vectors
- For example, for an arbitrary vector $\boldsymbol{v}$ with $v_{Bi} = \boldsymbol{v} \cdot \hat{\boldsymbol{B}}_i$, it is always true that $\boldsymbol{v} = \hat{\boldsymbol{B}}_i v_{Bi}$ (note that summation is implied over any repeated index)

- Similarly, a dyadic is a *second*-order Cartesian tensor
- Dyadics are *quadratic* forms of dextral unit vectors
- For example, consider the relationship of a dyadic $\underline{\boldsymbol{T}}$ and the matrix T_{ij} of its components in a mixed set of bases

$$\underline{\boldsymbol{T}} = \hat{\boldsymbol{B}}_i T_{ij} \hat{\boldsymbol{A}}_j$$

- The transpose of $\underline{\boldsymbol{T}}$ is simply

$$\underline{\boldsymbol{T}}^T = \hat{\boldsymbol{A}}_j T_{ij} \hat{\boldsymbol{B}}_i = \hat{\boldsymbol{A}}_i T_{ji} \hat{\boldsymbol{B}}_j$$

- For simplicity we will not carry names of associated base vectors in the symbol for a particular matrix of dyadic components

- An exception is the finite rotation tensor for which it is quite helpful to maintain basis association in naming matrices
- One can characterize the rotational motion of B in A in these two ways:

$$\hat{\boldsymbol{B}}_i = \underline{\boldsymbol{C}}^{BA} \cdot \hat{\boldsymbol{A}}_i = C_{ij}^{BA} \hat{\boldsymbol{A}}_j$$

- $\underline{\boldsymbol{C}}^{BA}$ is read as the finite rotation tensor of B in A, given by

$$\underline{\boldsymbol{C}}^{BA} = \hat{\boldsymbol{B}}_i \hat{\boldsymbol{A}}_i$$

- Both the tensor and its components depend on time
- Note the convention for naming the finite rotation tensor and its corresponding matrix of direction cosines

- The transpose is indicated by reversing the superscripts

$$\left(\underline{\boldsymbol{C}}^{BA}\right)^T = \underline{\boldsymbol{C}}^{AB}$$

so that

$$\hat{\boldsymbol{A}}_i = \underline{\boldsymbol{C}}^{AB} \cdot \hat{\boldsymbol{B}}_i$$

- $\underline{\boldsymbol{C}}^{BA}$ is an orthonormal tensor so that

$$\underline{\boldsymbol{C}}^{BA} \cdot \underline{\boldsymbol{C}}^{AB} = \underline{\boldsymbol{\Delta}}$$

where $\underline{\boldsymbol{\Delta}}$ is the identity tensor

- The matrix of direction cosines C^{BA} is given by

$$C_{ij}^{BA} = \hat{\boldsymbol{B}}_i \cdot \hat{\boldsymbol{A}}_j \quad \left(= C_{ji}^{AB}\right)$$

- C^{BA} is a matrix of the components of the transpose of the finite rotation tensor $\underline{\boldsymbol{C}}^{AB}$ so that

$$\begin{aligned} C_{ij}^{BA} &= \hat{\boldsymbol{A}}_i \cdot \underline{\boldsymbol{C}}^{AB} \cdot \hat{\boldsymbol{A}}_j \\ &= \hat{\boldsymbol{B}}_i \cdot \underline{\boldsymbol{C}}^{AB} \cdot \hat{\boldsymbol{B}}_j \end{aligned}$$

- The matrix of direction cosines is also orthonormal

$$C^{BA} C^{AB} = \Delta$$

where Δ is the 3×3 identity matrix

- When an intermediate frame N is involved,

$$C^{BA} = C^{BN} C^{NA} \qquad \underline{\boldsymbol{C}}^{BA} = \underline{\boldsymbol{C}}^{BN} \cdot \underline{\boldsymbol{C}}^{NA}$$

- Consider an arbitrary vector $\boldsymbol{v}$; it is always possible to write $v_{Z_i} = \boldsymbol{v} \cdot \hat{\boldsymbol{Z}}_i$ where Z is an arbitrary frame in which the dextral unit triad $\hat{\boldsymbol{Z}}_i$ is fixed
- With the column matrix notation

$$v_Z = \begin{Bmatrix} v_{Z1} \\ v_{Z2} \\ v_{Z3} \end{Bmatrix}$$

it is easily demonstrated that

$$v_B = C^{BA} v_A \qquad v_A = C^{AB} v_B$$

- For a rigid body B moving in frame A, there exist analogous vector-dyadic operations
  - the "push-forward" operation on $\boldsymbol{v}$ is defined by

$$\underline{\boldsymbol{C}}^{BA} \cdot \boldsymbol{v}$$

  - the "pull-back" operation on $\boldsymbol{v}$ is defined by

$$\underline{\boldsymbol{C}}^{AB} \cdot \boldsymbol{v}$$

- As can be demonstrated, these operations rotate the vector by an amount commensurate with the change in orientation from A to B and from B to A respectively.

- To visualize the pull-back operation
  - imagine the vector $\boldsymbol{v}$ frozen at some instant in time in a frame N which has a dextral triad $\hat{\boldsymbol{N}}_i$ that is coincident with $\hat{\boldsymbol{B}}_i$
  - rotate the frame N so that $\hat{\boldsymbol{N}}_i$ lines up with $\hat{\boldsymbol{A}}_i$
  - the rotated image of $\boldsymbol{v}$ is the result of the pull-back operation
- For the push-forward operation
  - imagine the vector $\boldsymbol{v}$ frozen at some instant in time in a frame N which has a dextral triad $\hat{\boldsymbol{N}}_i$ that is coincident with $\hat{\boldsymbol{A}}_i$
  - rotate the frame N so that $\hat{\boldsymbol{N}}_i$ lines up with $\hat{\boldsymbol{B}}_i$
  - the rotated image of $\boldsymbol{v}$ is the result of the push-forward operation
- These operations are useful for describing deformation

- Note that if a vector is moving in a frame then its time derivative in that frame will be nonzero
- However, if a vector is fixed in a frame then its time derivative in that frame will be zero
- Obviously, the concept of the derivative of a vector is then frame dependent
- Consider a vector $\boldsymbol{b}$ fixed in B where B is moving in A. Then,

$$\frac{{}^{A}d\boldsymbol{b}}{dt} \neq 0$$

But

$$\frac{{}^{B}d\boldsymbol{b}}{dt} = 0$$

- Thus, when the vector $\boldsymbol{v}$ is resolved along unit vectors that are fixed in the frame in which the derivative is being taken, the derivative of $\boldsymbol{v}$ is easily expressed as

$$\frac{{}^{Z}d\boldsymbol{v}}{dt} = \hat{\boldsymbol{z}}_i \dot{v}_{Zi}$$

where Z is an arbitrary frame as before

- However, as will be clear from material to follow, it is not always convenient to differentiate a vector in this way

- For differentiation of a vector in a different frame

$$\frac{{}^{A}d\boldsymbol{v}}{dt} = \frac{{}^{B}d\boldsymbol{v}}{dt} + \boldsymbol{\omega}^{BA} \times \boldsymbol{v}$$

where ω^{BA} is the angular velocity of B in A given by

$$\omega^{BA} = \hat{\boldsymbol{B}}_1 \frac{{}^{A}d\hat{\boldsymbol{B}}_2}{dt} \cdot \hat{\boldsymbol{B}}_3 + \hat{\boldsymbol{B}}_2 \frac{{}^{A}d\hat{\boldsymbol{B}}_3}{dt} \cdot \hat{\boldsymbol{B}}_1 + \hat{\boldsymbol{B}}_3 \frac{{}^{A}d\hat{\boldsymbol{B}}_1}{dt} \cdot \hat{\boldsymbol{B}}_2 = \omega_{Bi}^{BA} \hat{\boldsymbol{B}}_i$$

- The derivatives of the unit vectors are easily expressed in terms of the direction cosines

$$\frac{{}^{A}d\hat{\boldsymbol{B}}_i}{dt} = \dot{C}_{ij}^{BA} \hat{\boldsymbol{A}}_j = \dot{C}_{ij}^{BA} C_{jk}^{AB} \hat{\boldsymbol{B}}_k = -e_{ijk}\omega_{Bj}^{BA} \hat{\boldsymbol{B}}_k = \boldsymbol{\omega}^{BA} \times \hat{\boldsymbol{B}}_i$$

- Just as the time derivative of a vector depends on the frame in which the derivative is taken, so does the variation of a vector
- As Kane and others have shown, one can express the relationship between variations in two frames as

$$ {}^{A}\delta \boldsymbol{v} = {}^{B}\delta \boldsymbol{v} + \overline{\boldsymbol{\delta\psi}}^{BA} \times \boldsymbol{v} $$

where $\overline{\boldsymbol{\delta\psi}}^{BA}$ is the virtual rotation of B in A given by

$$ \overline{\boldsymbol{\delta\psi}}^{BA} = \hat{\boldsymbol{B}}_1 {}^{A}\delta \hat{\boldsymbol{B}}_2 \cdot \hat{\boldsymbol{B}}_3 + \hat{\boldsymbol{B}}_2 {}^{A}\delta \hat{\boldsymbol{B}}_3 \cdot \hat{\boldsymbol{B}}_1 + \hat{\boldsymbol{B}}_3 {}^{A}\delta \hat{\boldsymbol{B}}_1 \cdot \hat{\boldsymbol{B}}_2 = \overline{\delta\psi}_{Bi}^{BA} \hat{\boldsymbol{B}}_i $$

and

$$ {}^{Z}\delta \boldsymbol{v} = \hat{\boldsymbol{Z}}_i \delta v_{Zi} $$

- The variations of the unit vectors are easily expressed in terms of the direction cosines

$$ {}^{A}\delta\hat{\boldsymbol{B}}_i = \delta C_{ij}^{BA}\hat{\boldsymbol{A}}_j = \delta C_{ij}^{BA} C_{jk}^{AB}\hat{\boldsymbol{B}}_k = -e_{ijk}\overline{\delta\psi}_{Bj}^{BA}\hat{\boldsymbol{B}}_k = \overline{\boldsymbol{\delta\psi}}^{BA}\times\hat{\boldsymbol{B}}_i $$

  where $\delta(\ )$ is the usual Lagrangean variation
- It is evident that the vectors $\boldsymbol{\omega}^{BA}$ and $\overline{\boldsymbol{\delta\psi}}^{BA}$ can be regarded as operators which produce the time derivative and variation, respectively, in A of any vector fixed in B
- When an additional frame N is involved, Kane's addition theorem applies

$$ \boldsymbol{\omega}^{BA} = \boldsymbol{\omega}^{BN} + \boldsymbol{\omega}^{NA} \qquad \overline{\boldsymbol{\delta\psi}}^{BA} = \overline{\boldsymbol{\delta\psi}}^{BN} + \overline{\boldsymbol{\delta\psi}}^{NA} $$

- Note that to obtain the virtual rotation vector, one need only replace the dots in the angular velocity vector with δ's, ignoring any other terms
- The virtual work in A of an applied torque $\boldsymbol{T}$ acting on a body B is simply

$$\overline{\delta W} = \boldsymbol{T} \cdot \overline{\boldsymbol{\delta\psi}}^{BA}$$

- The velocity of a point P moving in A can be determined by time differentiation in A of the position vector $\boldsymbol{p}^{P/O}$ where O is any point fixed in A

$$\boldsymbol{v}^{PA} = \frac{^{A}d\boldsymbol{p}^{P/O}}{dt}$$

- Often the calculation of velocity in this way is complicated
- In such a case it is helpful to "step" one's way from the known to the unknown using the two chain rules:
  - 2 points fixed on a rigid body (or in a frame)
  - 1 point moving on a rigid body (or in a frame)

- For 2 points P and Q fixed on a rigid body B having an angular velocity ω^{BA} in A, the velocities of these points in A are related according to

$$\boldsymbol{v}^{PA} = \boldsymbol{v}^{QA} + \omega^{BA} \times \boldsymbol{p}^{P/Q}$$

- For a point P moving on a rigid body B while B is moving in A, the velocity of P in A is given by

$$\boldsymbol{v}^{PA} = \boldsymbol{v}^{PB} + \boldsymbol{v}^{\bar{B}A}$$

where $\boldsymbol{v}^{\bar{B}A}$ is the velocity of the point in B that is coincident with P at the instant under consideration (this can often be obtained by use of the other theorem)

- These relationships for the derivative and variation can be nicely expressed in matrix notation
- We've already seen how an arbitrary vector $\boldsymbol{v}$ can for an arbitrary frame Z be expressed in terms of

$$v_Z = \begin{Bmatrix} v_{Z1} \\ v_{Z2} \\ v_{Z3} \end{Bmatrix}$$

- The dual matrix $\widetilde{v_Z}_{ij} = -e_{ijk} v_{Zk}$ has the same measure numbers but arranged antisymmetrically

$$\widetilde{v_Z} = \begin{bmatrix} 0 & -v_{Z3} & v_{Z2} \\ v_{Z3} & 0 & -v_{Z1} \\ -v_{Z2} & v_{Z1} & 0 \end{bmatrix}$$

- With the tilde notation identities given in Hodges (1990) are helpful
- When Y and Z are 3×1 column matrices, it is easily shown that

$$
\begin{aligned}
(\widetilde{Z})^T &= -\widetilde{Z} \\
\widetilde{Z} Z &= 0 \\
\widetilde{Y} Z &= -\widetilde{Z} Y \\
Y^T \widetilde{Z} &= -Z^T \widetilde{Y} \\
\widetilde{Y} \widetilde{Z} &= Z Y^T - \Delta Y^T Z \\
\widetilde{\widetilde{Y} Z} &= \widetilde{Z} \widetilde{Y} + \widetilde{\widetilde{Y} Z}
\end{aligned}
$$

where Δ is the 3×3 identity matrix

- This notation also applies to vectors
- The *tensor* $\underline{\widetilde{\boldsymbol{v}}}$ has components in the Z basis given by the matrix $\widetilde{v}_Z$ such that

$$\begin{aligned}\underline{\widetilde{\boldsymbol{v}}} &= \boldsymbol{v} \times \underline{\boldsymbol{\Delta}} \\ &= \hat{\boldsymbol{Z}}_i \widetilde{v_Z}_{ij} \hat{\boldsymbol{Z}}_j\end{aligned}$$

  where $\underline{\boldsymbol{\Delta}}$ is the identity dyadic
- Note that for any vector $\boldsymbol{w}$, $\boldsymbol{v} \times \boldsymbol{w} = \underline{\widetilde{\boldsymbol{v}}} \cdot \boldsymbol{w}$
- Note that for any vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ with measure numbers expressed in some common basis Z, $\widetilde{v_Z} w_Z$ contains the measure numbers of the cross product $\boldsymbol{v} \times \boldsymbol{w}$ in the Z basis
- The $\widetilde{(\ )}$ operator is sometimes called a "cross product operator" for obvious reasons

- Now with these definitions in mind we note that

$$\widetilde{\omega_B^{BA}} = -\dot{C}^{BA} C^{AB}$$

and

$$\widetilde{\overline{\delta\psi}_B^{BA}} = -\delta C^{BA} C^{AB}$$

- In tensorial form these are

$$\widetilde{\boldsymbol{\omega}^{BA}} = {}^{A}\underline{\dot{\boldsymbol{C}}}^{BA} \cdot \underline{\boldsymbol{C}}^{AB}$$

and

$$\widetilde{\overline{\boldsymbol{\delta\psi}}^{BA}} = {}^{A}\delta \underline{\boldsymbol{C}}^{BA} \cdot \underline{\boldsymbol{C}}^{AB}$$

- Since C^{BA} is orthonormal, its elements are not independent
- Euler's theorem of rotation stipulates that any change of orientation can be characterized as a "simple rotation"
- A motion is a "simple rotation" of B in A if during the motion a line L maintains its orientation in B and in A
- Euler proved that at least four parameters are necessary for singularity free description of finite rotation:
  - the three measure numbers of a unit vector $\boldsymbol{e}$ along the line L ($e_i = e_{Ai} = e_{Bi}$) and
  - the magnitude of the rotation α
- With $e = \lfloor e_1 \;\; e_2 \;\; e_3 \rfloor^T$, the matrix of direction cosines is

$$C^{BA} = \Delta \cos\alpha + ee^T(1 - \cos\alpha) - \widetilde{e}\sin\alpha$$

- According to Kane, Likins, and Levinson (1983), the four Euler parameters denoted by ϵ_0 and

$$\epsilon = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{Bmatrix}$$

are defined as

$$\epsilon_i = e_i \sin(\alpha/2) \qquad \epsilon_0 = \cos(\alpha/2)$$

- They satisfy a constraint

$$\epsilon^T \epsilon + \epsilon_0^2 = 1$$

- The matrix of direction cosines is given by

$$C^{BA} = \left(1 - 2\epsilon^T\epsilon\right)\Delta + 2\left(\epsilon\epsilon^T - \epsilon_0\widetilde{\epsilon}\right)$$

while the angular velocity is

$$\omega_B^{BA} = 2\left[\left(\epsilon_0\Delta - \widetilde{\epsilon}\right)\dot{\epsilon} - \dot{\epsilon}_0\epsilon\right]$$

- It is tempting to eliminate the fourth parameter via the constraint
- This always introduces a singularity, but where the singularity is can be controlled
- Introduce Rodrigues parameters

$$\theta = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

where $\theta_i = 2\epsilon_i/\epsilon_0 = 2e_i \tan(\alpha/2)$

- Here the singularity is at $\epsilon_0 = 0$ or $\alpha = 180^\circ$

- Then the matrix of direction cosines is

$$C^{BA} = \frac{\left(1 - \frac{\theta^T\theta}{4}\right)\Delta + \frac{\theta\theta^T}{2} - \widetilde{\theta}}{1 + \frac{\theta^T\theta}{4}}$$

and the angular velocity is

$$\omega_B^{BA} = \frac{\left(\Delta - \frac{\widetilde{\theta}}{2}\right)\dot{\theta}}{1 + \frac{\theta^T\theta}{4}}$$

- Note that in the limit when α is very small, the parameters θ_i are components of the infinitesimal rotation vector

- In some cases it is useful to characterize rotation via the so-called "finite rotation vector" the components of which are $\Phi_i = e_i\alpha$ so that

$$\Phi = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}$$

- Then the matrix of direction cosines is

$$C^{BA} = \exp\left(-\widetilde{\Phi}\right) = \Delta - \widetilde{\Phi} + \frac{\widetilde{\Phi}\widetilde{\Phi}}{2} - \dots$$

- The angular velocity is

$$\begin{aligned}\omega_B^{BA} &= \left[\Delta\frac{\sin\alpha}{\alpha} - \widetilde{\phi}\frac{(1-\cos\alpha)}{\alpha^2} + \phi\phi^T\frac{(\alpha-\sin\alpha)}{\alpha^3}\right]\dot{\phi} \\ &= \left(\Delta - \frac{\widetilde{\phi}}{2}\right)\dot{\phi} + \dots\end{aligned}$$

where $\alpha^2 = \phi^T\phi$

- The material discussed so far has focused on fundamental rigid-body kinematics
- We must build on this foundation in the lectures following in order to understand the kinematical foundation of rigorous beam theory
- The next step is to develop a suitable method for determination of strain-displacement relations

- For describing the deformation, one needs to introduce frame a which is unaffected by beam deformation
- The following development is valid even if a is not an inertial frame
- Next one needs to specify the position vector from some arbitrary point O fixed in a to an arbitrary material point in the undeformed beam
- Denote this with $\hat{\mathbf{r}}(x_1, x_2, x_3, t) =$ where x_1 is arclength along the beam reference line, x_2 and x_3 are cross-sectional coordinates, and t is time (henceforth, explicit time dependence is ignored)

- Now introduce a infinite set of frames b along the undeformed beam with a dextral triad $\hat{\boldsymbol{b}}_i(x_1)$ fixed therein
- The unit vector $\hat{\boldsymbol{b}}_1(x_1)$ is tangent to the undeformed beam reference line at some arbitrary value of x_1
- In order to describe the geometry of the undeformed beam, we need to introduce
    - covariant basis vectors for the undeformed state

$$\boldsymbol{g}_i(x_1, x_2, x_3) = \frac{\partial \hat{\mathbf{r}}}{\partial x_i}$$

    - contravariant base vectors for the undeformed state

$$\boldsymbol{g}^i(x_1, x_2, x_3) = \frac{1}{2\sqrt{g}} e_{ijk} \boldsymbol{g}_j \times \mathbf{g}_k$$

- Note that $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta_{ij}$ and $\hat{\boldsymbol{b}}_i = \boldsymbol{g}_i(x_1, 0, 0)$
- Since $g = \det(\boldsymbol{g}_i \cdot \boldsymbol{g}_j) > 0$, one can always write

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \boldsymbol{r}(x_1) + \xi$$

  where $\xi = x_2\hat{\boldsymbol{b}}_2 + x_3\hat{\boldsymbol{b}}_3$ is the position vector to an arbitrary particle within the reference cross-section
- Consider the position vector $\hat{\mathbf{R}}(x_1, x_2, x_3)$ from O to the same particle in the deformed beam
- Reference cross-sections undergo two types of motion
  - *rigid-body* translations of the order of the beam length and large *rigid-body* rotations
  - small *deformation* of the reference cross-section

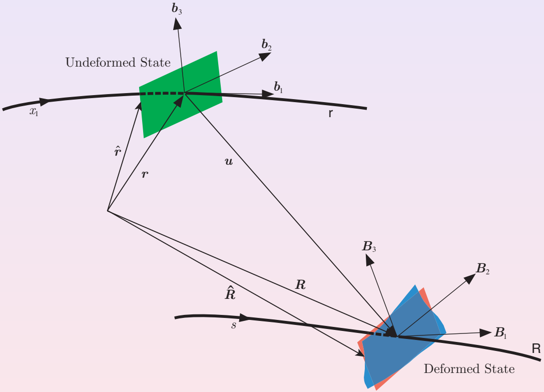

Figure: Schematic of Beam Kinematics

- To capture this behavior mathematically, introduce a set of frames B for the deformed beam analogous to b in the undeformed beam
- Global rotation (from b to B) is described by $\underline{\boldsymbol{C}} = \underline{\boldsymbol{C}}^{Bb}$
- Based on the above, we can express $\hat{\mathbf{R}}$ in the form

$$\hat{\mathbf{R}} = \boldsymbol{R} + \underline{\boldsymbol{C}} \cdot (\xi + \boldsymbol{w})$$

where

  - $\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{u}$
  - $\boldsymbol{u}$ describes the rigid-body translation
  - the “warping” $\boldsymbol{w}$ describes cross-sectional deformation

- Consider the plane of material points that make up the reference cross-section in the undeformed beam:
  - neither the planar form nor the section shape are preserved in general if $\boldsymbol{w}$ is nonzero
  - these points will lie very near a plane in the deformed beam, the orientation of which is determined by six constraints on $\boldsymbol{w}$
  - the orientation of B is determined by orientation of this plane
- Consider the set of material points that make up the reference line of the undeformed beam:
  - the reference line of the deformed beam is not the same set of material points
  - $\hat{\boldsymbol{B}}_1$ is not in general tangent to the deformed beam reference line

- Only the covariant basis vectors for deformed state are needed

$$\boldsymbol{G}_i(x_1, x_2, x_3) = \frac{\partial \hat{\mathbf{R}}}{\partial x_i}$$

- The deformation is most concisely described in terms of the deformation gradient tensor

$$\underline{\boldsymbol{A}} = \boldsymbol{G}_i \boldsymbol{g}^i$$

- The tensor $\underline{\boldsymbol{A}}$ has a meaning that is heuristically like $\frac{\partial \hat{\mathbf{R}}}{\partial \hat{\mathbf{r}}}$

- A very useful theorem in continuum mechanics called the polar decomposition theorem (PDT) states that the deformation gradient can always be decomposed as

$$\underline{\boldsymbol{A}} = \underline{\hat{\boldsymbol{C}}} \cdot \underline{\boldsymbol{U}}$$

where

  - $\underline{\hat{\boldsymbol{C}}}$ is a finite rotation tensor (corresponding to a pure rotation)
  - $\underline{\boldsymbol{U}}$ is the right stretch tensor (corresponding to a pure strain)
  - Thus, the PDT implies

$$\boldsymbol{G}_i = \underline{\boldsymbol{A}} \cdot \boldsymbol{g}_i = \underline{\hat{\boldsymbol{C}}} \cdot \underline{\boldsymbol{U}} \cdot \boldsymbol{g}_i$$

- That is, to arrive at the coordinate lines in the deformed beam, coordinate lines in the undeformed beam (the $\boldsymbol{g}_i$'s) are
  - first strained (pre-dot-multiplication by $\underline{\boldsymbol{U}}$)
  - then rotated (pre-dot-multiplication by $\underline{\hat{\boldsymbol{C}}}$)
- $\underline{\hat{\boldsymbol{C}}}$ rotates an infinitesimal material element at (x_1, x_2, x_3)
- Some of this rotation is due to global rotation and some is due to local deformation (i.e., warping)
- Supposing that the warping is small (in some sense), let's decompose the total rotation so that

$$\underline{\hat{\boldsymbol{C}}} = \underline{\boldsymbol{C}}(x_1) \cdot \underline{\boldsymbol{C}}^{\mathrm{local}}(x_1, x_2, x_3)$$

where $\underline{\boldsymbol{C}}^{\mathrm{local}}(x_1, x_2, x_3)$ is the *local rotation* tensor

- Recall that any finite rotation tensor can be written in the form

$$\exp(\widetilde{\phi}) = \underline{\mathbf{\Delta}} + \widetilde{\phi} + \frac{\widetilde{\phi}^2}{2} + \frac{\widetilde{\phi}^3}{6} + \dots$$

- Here we let $\underline{\boldsymbol{C}}^{\text{local}} = \exp(\widetilde{\phi})$
- Thus the total finite rotation tensor at some point in the cross-section is

$$\underline{\hat{\boldsymbol{C}}} = \underline{\boldsymbol{C}} \cdot \exp(\widetilde{\phi})$$

- Now let's introduce the Cauchy strain tensor (also known as the engineering strain tensor, the Jaumann strain tensor, and the Biot strain tensor)

$$\underline{\boldsymbol{\Gamma}} = \underline{\boldsymbol{U}} - \underline{\boldsymbol{\Delta}}$$

- This strain definition is nice because it does not contain the host of "strain squared" terms that are a well known part of other strain tensors (such as the Green strain)
- Now, from the PDT and the strain definition, we have

$$\underline{\boldsymbol{\Gamma}} = \underline{\hat{\boldsymbol{C}}}^T \cdot \underline{\boldsymbol{A}} - \underline{\boldsymbol{\Delta}} = \exp\left(-\widetilde{\phi}\right) \cdot \underline{\boldsymbol{C}}^T \cdot \underline{\boldsymbol{A}} - \underline{\boldsymbol{\Delta}}$$

- Note that $\underline{\boldsymbol{\Gamma}}$ is a Lagrangean strain tensor

- Thus, it is appropriate to resolve it along the *undeformed* beam reference triad $\hat{\boldsymbol{b}}_i$ yielding

$$\underline{\boldsymbol{\Gamma}} = \hat{\boldsymbol{b}}_i \Gamma_{ij} \hat{\boldsymbol{b}}_j$$

- It is also appropriate to resolve the tensor $\widetilde{\phi}$ along the undeformed beam triad $\hat{\boldsymbol{b}}_i$ yielding

$$\widetilde{\phi} = \hat{\boldsymbol{b}}_i \widetilde{\phi}_{ij} \hat{\boldsymbol{b}}_j$$

- It now follows that the deformation gradient tensor is resolved along the *mixed* bases

$$\underline{\boldsymbol{A}} = \hat{\boldsymbol{B}}_i A_{ij} \hat{\boldsymbol{b}}_j$$

or

$$A_{ij} = (\hat{\boldsymbol{B}}_i \cdot \boldsymbol{G}_k)(\boldsymbol{g}^k \cdot \hat{\boldsymbol{b}}_j)$$

- Things are now simple enough that we can finish up in matrix form
- Introducing

$$\phi = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

the matrix of strain components become

$$\Gamma = \exp\left(-\widetilde{\phi}\right) A - \Delta$$

- In general an expression for A can be found rather easily, but ϕ is unknown

- For the purpose of simplifying this expression for small local deformation, we let
  - $\max|\Gamma_{ij}(x_1, x_2, x_3)| = \epsilon << 1$
  - $\max|\widetilde{\phi}_{ij}(x_1, x_2, x_3)| = \varphi < 1$
- Then the strain becomes

$$\Gamma = E - \frac{\widetilde{\phi}^2}{2} + \frac{1}{2}\left(E\widetilde{\phi} - \widetilde{\phi}E\right) + O\left(\varphi^4, \varphi^2\epsilon\right)$$

where

$$E = \frac{A + A^T}{2} - \Delta$$

$$\widetilde{\phi} = \frac{A - A^T}{2}$$

- Assume that $\varphi = O(\epsilon^r)$
- Since ϵ (the strain) is small compared to unity, two cases are of interest:
  - *Small local rotation*: $r \geq 1$:

$$\Gamma = E$$

  - *Moderate local rotation*: $\frac{1}{2} \leq r < 1$.

$$\Gamma = E - \frac{\widetilde{\phi}^2}{2} + \frac{1}{2}(E\widetilde{\phi} - \widetilde{\phi}E)$$

- For most engineering beam problems, the small local rotation theory is adequate
- In most of the rest, incorporation of warping nonlinearities exhibited in moderate local rotation theory should be adequate

- The purpose of this example is to provide a simple illustration of the theory developed so far which will give us an explicit form of the strain
- Consider an initially straight beam that is undergoing planar deformation without warping ($\boldsymbol{w} = 0$)
- For the undeformed state, the position vector $\hat{\mathbf{r}}$ is given by

$$\hat{\mathbf{r}} = x_1 \hat{\boldsymbol{b}}_1 + x_2 \hat{\boldsymbol{b}}_2 + x_3 \hat{\boldsymbol{b}}_3$$

- The covariant and contravariant base vectors are very simple

$$\boldsymbol{g}_i = \frac{\partial \hat{\mathbf{r}}}{\partial x_i} = \hat{\boldsymbol{b}}_i = \boldsymbol{g}^i$$

- For the deformed state, the position vector $\hat{\mathbf{R}}$ is given by

$$\hat{\mathbf{R}} = (x_1 + u_{b1})\hat{\boldsymbol{b}}_1 + x_2\hat{\boldsymbol{B}}_2 + u_{b2}\hat{\boldsymbol{b}}_2 + x_3\hat{\boldsymbol{b}}_3$$

- Since the warping is zero, the reference cross-sectional plane in the deformed beam is made up of the same material points which make up the reference cross-sectional plane of the undeformed beam
- The covariant base vectors are $\boldsymbol{G}_i = \frac{\partial\hat{\mathbf{R}}}{\partial x_i}$ such that

$$\begin{aligned}
\boldsymbol{G}_1 =& (1 + u'_{b1})\hat{\boldsymbol{b}}_1 + x_2(\hat{\boldsymbol{B}}_2)' + u'_{b2}\hat{\boldsymbol{b}}_2 \\
\boldsymbol{G}_2 =& \hat{\boldsymbol{B}}_2 \\
\boldsymbol{G}_3 =& \hat{\boldsymbol{b}}_3 = \hat{\boldsymbol{B}}_3
\end{aligned}$$

- The matrix of direction cosines $C = C^{Bb}$ can be simply represented with a single angle ζ

$$\begin{Bmatrix} \hat{\boldsymbol{B}}_1 \\ \hat{\boldsymbol{B}}_2 \\ \hat{\boldsymbol{B}}_3 \end{Bmatrix} = \begin{bmatrix} \cos\zeta & \sin\zeta & 0 \\ -\sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{Bmatrix}$$

  where ζ is the cross-section rotation
- With this, the matrix of deformation gradient components in mixed bases can be written as

$$A = \begin{bmatrix} (1+u'_{b1})\cos\zeta + u'_{b2}\sin\zeta - x_2\zeta' & 0 & 0 \\ -(1+u'_{b1})\sin\zeta + u'_{b2}\cos\zeta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The strain for *small local rotation* applies here since there is no warping

$$\Gamma = \frac{1}{2}(A + A^T) - \Delta$$

so that

$$\begin{aligned} \Gamma_{11} =& (1 + u'_{b1})\cos\zeta + u'_{b2}\sin\zeta - x_2\zeta' - 1 \\ 2\Gamma_{12} =& 2\Gamma_{21} = u'_{b2}\cos\zeta - (1 + u'_{b1})\sin\zeta \end{aligned}$$

- Notice that these strains, when linearized, reduce to those of a Timoshenko beam

- For the undeformed state

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \boldsymbol{r}(x_1) + x_\alpha \hat{\boldsymbol{b}}_\alpha(x_1)$$

where $\boldsymbol{r}(x_1)$ is the position vector to points on the reference line

- If the reference line is chosen as the locus of cross-sectional centroids, then

$$\boldsymbol{r} = \langle \hat{\mathbf{r}} \rangle \;\text{ if and only if }\; \langle x_\alpha \rangle = 0$$

where the angle brackets stand for the average value over the cross-section

- The covariant base vectors are

$$\boldsymbol{g}_1 = \boldsymbol{r}' + x_\alpha \hat{\boldsymbol{b}}'_\alpha \qquad \boldsymbol{g}_\alpha = \hat{\boldsymbol{b}}_\alpha$$

- It is helpful to derive some special formulae to express $\boldsymbol{g}_1$ in a more recognizable form

$$\boldsymbol{r}' = \hat{\boldsymbol{b}}_1$$
$$(\hat{\boldsymbol{b}}_i)' = \boldsymbol{k} \times \hat{\boldsymbol{b}}_i = \underline{\tilde{\boldsymbol{k}}} \cdot \hat{\boldsymbol{b}}_i$$

where $\boldsymbol{k} = k_{bi}\hat{\boldsymbol{b}}_i$ ($\underline{\tilde{\boldsymbol{k}}}$) is the curvature vector (tensor) of the undeformed beam and

$$\underline{\tilde{\boldsymbol{k}}} = \left(\underline{\boldsymbol{C}}^{bA}\right)' \cdot \underline{\boldsymbol{C}}^{Ab} = \boldsymbol{k} \times \underline{\boldsymbol{\Delta}} = \hat{\boldsymbol{b}}_i \widetilde{k}_{ij} \hat{\boldsymbol{b}}_j = -\hat{\boldsymbol{b}}_i e_{ijl} k_{bl} \hat{\boldsymbol{b}}_j$$

- Normally the components of $\boldsymbol{k}$ are known in the b basis:

$$k_{b1} = \text{twist per unit length of the undeformed beam}$$
$$k_{b\alpha} = \text{components of undeformed beam curvature}$$

- With these definitions, the contravariant base vectors become

$$\boldsymbol{g}^1 = \frac{\hat{\boldsymbol{b}}_1}{\sqrt{g}}$$

$$\boldsymbol{g}^2 = \frac{x_3 k_{b1} \hat{\boldsymbol{b}}_1}{\sqrt{g}} + \hat{\boldsymbol{b}}_2$$

$$\boldsymbol{g}^3 = -\frac{x_2 k_{b1} \hat{\boldsymbol{b}}_1}{\sqrt{g}} + \hat{\boldsymbol{b}}_3$$

  where

$$\sqrt{g} = 1 - x_2 k_{b3} + x_3 k_{b2} > 0$$

- Normally, $\sqrt{g}$ is near unity

- Let us now use the more general displacement field referred to in the development

$$\hat{\mathsf{R}}(x_1, x_2, x_3) = \boldsymbol{R} + x_\alpha \hat{\boldsymbol{B}}_\alpha + w_i \hat{\boldsymbol{B}}_i$$

- Here $\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{u}$, $\boldsymbol{u} = u_{bi} \hat{\boldsymbol{b}}_i$ is the displacement vector of the beam, and $\boldsymbol{w} = w_i \hat{\boldsymbol{b}}_i$
- The push-forward operation altered the bases on the last two terms

- We must constrain the warping in order for $\hat{\boldsymbol{B}}_i$ and $\boldsymbol{R}$ to be well defined
  - The 1D position is the average position (i.e. average warping over the cross-section is zero)

$$\boldsymbol{R} = \left\langle \hat{\mathbf{R}} \right\rangle \text{ if and only if } \langle w_i \rangle = 0$$

  - Orientation of deformed beam cross-sectional frame

$$\hat{\boldsymbol{B}}_1 \cdot \left\langle x_\alpha \hat{\mathbf{R}} \right\rangle = 0 \text{ if and only if } \langle x_\alpha w_1 \rangle = 0$$

  - The average rotation about $\hat{\boldsymbol{B}}_1$ is zero

$$\hat{\boldsymbol{B}}_2 \cdot \left\langle \hat{\mathbf{R}}_{,2} \right\rangle = \hat{\boldsymbol{B}}_3 \cdot \left\langle \hat{\mathbf{R}}_{,3} \right\rangle \text{ if and only if } \langle w_{3,2} - w_{2,3} \rangle = 0$$

- The strain field can be conveniently expressed in terms of 1-D variables with the following definitions:

$$\begin{aligned} \boldsymbol{R}' &= (1+\gamma_{11})\hat{\boldsymbol{B}}_1 + 2\gamma_{1\alpha}\hat{\boldsymbol{B}}_\alpha \\ \hat{\boldsymbol{B}}_i' &= \boldsymbol{K} \times \hat{\boldsymbol{B}}_i \end{aligned}$$

- The force strain components are then

$$\gamma = C(e_1 + u_b' + \widetilde{k}_b u_b) - e_1$$

where

$$\gamma = \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix}; \qquad e_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

- The moment strains components are given by $\kappa = K_B - k_b$ so that

$$\tilde{\kappa} = -C'C^T + C\widetilde{k_b}C^T - \widetilde{k_b}$$

- In vector-dyadic form

$$\begin{aligned}\gamma &= \underline{\boldsymbol{C}}^{zB} \cdot \boldsymbol{R}' - \underline{\boldsymbol{C}}^{zb} \cdot \boldsymbol{r}' \\ \boldsymbol{\kappa} &= \underline{\boldsymbol{C}}^{zB} \cdot \boldsymbol{K} - \underline{\boldsymbol{C}}^{zb} \cdot \boldsymbol{k}\end{aligned}$$

  where z is an arbitrary frame

- Letting the z frame be b, γ and $\boldsymbol{\kappa}$ have γ and κ as measure numbers in the b basis

- The vector-dyadic form gives more insight as to what these expressions mean; see Hodges (1990) for this discussion
- Sometimes it is better to pull back to the *a* basis and regard measure numbers in the *a* basis as generalized strains
- Note that $\boldsymbol{K}$ is the curvature vector of the deformed beam defined by

$$\underline{\tilde{\boldsymbol{K}}} = \left(\underline{\boldsymbol{C}}^{Ba}\right)' \cdot \underline{\boldsymbol{C}}^{aB}$$

where

$$\begin{aligned} K_{B1} &= \text{the twist per unit length of the deformed beam} \\ K_{B\alpha} &= \text{components of deformed beam curvature} \end{aligned}$$

- The strain field for small local rotation is given by matrix E

$$
\begin{aligned}
\sqrt{g}E_{11} &= \gamma_{11} + x_3\kappa_2 - x_2\kappa_3 + w_1' \\
&\quad + k_{b1}(x_3 w_{1,2} - x_2 w_{1,3}) + k_{b2}w_3 - k_{b3}w_2 + \underline{\kappa_2 w_3 - \kappa_3 w_2} \\
2\sqrt{g}E_{12} = 2\sqrt{g}E_{21} &= 2\gamma_{12} - x_3\kappa_1 + w_2' + \sqrt{g}w_{1,2} \\
&\quad + k_{b1}(x_3 w_{2,2} - x_2 w_{2,3}) + k_{b3}w_1 - k_{b1}w_3 + \underline{\kappa_3 w_1 - \kappa_1 w_3} \\
2\sqrt{g}E_{13} = 2\sqrt{g}E_{31} &= 2\gamma_{13} + x_2\kappa_1 + w_3' + \sqrt{g}w_{1,3} \\
&\quad + k_{b1}(x_3 w_{3,2} - x_2 w_{3,3}) + k_{b1}w_2 - k_{b2}w_1 + \underline{\kappa_1 w_2 - \kappa_2 w_1}
\end{aligned}
$$

$$
E_{22} = w_{2,2} \qquad 2E_{23} = 2E_{32} = w_{2,3} + w_{3,2} \qquad E_{33} = w_{3,3}
$$

- Notice that if $O(\epsilon^2)$ terms (underlined) are neglected, then the strain field is linear in generalized strains γ and κ

- An example is worked out for a specified warping
- In general one needs to solve for the warping, which is affected by cross-sectional geometry and material properties as well as initial curvature and twist
- Warping is typically a linear function of the 1-D strain measures, leading to a strain energy density of the form $U = U(\gamma, \kappa)$
- In other words, warping disappears from the problem, affecting only the elastic constants of the beam (as in the St. Venant problem)
- Now we will outline a finite element approach to cross-sectional analysis and then proceed as if section constants are known

- Real rotor blades are
  - internally complex, built-up structures
  - more and more likely to be made of composite materials
- These provide certain well-known advantages
  - High strength-to-weight ratio
  - Long fatigue life
  - Damage tolerance
  - Directional nature with potential for tailoring
- However, they also introduce well-known complexities
  - Anisotropic
  - Inhomogeneous

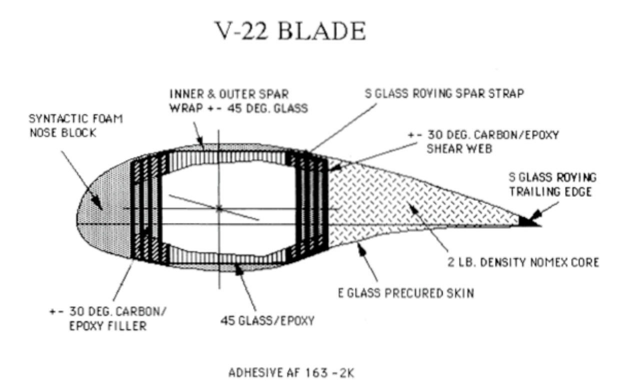

Figure: V-22 Blade Section – Courtesy Bell Helicopters

- Rotor blades are 3-D bodies and demand a 3-D approach
- Consider a 3-D representation of the strain energy

$$\mathcal{U} = \frac{1}{2}\int\int\int \Gamma^T D\Gamma \, dx_2 dx_3 dx_1$$

where $\Gamma = \Gamma(\hat{u})$ and $\hat{u} = \hat{u}(x_1, x_2, x_3)$
- This offers:
  - a complete 3-D description of the problem
  - inhomogeneous, anisotropic, nonlinear – no problem to represent, but $O(10^6)$ degrees of freedom may be required!
  - more than adequate motivation to attempt to use beam theory

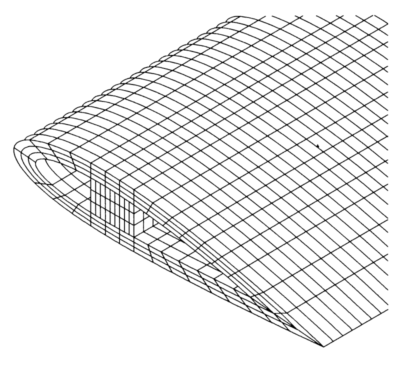

Figure: Schematic of discretized wing

- To apply beam theory to composite rotor blades, and expect an accurate answer, requires us to capture 3-D behavior with a 1-D model!
- Problem: as we've seen, there is a 3-D quantity (warping) present in the energy functional
- Therefore, one must start from a general 3-D representation, and solve the problem including
    - inhomogeneous, anisotropic materials
    - all possible deformation in the 3-D representation
    - determination of the warping as a function of 1-D variables

- Beams have one dimension much larger than the other two
- Dimensional reduction takes the 3-D body and represents it as a 1-D body
- This implies that small parameters must be exploited
  - maximum magnitude of the strain $\epsilon << 1$
  - $a < \ell$ (a is a typical cross-sectional diameter and ℓ is the characteristic length of the deformation along the beam)
  - $a < R$ ($R = 1/\sqrt{k_b^T k_b}$)
- The result is the strain energy per unit length
  - in terms of 1-D measures of strain
  - with asymptotically exact cross-sectional elastic constants
  - with asymptotically exact recovering relations

- We follow this procedure when modeling a beam:
  - Find 2-D (sectional) elastic constants for use in 1-D (beam) theory
  - Find 1-D (beam) deformation parameters from loading and sectional constants
  - Find 3-D displacement, strain, and stress in terms of 1-D (beam) deformation parameters
- Analogy from elementary beam theory:
  - Constitutive relation: $M_2 = EI_{22}u_3{}''$
  - Equilibrium equation: $M_2{}'' = q(x_1)$
  - Recovery relation: $\sigma_{11} = -\frac{M_2 x_3}{I_{22}}$
- Approach based on variational-asymptotic method

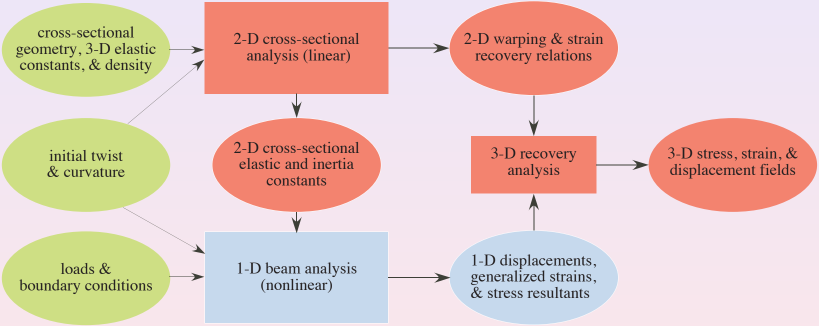

Figure: Process of Beam Analysis

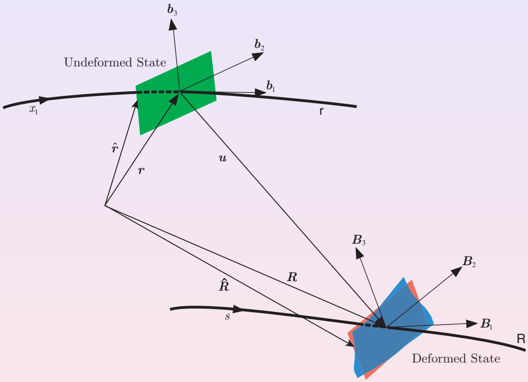

Figure: Beam kinematics allows for large displacement and rotation with small strain and local rotation

Undeformed state:

$$\hat{\boldsymbol{r}}(x_1, x_2, x_3) = \boldsymbol{r}(x_1) + x_\alpha \hat{\boldsymbol{b}}_\alpha = \boldsymbol{r}(x_1) + h\zeta_\alpha \hat{\boldsymbol{b}}_\alpha$$

$$\boldsymbol{r}' = \hat{\boldsymbol{b}}_1 \qquad (\hat{\boldsymbol{b}}_i)' = \boldsymbol{k} \times \hat{\boldsymbol{b}}_i = \underline{\tilde{\boldsymbol{k}}} \cdot \hat{\boldsymbol{b}}_i \qquad \langle\langle x_\alpha \rangle\rangle = 0$$

$$\langle\langle \bullet \rangle\rangle = \frac{1}{|\mathcal{A}|}\int_S \bullet dx_2 dx_3 \quad \langle \bullet \rangle = \frac{1}{|\mathcal{A}|}\int_S \bullet \sqrt{g} dx_2 dx_3$$

$$\sqrt{g} = 1 - x_2 k_{b3} + x_3 k_{b2} > 0 \qquad \boldsymbol{g}^1 = \frac{\hat{\boldsymbol{b}}_1}{\sqrt{g}}$$

$$\boldsymbol{g}^2 = \frac{x_3 k_{b1} \hat{\boldsymbol{b}}_1}{\sqrt{g}} + \hat{\boldsymbol{b}}_2 \qquad \boldsymbol{g}^3 = -\frac{x_2 k_{b1} \hat{\boldsymbol{b}}_1}{\sqrt{g}} + \hat{\boldsymbol{b}}_3$$

- Deformed state

$$\hat{\boldsymbol{R}}(x_1, x_2, x_3) = \boldsymbol{R}(x_1) + x_\alpha \hat{\boldsymbol{B}}_\alpha(x_1) + w_n(x_1, \zeta_2, \zeta_3)\hat{\boldsymbol{B}}_n(x_1)$$
$$(\hat{\boldsymbol{B}}_i)' = \boldsymbol{K} \times \hat{\boldsymbol{B}}_i = \underline{\tilde{\boldsymbol{K}}} \cdot \hat{\boldsymbol{B}}_i$$

- Constraints

$$\begin{aligned} \mathbf{R}' =& (1 + \gamma_{11})\hat{\boldsymbol{B}}_1 \\ \langle\langle w_n(x_1, \zeta_2, \zeta_3)\rangle\rangle =& 0 \\ \Big\langle\Big\langle w_{2,3}(x_1, \zeta_2, \zeta_3)\Big\rangle\Big\rangle =& \Big\langle\Big\langle w_{3,2}(x_1, \zeta_2, \zeta_3)\Big\rangle\Big\rangle \end{aligned}$$

- Under the condition of small local rotation, Jaumann-Biot-Cauchy strain measures are

$$\Gamma^* = \frac{1}{2}(\chi + \chi^T) - \Delta$$

$$\chi_{mn} = \hat{\boldsymbol{B}}_m \cdot \frac{\partial \hat{\boldsymbol{R}}}{\partial x_k} \boldsymbol{g}^k \cdot \hat{\boldsymbol{b}}_n$$

- The matrix χ contains components of the deformation gradient tensor in mixed bases
- In column matrix form they are arranged as

$$\Gamma = \lfloor \Gamma^*_{11} \; 2\Gamma^*_{12} \; 2\Gamma^*_{13} \; \Gamma^*_{22} \; 2\Gamma^*_{23} \; \Gamma^*_{33} \rfloor^T$$

- The 3D strain is linear in γ_{11}, κ, the warping w, and its derivatives

$$\Gamma = \frac{1}{a}\Gamma_a w + \Gamma_\epsilon \overline{\epsilon} + \Gamma_R w + \Gamma_\ell w'$$

$$\overline{\epsilon} = \begin{Bmatrix} \gamma_{11} \\ \kappa \end{Bmatrix}$$

$$\begin{aligned} \gamma_{11} &= {e_1}^T C(e_1 + u' + \widetilde{k}u) - 1 \\ 0 &= {e_\alpha}^T C(e_1 + u' + \widetilde{k}u) \\ \widetilde{\kappa} &= -C'C^T + C\widetilde{k}C^T - \widetilde{k} \end{aligned}$$

where

$$\Gamma_a = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial \zeta_2} & 0 & 0 \\ \frac{\partial}{\partial \zeta_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial \zeta_2} & 0 \\ 0 & \frac{\partial}{\partial \zeta_3} & \frac{\partial}{\partial \zeta_2} \\ 0 & 0 & \frac{\partial}{\partial \zeta_3} \end{bmatrix} \qquad \Gamma_\epsilon = \frac{1}{\sqrt{g}} \begin{bmatrix} 1 & 0 & -\zeta_3 & \zeta_2 \\ 0 & \zeta_3 & 0 & 0 \\ 0 & -\zeta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_R = \frac{1}{\sqrt{g}} \begin{bmatrix} \tilde{k} + k_1 \left(\zeta_3 \frac{\partial}{\partial \zeta_2} - \zeta_2 \frac{\partial}{\partial \zeta_3} \right) \Delta \\ 0 \end{bmatrix} \qquad \Gamma_\ell = \frac{1}{\sqrt{g}} \begin{Bmatrix} \Delta \\ 0 \end{Bmatrix}$$

- The strain energy density for a beam per unit length

$$U = \frac{1}{2}\left\langle \Gamma^T D \Gamma \right\rangle$$

- The 3-D Jaumann stress σ, which is conjugate to the Jaumann strain Γ is

$$\sigma = D\Gamma$$

- The basic 3-D problem can be now represented as the following minimization problem

$$\int U\left[\overline{\epsilon}(x_1), w(x_1,\zeta_2,\zeta_3), \frac{\partial w(x_1,\zeta_2,\zeta_3)}{\partial \zeta_\alpha}, w'(x_1,\zeta_2,\zeta_3)\right] dx_1$$
$$+ \text{potential energy of external forces} \rightarrow \min$$

- In constructing a 1-D beam theory from 3-D elasticity, one attempts to represent the strain energy stored in the 3-D body by finding the strain energy which would be stored in an imaginary 1-D body
- The warping displacement components $w_n(x_1, \zeta_2, \zeta_3)$ must be written as functions of the 1-D functions $\boldsymbol{R}(x_1)$ and $\hat{\boldsymbol{B}}_n(x_1)$
- This is too complicated to do exactly due to nonlocal dependence
- One can and must take advantage of small parameters

- Two small parameters
  - The ratio $\frac{a}{\ell}$ of the maximum dimension of the cross-section (a) divided by the characteristic wavelength of the deformation along the beam (ℓ)
  - The maximum dimension of the cross section times the maximum magnitude of initial curvature or twist $\frac{a}{R}$
- Since both of them have the same numerator, expansion in $\frac{a}{\ell}$ and $\frac{a}{R}$ is equivalent to the expansion in a only
- Note the following concerning the maximum strain magnitude ε is
  - necessary for determining the strain field
  - only needed in the cross-sectional analysis when analyzing the trapeze effect

- The *variational-asymptotic method*
  - is essentially the work of Berdichevsky and co-workers (1976, 1979, 1981, 1983, 1985, etc.)
  - considers small parameters applied to energy functionals rather than to differential equations
- If μ is a typical material modulus, the strain energy is of the form

$$\mu\varepsilon^2\left[O(1)+O\left(\frac{a}{\ell}\right)+O\left(\frac{a}{R}\right)\right.$$
$$\left.+O\left(\frac{a}{\ell}\right)^2+O\left(\frac{a}{R}\right)^2+O\left(\frac{a^2}{\ell R}\right)+\dots\right]$$

- Finite element discretization of the warping

$$w(x_1, \zeta_2, \zeta_3) = S(\zeta_2, \zeta_3) W(x_1)$$

- Strain energy density

$$\begin{aligned} 2U = & \left(\frac{1}{a}\right)^2 W^T E W \\ & + \left(\frac{1}{a}\right) 2W^T (D_{a\epsilon}\bar{\epsilon} + D_{aR} W + D_{a\ell} W') \\ & + (1)(\bar{\epsilon}^T D_{\epsilon\epsilon}\bar{\epsilon} + W^T D_{RR} W + W'^T D_{\ell\ell} W' \\ & + 2W^T D_{R\epsilon}\bar{\epsilon} + 2W'^T D_{\ell\epsilon}\bar{\epsilon} + 2W^T D_{R\ell} W') \end{aligned}$$

- The following definitions were introduced

$$E = \left\langle\left\langle [\Gamma_a S]^T \mathcal{D} [\Gamma_a S] \right\rangle\right\rangle \qquad D_{a\epsilon} = \left\langle\left\langle [\Gamma_a S]^T \mathcal{D} [\Gamma_\epsilon] \right\rangle\right\rangle$$

$$D_{aR} = \left\langle\left\langle [\Gamma_a S]^T \mathcal{D} [\Gamma_R S] \right\rangle\right\rangle \qquad D_{a\ell} = \left\langle\left\langle [\Gamma_a S]^T \mathcal{D} [\Gamma_\ell S] \right\rangle\right\rangle$$

$$D_{\epsilon\epsilon} = \left\langle\left\langle [\Gamma_\epsilon]^T \mathcal{D} [\Gamma_\epsilon] \right\rangle\right\rangle \qquad D_{RR} = \left\langle\left\langle [\Gamma_R S]^T \mathcal{D} [\Gamma_R S] \right\rangle\right\rangle$$

$$D_{\ell\ell} = \left\langle\left\langle [\Gamma_\ell S]^T \mathcal{D} [\Gamma_\ell S] \right\rangle\right\rangle \qquad D_{R\epsilon} = \left\langle\left\langle [\Gamma_R S]^T \mathcal{D} [\Gamma_\epsilon] \right\rangle\right\rangle$$

$$D_{\ell\epsilon} = \left\langle\left\langle [\Gamma_\ell S]^T \mathcal{D} [\Gamma_\epsilon] \right\rangle\right\rangle \qquad D_{R\ell} = \left\langle\left\langle [\Gamma_R S]^T \mathcal{D} [\Gamma_\ell S] \right\rangle\right\rangle$$

- Note: these matrices carry information about the material properties and geometry of a given cross section

- Warping field is expanded

$$W = W_0 + aW_1 + a^2 W_2$$

- Constraints are discretized

$$W^T H \Psi_{c\ell} = 0$$

where

$$H = \left\langle\left\langle S^T S \right\rangle\right\rangle \qquad E\Psi_{c\ell} = 0 \qquad \Psi_{c\ell}^T H \Psi_{c\ell} = \Delta$$

- Strain energy up to order a^2

$$
\begin{aligned}
2U = {} & (1)[\bar{\epsilon}^T D_{\epsilon\epsilon}\bar{\epsilon} + {W_0}^T E W_0 + 2{W_0}^T D_{a\epsilon}\bar{\epsilon}] + \\
& 2(a)[\underline{{W_0}^T E W_1} + \underline{{W_1}^T D_{a\epsilon}\bar{\epsilon}} + {W_0}^T D_{aR} W_0 + {W_0}^T D_{R\epsilon}\bar{\epsilon}] + \\
& (a^2)[\underline{\underline{2{W_0}^T E W_2}} + {W_1}^T E W_1 + 2{W_0}^T (D_{aR} + D_{aR}^T) W_1 \\
& + \underline{\underline{2{W_2}^T D_{a\epsilon}\bar{\epsilon}}} + {W_0}^T D_{RR} W_0 + 2{W_1}^T D_{R\epsilon}\bar{\epsilon}]
\end{aligned}
$$

- Evaluation of W_0
  - According to the variational-asymptotic method, one keeps only the dominant interaction term between W and $\overline{\epsilon}$ and the dominant quadratic term in W

$$2U_0 = \left(\frac{1}{a}\right)^2 W^T E W + \left(\frac{1}{a}\right) 2 W^T D_{a\epsilon}\overline{\epsilon}$$

  - The Euler-Lagrange equation (including constraints) is

$$\left(\frac{1}{a}\right) E W + D_{a\epsilon}\overline{\epsilon} = H \Psi_{c\ell}\mu$$

  - The Lagrange multiplier becomes

$$\mu = \Psi_{c\ell}^T D_{a\epsilon}\overline{\epsilon}$$

- Evaluation of W_0 (continued)
  - The warping becomes

$$\left(\frac{1}{a}\right) EW = -(\Delta - H\Psi_{c\ell}\Psi_{c\ell}^T) D_{a\epsilon}\overline{\epsilon}$$

  - Note the generalized inverse

$$EE_{c\ell}^+ = \Delta - H\Psi_{c\ell}\Psi_{c\ell}^T$$
$$E_{c\ell}^+ E = \Delta - \Psi_{c\ell}\Psi_{c\ell}^T H$$
$$E_{c\ell}^+ EE_{c\ell}^+ = E_{c\ell}^+$$

- Evaluation of W_0 (continued)

$$W = -aE_{c\ell}^{+}D_{a\epsilon}\overline{\epsilon} = W_0$$

- Prismatic beam stiffness matrix

$$2U = \overline{\epsilon}^T A \overline{\epsilon}$$

$$A = D_{\epsilon\epsilon} - [D_{a\epsilon}]^T E_{c\ell}^{+}[D_{a\epsilon}]$$

- Evaluation of W_1
  - Perturbing the warping W, one obtains

$$W = W_0 + aW_1$$

  - The perturbation of the energy then becomes

$$\begin{aligned} 2U_1 =& a^2 W_1{}^T E W_1 + \\ &+ 2aW_1{}^T (D_{a\epsilon}\overline{\epsilon} + EW_0 + a^2 D_{aR} W_1) \\ &+ 2a^2 W_1{}^T (D_{aR} + D_{aR}^T) W_0 \end{aligned}$$

- Evaluation of W_1 (continued)
  - Minimization of the perturbed energy (including constraints) yields the Euler-Lagrange equation

$$EW_1 + \left(\frac{1}{a}\right) H\Psi_{c\ell}\Psi_{c\ell}^T D_{a\epsilon}\overline{\epsilon} + (D_{aR} + D_{aR}^T)W_0 = H\Psi_{c\ell}\mu$$

  - The Lagrange multiplier becomes

$$\mu = \left(\frac{1}{a}\right) \Psi_{c\ell}^T D_{a\epsilon}\overline{\epsilon} + \Psi_{c\ell}^T(D_{aR} + D_{aR}^T)W_0$$

  - The warping then becomes

$$EW_1 = -(\Delta - H\Psi_{c\ell}\Psi_{c\ell}^T)(D_{aR} + D_{aR}^T)W_0$$

- Evaluation of W_1 (continued)
  - The strain energy corrected to first order in a/R is found (all influence of the perturbed warping cancels out)

$$2U = \overline{\epsilon}^T A_r \overline{\epsilon}$$

where

$$\begin{aligned} A_r = & D_{\epsilon\epsilon} - (D_{a\epsilon})^T(\Psi_{c\ell} D_{a\epsilon}) \\ & + a[(\Psi_{c\ell} D_{a\epsilon})^T(D_{aR} + D_{aR}^T)(\Psi_{c\ell} D_{a\epsilon}) \\ & - (\Psi_{c\ell} D_{a\epsilon})^T D_{R\epsilon} - D_{R\epsilon}^T(\Psi_{c\ell} D_{a\epsilon})] \end{aligned}$$

  - Higher-order corrections in a are presented in detail in the book by Hodges (2006)

- The 1-D constitutive law that follows from the energy is of the form

$$\begin{Bmatrix} F_{B_1} \\ M_B \end{Bmatrix} = [S] \begin{Bmatrix} \gamma_{11} \\ \kappa \end{Bmatrix}$$

- For isotropic, prismatic beams when x_2 and x_3 are principal axes

$$\begin{Bmatrix} F_{B_1} \\ M_{B_1} \\ M_{B_2} \\ M_{B_3} \end{Bmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & EI_3 \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}$$

- Adding modest pretwist and initial curvature, and letting $D = I_2 + I_3 - J$ and $\beta = 1 + \nu$, one obtains

$$\begin{Bmatrix} F_{B_1} \\ M_{B_1} \\ M_{B_2} \\ M_{B_3} \end{Bmatrix} = \begin{bmatrix} EA & EDk_1 & -\beta EI_2 k_2 & -\beta EI_3 k_3 \\ EDk_1 & GJ & 0 & 0 \\ -\beta EI_2 k_2 & 0 & EI_2 & 0 \\ -\beta EI_3 k_3 & 0 & 0 & EI_3 \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}$$

- For generally anistropic beams, the matrix S becomes fully populated (while remaining symmetric and positive definite, of course)

- Taking into account the effects of $\frac{a}{\ell}$, one finds a model for prismatic, isotropic beams of the form

$$\begin{Bmatrix} F_{B_1} \\ F_{B_2} \\ F_{B_3} \\ M_{B_1} \\ M_{B_2} \\ M_{B_3} \end{Bmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GK_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & GK_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & GJ & 0 & 0 \\ 0 & 0 & 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & EI_3 \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}$$

- Initial twist and curvature add coupling which shifts the neutral and/or shear centers
- For generally anistropic beams, the matrix S becomes fully populated (while remaining symmetric and positive definite, of course)

- Cross-sectional analysis can be undertaken by the computer program VABS (commercially available)
- VABS produces
  - the 6×6 matrices for stiffness and inertia properties
  - recovery relations that allow one to use results from the beam analysis to find 3D stresses, strains and displacements

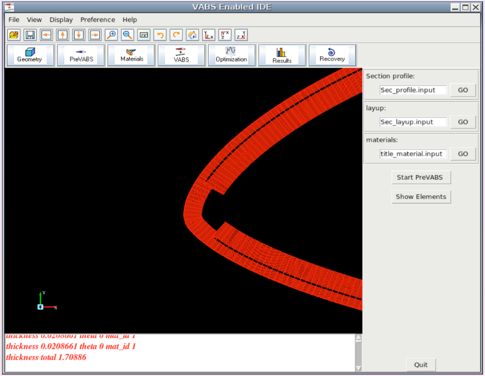

Figure: Sample cross-sectional PreVABS mesh

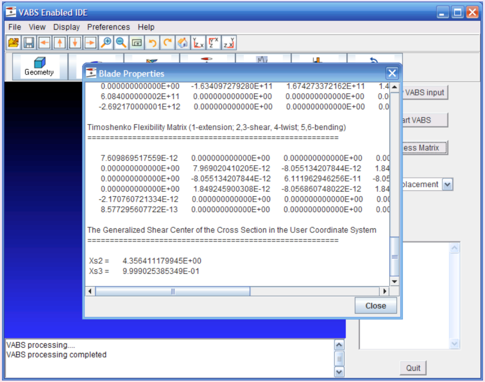

Figure: Sample stiffness output from cross-sectional analysis VABS

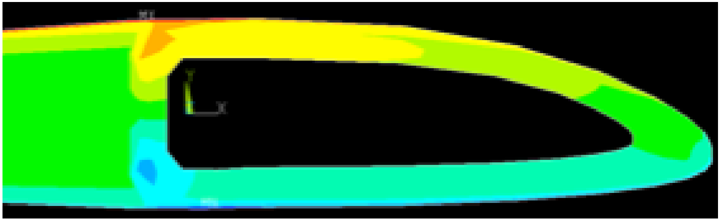

Figure: Sample stress output from cross-sectional analysis VABS

- A comprehensive beam modeling scheme is presented which
  - allows for systematic treatment of *all possible types of deformation* in composite beams
  - is ideal for 2-D finite element sectional analysis
  - gives asymptotically exact section constants
  - leads to the geometrically exact beam equations, found in Hodges (2006)
  - gives formulae needed for recovering strain and stress distributions
  - is the basis for the commercial computer code VABS

- Present an exact set of equations for the dynamics of beams in a moving frame suitable for rotorcraft applications
- Present the equations in a matrix notation so that the entire formulation can be written most concisely
- Present a unified framework in which other less general developments can be checked for consistency
- Show how differences that normally arise in beam analyses can be interpreted in light of this unified framework
- Show how the present unified framework can be used to develop an elegant basis for accurate, efficient, and robust computational solution techniques

- Published work reveals differences in the way the displacement field is represented
  - different numbers of kinematical variables to describe motion of the cross sectional frame
  - different variables to describe finite rotation of this frame
  - different orthogonal base vectors for measurement of displacement
- In the present approach the kinematical equations are exact and separate from equilibrium and constitutive law developments
  - generalized strains are written in simple matrix notation
  - change of displacement and orientation variables affects only kinematics (a relatively small portion of the analysis)

- Reissner (1973) derived exact intrinsic equations for beam static equilibrium (limited to unrestrained warping)
  - no displacement or orientation variables ("intrinsic")
  - reduce to the Kirchhoff-Clebsch-Love equations when shear deformation is set equal to zero
  - geometrically exact – all correct beam equations can be derived from these
  - intrinsic generalized strains were derived from virtual work
- Asymptotic analysis shows that for slender, closed-section beams, a linear 2-D cross-sectional analysis determines elastic constants for use in nonlinear 1-D beam analysis

- Thus, in the present work we presuppose that an elastic law is given as a 1-D strain energy function
- Present analysis is based on *exact* kinematics and kinetics but an *approximate* constitutive law (no "ordering scheme")
- Here the exact equilibrium equations are
  - extended to account for dynamics
  - derived from Hamilton's weak principle (HWP) by Hodges (1990) to facilitate development of a finite element method
- A mixed finite element approach can be developed from HWP in which there are many computational advantages
- Widely available codes RCAS and DYMORE are based on these equations, albeit in displacement form in the latter

- Recall that the displacement field is represented by

$$\begin{aligned}\hat{\boldsymbol{r}} &= \boldsymbol{r} + \xi = \boldsymbol{r} + x_2\hat{\boldsymbol{b}}_2 + x_3\hat{\boldsymbol{b}}_3 \\ \hat{\boldsymbol{R}} &= \boldsymbol{R} + \underline{\boldsymbol{C}} \cdot (\xi + \boldsymbol{w}) = \boldsymbol{r} + \boldsymbol{u} + x_2\hat{\boldsymbol{B}}_2 + x_3\hat{\boldsymbol{B}}_3 + w_i\hat{\boldsymbol{B}}_i\end{aligned}$$

with $\underline{\boldsymbol{C}}$ as the global rotation tensor

- Force strains

$$\gamma = \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix} = C\left(e_1 + u_b' + \widetilde{k}_b u_b\right) - e_1$$

where u_b is the column matrix whose elements are the measure numbers of the displacement along $\hat{\boldsymbol{b}}_i$

- Moment strains

$$\kappa = \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = K_B - k_b$$

where $\widetilde{K}_B = -C'C^T + C\widetilde{k}_b C^T$

- Introduce column matrices of known measure numbers along $\hat{\boldsymbol{b}}_i$ for inertial
  - velocity of the undeformed beam reference axis v_b
  - angular velocity of the undeformed beam cross sectional frame ω_b
- Generalized speeds in the sense of Kane and Levinson (1985) are elements of column matrices that contain measure numbers along $\hat{\boldsymbol{B}}_i$ for inertial
  - velocity of deformed beam reference axis

$$V_B = C(v_b + \dot{u}_b + \widetilde{\omega}_b u_b)$$

  - angular velocity of deformed beam cross-sectional frame

$$\widetilde{\Omega}_B = -\dot{C}C^T + C\widetilde{\omega}_b C^T$$

- As developed by Borri et al. (1985), HWP for the present problem is

$$\int_{t_1}^{t_2}\int_0^{\ell}\left[\delta(K-U)+\overline{\delta W}\right]dx_1\,dt=\overline{\delta\mathcal{A}}$$

- Here
  - t_1 and t_2 are arbitrary fixed times
  - K and U are the kinetic and strain energy densities per unit length, respectively
  - $\overline{\delta\mathcal{A}}$ is the virtual action at the ends of the beam and at the ends of the time interval
  - $\overline{\delta W}$ is the virtual work of applied loads per unit length

- St. Venant warping influences the elastic constants in a beam constitutive law written in terms of γ and κ
- Regarding the strain energy per unit length as $U = U(\gamma, \kappa)$, one can obtain the variations required in HWP as

$$\int_0^\ell \delta U dx_1 = \int_0^\ell \left[\delta\gamma^T \left(\frac{\partial U}{\partial \gamma} \right)^T + \delta\kappa^T \left(\frac{\partial U}{\partial \kappa} \right)^T \right] dx_1$$

- The partial derivatives are section force and moment measure numbers along $\hat{\boldsymbol{B}}_i$

$$F_B = \left(\frac{\partial U}{\partial \gamma} \right)^T ; \qquad M_B = \left(\frac{\partial U}{\partial \kappa} \right)^T$$

- Introduce a column matrix of virtual displacements defined as $\overline{\delta q}_B = C\delta u_b$
- Similarly, let the antisymmetric matrix of virtual rotations be $\widetilde{\overline{\delta\psi}}_B = -\delta C C^T$
- Now, one can show that

$$\begin{aligned}\delta\gamma =& \overline{\delta q}'_B + \widetilde{K}_B\overline{\delta q}_B + (\widetilde{e}_1 + \widetilde{\gamma})\overline{\delta\psi}_B \\ \delta\kappa =& \overline{\delta\psi}'_B + \widetilde{K}_B\overline{\delta\psi}_B\end{aligned}$$

so that there are neither displacement nor orientation variables present in the variations

- The kinetic energy per unit length is $K = K\left(V_B, \Omega_B\right)$
- Thus, the variation required in HWP is

$$\int_0^\ell \delta K dx_1 = \int_0^\ell \left[\delta V_B^T \left(\frac{\partial K}{\partial V_B}\right)^T + \delta \Omega_B^T \left(\frac{\partial K}{\partial \Omega_B}\right)^T\right] dx_1$$

- Introduce sectional linear and angular momenta, P_B and H_B, that are conjugate to the generalized speeds

$$P_B = \left(\frac{\partial K}{\partial V_B}\right)^T = m(V_B - \widetilde{\overline{\xi}}_B \Omega_B)$$

$$H_B = \left(\frac{\partial K}{\partial \Omega_B}\right)^T = i_B \Omega_B + m\widetilde{\overline{\xi}}_B V_B$$

- With the above definitions of virtual displacement and virtual rotation, the variations become

$$\delta V_B = \dot{\overline{\delta q}}_B + \widetilde{\Omega}_B \overline{\delta q}_B + \widetilde{V}_B \overline{\delta \psi}_B$$

$$\delta \Omega_B = \dot{\overline{\delta \psi}}_B + \widetilde{\Omega}_B \overline{\delta \psi}_B$$

which are, as with generalized strain variations, independent of displacement or orientation variables

- These are needed so that contributions of kinetic energy to equilibrium equations can be obtained without displacement or orientation variables in them

- The virtual work of external forces per unit length is

$$\overline{\delta W} = \int_0^\ell \left(\overline{\delta q}_B^T f_B + \overline{\delta \psi}_B^T m_B \right) dx_1$$

- The virtual action at the ends of the beam and of the time interval is

$$\overline{\delta \mathcal{A}} = \int_0^\ell \left(\overline{\delta q}_B^T \hat{P}_B + \overline{\delta \psi}_B^T \hat{H}_B \right) \Big|_{t_1}^{t_2} dx_1 - \int_{t_1}^{t_2} \left(\overline{\delta q}_B^T \hat{F}_B + \overline{\delta \psi}_B^T \hat{M}_B \right) \Big|_0^\ell dt$$

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_0^{\ell} \Bigg\{ & \left(\dot{\overline{\delta q}}_B^T - \overline{\delta q}_B^T \widetilde{\Omega}_B - \overline{\delta\psi}_B^T \widetilde{V}_B \right) P_B \\
& + \left(\dot{\overline{\delta\psi}}_B^T - \overline{\delta\psi}_B^T \widetilde{\Omega}_B \right) H_B \\
& - \left[\left(\overline{\delta q}_B' \right)^T - \overline{\delta q}_B^T \widetilde{K}_B - \overline{\delta\psi}_B^T \left(\widetilde{e}_1 + \widetilde{\gamma} \right) \right] F_B \\
& - \left[\left(\overline{\delta\psi}_B' \right)^T - \overline{\delta\psi}_B^T \widetilde{K}_B \right] M_B + \overline{\delta q}_B^T f_B + \overline{\delta\psi}_B^T m_B \Bigg\} dx_1\, dt \\
& = \int_0^{\ell} \left(\overline{\delta q}_B^T \hat{P}_B + \overline{\delta\psi}_B^T \hat{H}_B \right) \Big|_{t_1}^{t_2} dx_1 - \int_{t_1}^{t_2} \left(\overline{\delta q}_B^T \hat{F}_B + \overline{\delta\psi}_B^T \hat{M}_B \right) \Big|_0^{\ell} dt
\end{aligned}
$$

- The Euler-Lagrange equations from HWP are the geometrically-exact partial differential equations of motion

$$
\begin{aligned}
F_B' + \widetilde{K}_B F_B + f_B &= \dot{P}_B + \widetilde{\Omega}_B P_B \\
M_B' + \widetilde{K}_B M_B + \left(\widetilde{e}_1 + \widetilde{\gamma}\right) F_B + m_B &= \dot{H}_B + \widetilde{\Omega}_B H_B + \widetilde{V}_B P_B
\end{aligned}
$$

- HWP also leads to a consistent set of boundary conditions in which either force or moment can be specified or found at the ends of the beam
- These equations are
  - geometrically exact equations for the dynamics of a beam in a frame A whose inertial motion is arbitrary and known
  - identical to those of Reissner (1973) when specialized to the static case

- In order to finalize the development, we use the column matrix of Rodrigues parameters θ as orientation variables
- Now the direction cosine matrix can easily be expressed as

$$C = \frac{(1 - \frac{\theta^T\theta}{4})\Delta - \widetilde{\theta} + \frac{\theta\theta^T}{2}}{1 + \frac{\theta^T\theta}{4}}$$

- Similarly, κ and Ω are

$$\kappa = \left(\frac{\Delta - \frac{\widetilde{\theta}}{2}}{1 + \frac{\theta^T\theta}{4}}\right)\theta' + Ck_b - k_b$$

$$\Omega_B = \left(\frac{\Delta - \frac{\widetilde{\theta}}{2}}{1 + \frac{\theta^T\theta}{4}}\right)\dot{\theta} + C\omega_b$$

- The last two equations should be inverted

$$\theta' = \left(\Delta + \frac{1}{2}\widetilde{\theta} + \frac{1}{4}\theta\theta^T\right)(K_B - Ck_b)$$

and

$$\dot{\theta} = \left(\Delta + \frac{1}{2}\widetilde{\theta} + \frac{1}{4}\theta\theta^T\right)(\Omega_B - C\omega_b)$$

- Also, force strain and velocity equations should be inverted

$$u_b' = C^T(e_1 + \gamma) - e_1 - \widetilde{k}_b u_b$$

and

$$\dot{u}_b = C^T V_B - v_b - \widetilde{\omega}_b u_b$$

- The column matrices u_b, θ, γ, κ, V_B, Ω_B, F_B, M_B, P_B, and H_B are regarded as independent quantities
- Central differencing in the spatial variable x_1 has been shown to be equivalent to the mixed finite element formulation of Hodges (2006)
- Coefficient matrices are *very* sparse

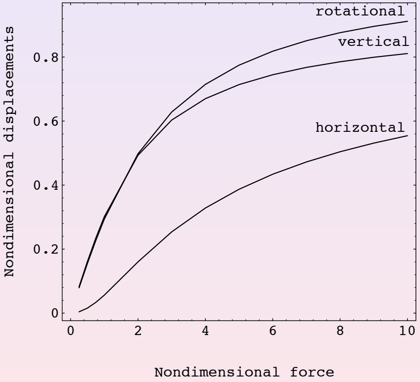

Figure: Large displacements of cantilever beam

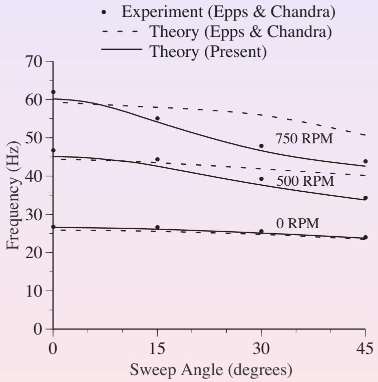

Figure: Frequency of the third bending mode for swept-tip beam

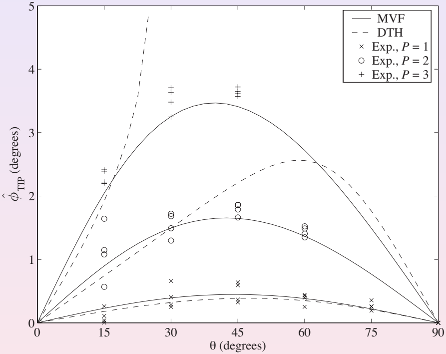

Figure: Princeton beam experiment, torsion for various P versus setting angle θ

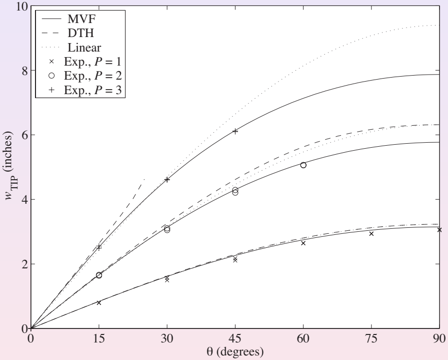

Figure: Princeton beam experiment, out-of-plane bending for various P versus setting angle θ

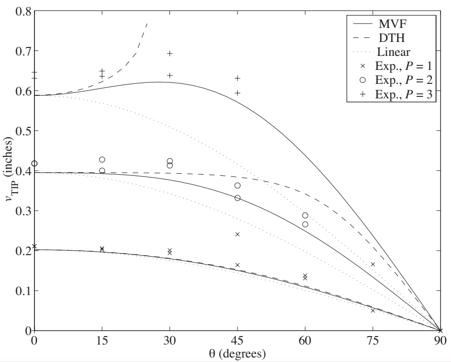

Figure: Princeton beam experiment, in-plane bending for various P versus setting angle θ

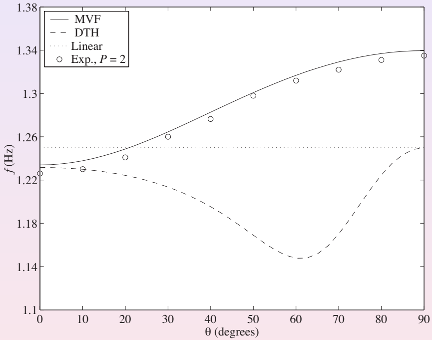

Figure: Princeton beam experiment, out-of-plane bending frequency versus P for various setting angles θ

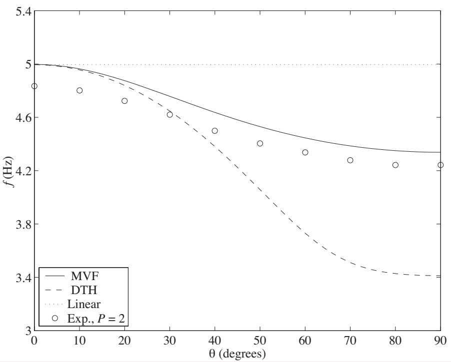

Figure: Princeton beam experiment, in-plane bending frequency versus P for various setting angles θ

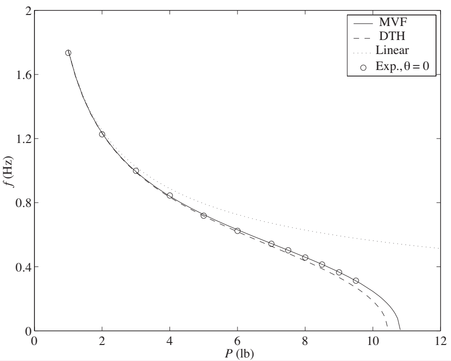

Figure: Princeton beam experiment, out-of-plane frequency versus P for setting angle $\theta = 0$

- The Euler-Lagrange equations are the kinematical, constitutive, and equations of motion
  - The kinematical equations are written exactly utilizing so-called intrinsic strain measures
  - The equations of motion are written exactly in their intrinsic form
  - The constitutive law is presumed given and is left in a generic form
  - The choice of displacement and rotational variables is localized in a relatively small portion of the analysis

- When specialized, the equations reduce to less general treatments in the literature
- Although the resulting equilibrium equations are identical to those derived from a Newtonian method, the formulation is variationally consistent
- The present development provides substantial insight into relationships among variational formulations as well as between these and Newtonian ones
- Coefficient matrices are very sparse and the computational efficiency can be improved by taking advantage thereof